ON THE ASYMPTOTIC BEHAVIOUR OF NONLINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction. In this paper we study the asymptotic behaviour of the following systems of ordinary differential equations:

$$\dot{x} = f(t, x), \qquad (N)$$

$$\dot{x} = f(t, x) + g(t, x), \qquad (P)$$

$$\dot{x} = f(t, x) + g(t, x) + h(t, x), \qquad (P_1)$$

where the identically zero function is a solution of (N), i.e. $f(t, 0) = 0$ for all time t. Suppose one knows that all the solutions of (N) which start near zero remain near zero for all future time, or even that they approach zero as time increases. For the perturbed systems (P) and (P_1) the above property concerning the solutions near zero may or may not remain true. A more precise formulation of this problem is as follows: if zero is stable or asymptotically stable for (N), and if the functions $g(t, x)$ and $h(t, x)$ are small in some sense, give conditions on $f(t, x)$ so that zero is (eventually) stable or asymptotically stable for (P) and (P_1).

A great deal of work has been done in an attempt to provide positive answers to this problem. Historically, there have been two approaches. One approach is to set conditions on f, such as being uniformly Lipschitz, and find out what kind of perturbations $g(t, x)$ and $h(t, x)$ preserve stability (e.g., [**5**, Chap. 13], [**14**]). The second approach is to set the kind of perturbations $g(t, x)$ and $h(t, x)$ that will be allowed and find out which differential equations (N) will have their asymptotic behaviour preserved by all such perturbations ([**9**], [**15**]). Most of the classical results usually assume that the perturbations $g(t, x)$ and $h(t, x)$ are either small as compared to x for sufficiently small x, or small for sufficiently large t and all $|x| < \infty$. One of the best standard results is Theorem 24.1 in Yoshizawa's book [**15**].

An indispensable tool for the study of perturbations is Liapunov's second method. The type of results obtained by using Liapunov functions is necessarily qualitative in nature. Quantitative estimates for solutions of perturbed systems can be obtained using other methods. For example, the practice of approximating nonlinear differential systems by a linear system which can be explicitly solved leads to the theory of perturbations of linear equations. For results of this type see Brauer and Wong [**4**] and Coddington and Levinson [**5**].

Brauer ([**2**], [**3**]) has obtained results on the asymptotic behaviour of solutions of nonlinear systems and their perturbations by means of an analogue of the variation of constants formula for nonlinear systems due to Alekseev. Also see [**1**] for other results of this type and recent references.

The purpose of this paper is to give sufficient conditions for asymptotic stability of solutions of (P) and (P_1) in terms of a concept known as total stability [**11**], or alternatively, stability under the constantly acting disturbances of the Soviet mathematicians [**9**]. We use Liapunov theory to obtain admissible classes of perturbations for which the solutions of (P) and (P_1) are asymptotically stable. More precisely, we prove: if the zero solution of (N) is totally stable, $g(t, x)$ tends to zero as $t \to \infty$ uniformly on any compact subset of R^n and $h(t, x)$ is integrable for any bounded x, then the solutions of (P) and (P_1) tend to zero as $t \to \infty$. Furthermore, we give a sufficient condition which will guarantee, in addition to total stability, that the zero solution of (N) be uniformly asymptotically stable. As a consequence of this result two corollaries follow: one concerning the reciprocity between the total and uniform asymptotic stability of perturbed linear systems, and another concerning this phenomenon of linear systems (Theorem 28 of Massera's famous paper [**11**]).

2. Notations and preliminaries. Let R^n denote Euclidean n-space and let $|\cdot|$ denote any n-dimensional norm. In (N), x and f are elements of R^n, t is a real scalar and $f(t, 0) = 0$ for all $t \geq 0$. The following convention is adopted in the paper: every differential equation which we consider shall have a right-hand side which is continuous and sufficiently smooth on the semi-cylinder

$$D_r = \{(t, x) : t \geq 0,\ x \in R^n,\ |x| < r,\ r > 0\}$$

for local existence and uniqueness of all solutions. For $(t_0, x_0) \in D_r$ let $x(t, t_0, x_0)$ be that solution (of the equation being considered) for which $x(t_0, t_0, x_0) = x_0$. It is interesting to note that the mere existence of a Liapunov function for (N) actually implies that the zero function must be a solution and that it is unique in the following sense: if a solution becomes zero at some point, it remains zero thereafter. See [**15**, Chap. 1]. We assume the following conditions throughout this paper:

(H1) $f(t, x)$, $g(t, x)$ and $h(t, x)$ are defined and continuous on D_r and $f(t, 0) = 0$ for all $t \geq 0$.

(H2) $|f(t, x)|$ is bounded for all $t \geq 0$ and x in any compact subset of the set $\{x \in R^n : |x| < r,\ r > 0\}$.

(H3) $g(t, x) \to 0$ as $t \to \infty$ uniformly for x in any compact subset of the set $\{x \in R^n : |x| < r, r > 0\}$.

(H4) $|h(t, x)| \leq \theta(t)$ for all $(t, x) \in D_r$ where $\theta \in L_1[0, \infty)$.

In order for this paper to be self-contained, a few necessary definitions shall be stated here.

DEFINITION 2.1. The zero solution of (N) is

(a) stable if for every $\varepsilon > 0$ and every $t_0 \geq 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that $|x(t, t_0, x_0)| < \varepsilon$ for all $t \geq t_0$ whenever $|x_0| < \delta(\varepsilon, t_0)$,

(b) uniformly stable if (a) holds with $\delta = \delta(\varepsilon)$,

(c) asymptotically stable if (a) holds and for every $t_0 \geq 0$ there exists $\eta(t_0) > 0$ such that $\lim x(t, t_0, x_0) = 0$ as $t \to \infty$ whenever $|x_0| < \eta$,

(d) uniformly asymptotically stable if (b) holds and there is $\delta_0 > 0$ such that for every $\varepsilon > 0$ and every $t_0 \geq 0$, there exists $T(\varepsilon) > 0$, independent of t_0, such that $|x(t, t_0, x_0)| < \varepsilon$ for all $t \geq t_0 + T(\varepsilon)$ whenever $|x_0| < \delta_0$,

(e) totally stable if for every $\varepsilon > 0$ and every $t_0 \geq 0$ two positive numbers $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ can be found such that for every solution $x(t, t_0, x_0)$ of (P) the inequality $|x(t, t_0, x_0)| < \varepsilon$ holds for all $t \geq t_0$ whenever $|x_0| < \delta_1$ and $|g(t, x)| < \delta_2$.

DEFINITION 2.2. A continuous function φ defined on $[0, r)$ is said to belong to class K if $\varphi(0) = 0$ and if φ is strictly increasing.

DEFINITION 2.3. A real function $V(t, x)$ defined on D_r is positive definite if there exists $\varphi \in K$ such that $V(t, x) \geq \varphi(|x|)$ on D_r. V is negative definite if $-V$ is positive definite.

DEFINITION 2.4. $V(t, x)$ has an infinitely small upper bound (is decrescent) if there exists $\psi \in K$ such that $V(t, x) \leq \psi(|x|)$ on D_r.

DEFINITION 2.5. $V(t, x)$ is a Liapunov function for (N) on D_r if and only if

(1) $V(t, x)$ is positive definite and C^1 on D_r,
(2) $V(t, 0) = 0$ for all $t \geq 0$,
(3) $\dot{V}_{(N)}(t, x) = (\partial/\partial t) V(t, x) + \operatorname{grad} V(t, x) \,.\, f(t, x)$ is negative definite on D_r,
(4) $|\operatorname{grad} V(t, x)|$ is bounded on D_r.

Notice that from (2), (4), and by the mean value theorem it follows that $V(t, x)$ is decrescent on D_r.

In order to illustrate and motivate the nature of results obtained as well as for later use we reproduce some known results. Markus **[10]** considered the differential systems

$$\dot{x} = f(x), \tag{2.1}$$

$$\dot{x} = f(x) + g(t, x), \tag{2.2}$$

where f is continuous and has continuous first partial derivatives in the domain $\{x \in R^n : |x| < r,\ r > 0\}$, and g is continuous on D_r and satisfies (H3). He introduced the concepts of "limiting equations" and "asymptotically autonomous equations" and proved, among other results, the following theorem.

THEOREM 2.6. *Let x^* be a critical point of* (2.1) *and assume that the variational equations of* (2.1) *based on x^* have their eigenvalues all with negative real parts. Then there exist positive numbers η and T such that for any solution $x(t, t_0, x_0)$ of* (2.2) *for which $|x_0 - x^*| < \eta$ for $t_0 \geq T$, one has $x(t, t_0, x_0) \to x^*$ as $t \to \infty$.*

The total stability problem of the zero solution of (N) was first examined in a paper of Duboshin **[6]** and in subsequent works of Malkin **[8]**, **[9]** and Gorshin **[7]**. The following theorem is due to Malkin **[8]**, **[9]**.

THEOREM 2.7. *Let $f(t, x)$ satisfy* (H1) *and assume there exists a Liapunov function $V(t, x)$ for (N) on D_r. Then the zero solution of (N) is totally stable.*

3. Main results. It is well known that the techniques which are used for the study of the behaviour of solutions of ordinary differential equations in the vicinity of a given solution fall roughly into three categories: the method of linearization, Liapunov functions, and the method of isolating blocks. It is the second of these methods which is of interest here. In this section we first generalize the aforesaid result of Markus to the system (P) and then prove an analogous result for the system (P_1) which has an additional integrable perturbation.

Let us analyze the result of Markus to see how it might be generalized. Let x^* be a critical point of (2.1) satisfying the hypothesis of Theorem 2.6. The variational equation of (2.1) based on x^* has the form $\dot{y} = Ay$, $A = f_x(x^*)$, where $f_x(x^*)$ is the constant matrix whose element in the ith row and jth column is $\partial f_i/\partial x_j(x^*)$. Since all eigenvalues of A have negative real parts then the zero solution of $\dot{y} = Ay$ is asymptotically stable and by Liapunov theory [**11**] there exists a positive definite Liapunov function $V(x)$ whose derivative $\dot{V}_{(2.1)}(x)$ is negative definite; V may be taken as an algebraic form of any given even degree. Clearly, if $y \equiv 0$ is a stable or asymptotically stable solution of $\dot{y} = Ay$, then the same will be true of the critical point x^* of (2.1), as can be seen by applying the definition of stability or asymptotic stability to the solution $x \equiv x^*$ of (2.1). Since $\dot{y} = Ay$ is autonomous, stability implies uniform stability and asymptotic stability implies uniform asymptotic stability [**11**].

THEOREM 3.1. *Let $f(t, x)$ and $g(t, x)$ satisfy hypotheses* (H1) *and* (H3). *Let $V(t, x)$ be a Liapunov function for (N) on D_r. Then there exist $T_0 \geq 0$ and $\delta_0 > 0$ such that the solution $x(t, t_0, x_0)$ of (P) satisfies $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$ whenever $t_0 \geq T_0$ and $|x_0| < \delta_0$.*

Proof. Define

$$M = \max_{1 \leq i \leq n} \sup\left\{\left|\frac{\partial V}{\partial x_i}(t, x)\right|, (t, x) \in D_r\right\}.$$

For $\varepsilon = r/2$ choose numbers $\delta_1 > 0$ and $\delta_2 > 0$ by Theorem 2.7. Using (H3) we may pick a number $T_0 > 0$ so large that $|g(t, x)| < \delta_2$ for all $t \geq T_0$ and $|x| \leq r/2$. We assert that with this T_0 and $\delta_0 = \delta_1$ the conclusion of the theorem is true.

Let $x(t, t_0, x_0)$ be a local solution of (P) such that $|x_0| < \delta_0$ for some $t_0 \geq T_0$. From the definition of T_0 and δ_0 it follows that this solution exists and $|x(t, t_0, x_0)| < r/2$ for all $t \geq t_0$. We now show that $\liminf |x(t, t_0, x_0)| = 0$ as $t \to \infty$. Let us assume that this is false. Then there is $\varepsilon \in (0, r/2)$ and $T \geq 0$ such that $|x(t, t_0, x_0)| \geq \varepsilon$ for all $t \geq T$. Since $\dot{V}_{(N)}(t, x)$ is negative definite we have

$$m = \inf\{-\dot{V}_{(N)}(t, x) : \varepsilon \leq |x| \leq r/2\} > 0.$$

The number T may be taken so large that $|g(t, x)| < m/(2nM)$ for all $t \geq T$ and $|x| \leq r/2$.

Then, for $t \geq T$, we get

$$\dot{V}_{(P)}(t, x(t)) = \dot{V}_{(N)}(t, x(t)) + \operatorname{grad} V(t, x(t)) \,.\, g(t, x(t))$$
$$\leq -m + nmM/(2nM) = -\frac{m}{2}.$$

Hence, if $t \geq T$ and t is sufficiently large, we would have after integrating

$$V(t, x(t)) \leq V(T, x(T)) - \frac{m}{2}(t - T) < 0.$$

This yields a contradiction to the assumed positive definiteness of $V(t, x)$ and proves the assertion.

We now show that $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$. Let ε be any positive number and pick the positive numbers $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ by Malkin's theorem. Since $\liminf |x(t, t_0, x_0)| = 0$ as $t \to \infty$ we may pick a positive number $t_1 = t_1(\varepsilon)$ such that $|x(t_1)| < \delta_1(\varepsilon)$ and $|g(t, x)| < \delta_2(\varepsilon)$ for all $t \geq t_1$ and $|x| \leq \varepsilon$. From the definition of $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ we see that $|x(t, t_0, x_0)| < \varepsilon$ for all $t \geq t_1$. Since ε is arbitrarary we obtain that $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$, and the proof is complete.

We shall now prove an analogous result for systems of the form (P_1).

THEOREM 3.2. *Let $f(t, x)$, $g(t, x)$ and $h(t, x)$ satisfy hypotheses* (H1) *through* (H4). *Let $V(t, x)$ be a Liapunov function for (N) on D_r. Then there exist $T_0 \geq 0$ and $\delta_0 > 0$ such that if $t_0 \geq T_0$ and $|x_0| < \delta_0$, the solution $x(t, t_0, x_0)$ of (P_1) satisfies $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$.*

Proof. For $0 < r_0 < r/2$ we define

$$2k = \inf\{V(t, x) : t \geq 0,\ r_0 \leq |x| \leq r/2\}.$$

Since $V(t, x)$ is positive definite on D_r it follows that $k > 0$. On the other hand, $V(t, x)$ admits an infinitely small upper bound on D_r (see the sentence after Definition 2.5) and then we may choose a positive number $\rho < r_0$ such that $V(t, x) < k$ on the semi-cylinder $D_\rho = \{(t, x) : t \geq 0,\ |x| < \rho\}$. Further we define

$$m = \inf\{-\dot{V}_{(N)}(t, x) : t \geq 0,\ \rho \leq |x| \leq r_0\}$$

and

$$M = \max_{1 \leq i \leq n} \sup\left\{\left|\frac{\partial V}{\partial x_i}(t, x)\right| : (t, x) \in D_r\right\}.$$

Choose $T > 0$ so large that $|g(t, x)| < m/(nM)$ for all $t \geq T$ and $|x| \leq r_0$. By hypothesis (H4) there is a function $\theta(t) \in L_1[0, \infty)$ such that $|h(t, x)| \leq \theta(t)$ on D_{r_0}. The number T can be chosen so large that $nM \int_T^\infty \theta(t)\, dt < k$. We claim that if $x(t, t_0, x_0)$ is a local solution of (P_1) with $|x_0| < \rho$ for $t_0 \geq T$ then this solution exists for all $t \geq t_0$ and, moreover, $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$.

We shall show that $x(t, t_0, x_0)$ exists and $|x(t, t_0, x_0)| < r_0$ for all $t \geq t_0$. Suppose that this is not true. Then there are positive numbers t_1 and t_2, $t_0 < t_1 < t_2$, such that $|x(t_1, t_0, x_0)| =$

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ρ, $|x(t_2, t_0, x_0)| = r_0$, and $\rho < |x(t, t_0, x_0)| < r_0$ for $t \in (t_1, t_2)$. We have

$$\dot{V}_{(P_1)}(t, x(t)) = \dot{V}_{(N)}(t, x(t)) + \operatorname{grad} V(t, x(t)).\ (g(t, x(t)) + h(t, x(t)))$$

from which it follows that

$$\dot{V}_{(P_1)}(t, x(t)) \le -m + nM(|g(t, x(t))| + \theta(t)) < nM\theta(t).$$

By integration over the interval $[t_1, t]$ we get

$$\begin{aligned} V(t, x(t)) &\le V(t_1, x(t_1)) + nM\int_{t_1}^{t} \theta(\tau)\, d\tau \\ &< V(t_1, x(t_1)) + k \end{aligned}$$

which by the continuity of $V(t, x(t))$ at $t = t_2$ implies

$$k \le V(t_2, x(t_2)) - V(t_1, x(t_1)) < k,$$

a contradiction. Therefore $|x(t, t_0, x_0)| < r_0$ for $t \ge t_0$.

We shall now show that $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$. If the assertion is false then there is $\varepsilon > 0$ and a sequence $\{t_j\} \to \infty$ as $j \to \infty$ such that $|x(t_j, t_0, x_0)| \ge 2\varepsilon$. Define

$$R = \sup\{|f(t, x)| + |g(t, x)| : (t, x) \in D_{r_0}\}.$$

From

$$x(t) = x(t_j) + \int_{t_j}^{t} \{f(\tau, x(\tau)) + g(\tau, x(\tau)) + h(\tau, x(\tau))\}\, d\tau$$

we obtain

$$|x(t) - x(t_j)| \le R(t - t_j) + \int_{t_j}^{t} \theta(\tau)\, d\tau.$$

Let $t'_j = t_j + \varepsilon/(2R)$, and assume that $t'_j < t_{j+1}$, for $j = 1, 2, \ldots$, and $\int_{t_1}^{\infty} \theta(\tau)\, d\tau < \varepsilon/2$. Then, if $t_j \le t \le t'_j$, from the above inequality it follows that $|x(t) - x(t_j)| < \varepsilon$ and $|x(t)| \ge \varepsilon$. Define

$$m_1 = \inf\{-\dot{V}_{(N)}(t, x) : t \ge 0, |x| \ge \varepsilon\},$$

and suppose t_1 large enough so that $|g(t, x)| < m_1/(2nM)$ for all $t \ge t_1$ and $|x| \le r_0$. Therefore, for $t_j \le t \le t'_j$, we obtain

$$\dot{V}_{(P_1)}(t, x(t)) \le -\frac{m_1}{2} + nM\theta(t).$$

Integrating both sides over the interval $[t_j, t'_j]$ we get

$$\begin{aligned} V(t'_j, x(t'_j)) &= V(t_1, x(t_1)) + \int_{t_1}^{t'_j} V_{(N)}(\tau, x(\tau))\, d\tau \\ &\le V(t_1, x(t_1)) + \sum_{i=1}^{j} \int_{t_i}^{t'_i} \left(-\frac{m_1}{2}\right) d\tau + \int_{t_1}^{\infty} nM\theta(\tau)\, d\tau \\ &\le V(t_1, x(t_1)) - jm_1\varepsilon/(4R) + nM\int_{t_1}^{\infty} \theta(\tau)\, d\tau < 0 \end{aligned}$$

for j sufficiently large. We have contradicted $V(t, x)$ positive definite and the proof of the theorem is complete.

EXAMPLE. As an example of an application of Theorem 3.2 we consider the following scalar differential equation

$$\dot{x} = -x^{2k+1} + g(t, x) + h(t, x) \tag{3.1}$$

where $k = 0, 1, 2, \ldots, g(t, x) \to 0$ as $t \to \infty$ uniformly on compact subsets of R^1 and $h(t, x)$ satisfies (H4). The function $V(x) = x^2$ is a Liapunov function for the equation

$$\dot{x} = -x^{2k+1}. \tag{3.2}$$

From Theorem 3.2 it follows that there exist positive numbers δ_0 and T_0 such that any solution $x(t, t_0, x_0)$ of (3.1) with $|x_0| < \delta_0$ and $t_0 \geq T_0$ must tend to zero as $t \to \infty$. In fact, from the proof of this theorem we see that for each $l > 0$ there exists a number $t(l) > 0$ such that for any solution $x(t, t_0, x_0)$ of (3.1) with $|x_0| < l$ for any $t_0 \geq t(l)$ one has $x(t, t_0, x_0) \to 0$ as $t \to \infty$.

Some remarks concerning the above theorems are in order here.

REMARK 1. The hypotheses of Theorem 2.7 do not imply that any solution $x(t, t_0, x_0)$ of (N) tends to zero as $t \to \infty$. In fact, $g(t, x)$ does not vanish, nor does it diminish as $t \to \infty$ (see Definition 2.1(e)). However, some kind of asymptotic property can be proved. Namely, Malkin [**9**, Chap. 6] has showed that in the hypotheses of Theorem 2.7 if $|x_0|$ and $|g(t, x)|$ are sufficiently small then the solutions of (P) remain inside an arbitrarily small neighbourhood of the origin although they do not tend to zero asymptotically. Under stronger hypotheses (H3) (Theorem 3.1) and (H4) (Theorem 3.2) we were able to prove that the solutions of (P) and (P_1), respectively, tend to zero as $t \to \infty$.

REMARK 2. Theorems 3.1 and 3.2 should be compared with Theorem 24.1 of Yoshizawa [**15**] which requires exponential asymptotic stability of the zero solution of (N) and with Theorem 4.1 of Strauss and Yorke [**14**] which requires uniform asymptotic stability of the zero solution of (N). Each of their results requires a stronger condition on the function $f(t, x)$ of (N) than do Theorems 3.1 and 3.2. Our hypotheses on $f(t, x)$ are rather weak conditions, and the hypotheses of Theorems 3.1 and 3.2 require the total stability of the zero solution of (N) but not exponentially or uniform asymptotic stability. However, the results of Yoshizawa, Strauss and Yorke allow a slightly larger class of perturbation terms than is permitted by Theorems 3.1 and 3.2.

REMARK 3. The most significant theorem on total stability, found independently by Malkin [**8**] and Gorshin [**7**], states that, if $f(t, x)$ in (N) is Lipschitzian in x uniformly with respect to t on D_r and if the zero solution of (N) is uniformly asymptotically stable, then it is totally stable. Unfortunately, this theorem does not admit a reciprocal. For example, consider the scalar differential equation

$$\dot{x} = f(x),$$

where $f(0)=0$, and $f(x)=-x^2\sin(1/x)$ for $x\neq 0$. The zero solution of this equation is totally stable but not uniformly asymptotically stable. For another example, see Massera [**11**, Erratum]. It is worth noting that Seibert [**13**] proved that in the autonomous case, total stability occurs if and only if the origin of R^n has a fundamental system of compact contracting neighbourhoods. This result was extended to periodic equations and to equations that possess a periodic limiting equation by Salvadori and Schiaffino [**12**].

If the zero solution of the system

$$\dot{x}=A(t)x, \tag{3.4}$$

where $A(t)$ is $n\times n$ continuous matrix for $t\geq 0$, is uniformly asymptotically stable, that is, exponentially asymptotically stable, then there exists a Liapunov function $V(t,x)$ satisfying the conditions in Definition 2.5 [**11**]. Therefore, by Theorem 2.7, the zero solution is totally stable.

Finally in this paper, under some additional mild condition on $f(t,x)$ in (N), we are able to obtain a partially converse theorem to the theorem of Malkin and Gorshin. More precisely, we have the following.

THEOREM 3.3. *Let* $f(t,x)$ *satisfy* (H1) *and let*

$$\lim_{\alpha\to 0^+} e^{\alpha(t-t_0)}f(t, e^{-\alpha(t-t_0)}x)=f(t,x) \tag{3.5}$$

uniformly for $t_0\geq 0$, $t\geq t_0$ *and* $|x|<r$. *Then the zero solution of* (N) *is uniformly asymptotically stable if it is totally stable.*

Proof. Let the zero solution of (N) be totally stable. Then, for every $\varepsilon>0$ and $t_0\geq 0$, we can pick $\delta_1(\varepsilon)>0$ and $\delta_2(\varepsilon)>0$ such that every solution $x(t,t_0,x_0)$ of (P) satisfies $|x(t,t_0,x_0)|<\varepsilon$ for $t\geq t_0$, whenever $|x_0|<\delta_1$ and $|g(t,x)|<\delta_2$. Let $\alpha=\alpha(\varepsilon)$ be such that $0<\alpha<\delta_2/(2\varepsilon)$ and

$$|e^{\alpha(t-t_0)}f(t,e^{-\alpha(t-t_0)}x)-f(t,x)|<\frac{\delta_2}{2} \tag{3.6}$$

for $t\geq t_0\geq 0$ and $|x|\leq r$. This is possible because of (3.5). Define

$$g(t,x)=e^{\alpha(t-t_0)}f(t,e^{-\alpha(t-t_0)}x)-f(t,x)+\alpha x. \tag{3.7}$$

Then, from (3.6) and by the definition of α, it follows that $|g(t,x)|<\delta_2$ for $t\geq t_0$ and $|x|<\varepsilon$. Further, for $t_0\geq 0$, we consider the system

$$\dot{x}=e^{\alpha(t-t_0)}f(t,e^{-\alpha(t-t_0)}x)+\alpha x \tag{P_α}$$

which because of (3.7) can be written in the form

$$\dot{x}=f(t,x)+g(t,x). \tag{P}$$

But the solutions $x(t,t_0,x_0)$ of (N) and $x_\alpha(t,t_0,x_0)$ of (P_α) are related by $x_\alpha(t,t_0,x_0)=e^{\alpha(t-t_0)}x(t,t_0,x_0)$, where $|x_\alpha(t_0)|=|x_0|<\delta_1$. Thus we have $|x(t,t_0,x_0)|\leq\varepsilon e^{-\alpha(t-t_0)}$, which proves the uniform asymptotic stability of the zero solution of (N), completing the proof.

As a consequence of the above theorem, we infer a useful corollary, whose proof is straightforward.

COROLLARY 3.4. *Assume* $f(t, x) = A(t)x + p(t, x)$, *in* (N) *where* $A(t)$ *is as in* (3.4), $p(t, x)$ *is defined and continuous on* D_r *and, moreover, there exists a continuous and nonnegative function* $\lambda(t)$ *defined for* $t \geq 0$, *such that* $\lambda(t) \to 0$ *as* $t \to \infty$, *and* $|p(t, x)| \leq \lambda(t)|x|$ *on* D_r. *Then the zero solution of*

$$\dot{x} = A(t)x + p(t, x)$$

is uniformly asymptotically stable if it is totally stable.

From this corollary we obtain a well known result [**11**, Theorem 28]; namely, we have

COROLLARY 3.5. *If the zero solution of* (3.4) *is totally stable, then it is uniformly asymptotically stable.*

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REFERENCES

1. Zh. S. Athanassov, Perturbation theorems for nonlinear systems of ordinary differential equations. *J. Math. Anal. Appl.* **86** (1982), 194–207.

2. F. Brauer, Perturbations of nonlinear systems of differential equations, I. *J. Math., Anal. Appl.* **14** (1967), 198–206.

3. F. Brauer, Perturbations of nonlinear systems of differential equations, II. *J. Math. Anal. Appl.* **17** (1967), 418–434.

4. F. Brauer and J. S. W. Wong, On the asymptotic relationships between solutions of two systems of ordinary differential equations, *J. Differential Equations* **6** (1969), 527–543.

5. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations* (McGraw-Hill, 1965).

6. G. N. Duboshin, On the problem of stability of a motion under constantly acting perturbations. *Trudy Gos. Astron. Inst. P. K. Sternberg* **14** (1940).

7. S. I. Gorshin, On the stability of motion under constantly acting perturbations. *Izv. Akad. Nauk Kazakh. SSR. Ser. Mat. Mekh.* (2), **56** (1948), 46–73.

8. I. G. Malkin, Stability in the case of constantly acting disturbances. *Prikl. Mat. Mekh.* **8** (1944), 241–245.

9. I. G. Malkin, *Theory of stability of motion, second edition* (Nauka, 1966).

10. L. Markus, Asymptotically autonomous differential systems. *Contributions to the theory of nonlinear oscillations*, Annals of Mathematics Studies No. 36 (Princeton University Press, 1956), 17–29.

11. J. L. Massera, Contributions to stability theory. *Ann. of Math.* **64** (1956), 182–206; Erratum, **68** (1956), 202.

12. L. Salvadori and A. Schiaffino, On the problem of total stability. *Nonlinear Anal.* **1** (1977), 207–213.

13. P. Seibert, Estabilidad bajo perturbaciones sostenidas y su generalizacion en flujos continuos. *Acta Mexicana Cienc. Tecn.* **11** (1968), 154–165.

14. A. Strauss and J. A. Yorke, Perturbation theorems for ordinary differential equations. *J. Differential Equations* **3** (1967), 15–30.

15. T. Yoshizawa, *Stability theory by Liapunov's second method* (Math. Soc. Japan, 1966).

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