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RESEARCH ARTICLE

KMS states on $C_c^*(\mathbb{N}^2)$

Anbu Arjunan[1], Sruthymurali[2] and S. Sundar[3,*]

[1]Indian Statistical Institute, Delhi Centre, NewDelhi, 110016, India,
[2]Indian Statistical Institute, Bangalore Centre, Bangalore, 560059, India,
[3]Institute of Mathematical Sciences (HBNI), Chennai, 600113, India
*Corresponding author. E-mail: sundarsobers@gmail.com

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Abstract
Let $C_c^*(\mathbb{N}^2)$ be the universal C^*-algebra generated by a semigroup of isometries $\{v_{(m,n)} : m, n \in \mathbb{N}\}$ whose range projections commute. We analyse the structure of KMS states on $C_c^*(\mathbb{N}^2)$ for the time evolution determined by a homomorphism $c : \mathbb{Z}^2 \to \mathbb{R}$. In contrast to the reduced version $C_{red}^*(\mathbb{N}^2)$, we show that the set of KMS states on $C_c^*(\mathbb{N}^2)$ has a rich structure. In particular, we exhibit uncountably many extremal KMS states of type I, II and III.

1. Introduction

In the recent years, there has been a flurry of activity centred around understanding the inner structure (like K-theory, KMS states *etc*.) of C^*-algebras associated to semigroups. The revival of the subject of semigroup C^*-algebras, especially in the last decade, can be attributed to the works of Cuntz and Li on C^*-algebras associated with rings ([6], [9], [10]). Ring C^*-algebras are defined exactly like group C^*-algebras taking into account both the addition and the multiplication rule of the ring, where the ring involved is usually assumed to be an integral domain. Soon, it was realised that it is only appropriate to view these algebras as boundary quotients of semigroup C^*-algebras.

Let P be a cancellative semigroup. Let the left regular representation of P on $\ell^2(P)$ be denoted by $V := \{V_a\}_{a\in P}$. The reduced C^*-algebra of the semigroup P, denoted $C_{red}^*(P)$, is the C^*-algebra generated by $\{V_a : a \in P\}$. Li also defines a universal version, denoted $C^*(P)$, generated by isometries $\{v_a : a \in P\}$ and projections corresponding to 'certain ideals of P' satisfying relations that reflect the relations in the regular representation. The study of $C_{red}^*(P)$ has received much attention in the recent years. Questions concerning its nuclearity, K-theory and the structure of KMS states on $C_{red}^*(P)$ were investigated intensively and satisfactory answers were obtained for a large class of semigroups. Some of the notable papers that explore the above mentioned issues are [8], [17], [16], [7], [15], [14] and [4].

However, isometric representations of semigroups other than the regular representation were also considered in the literature, and the associated C^*-algebras were analysed. Let G be a discrete, countable, abelian group, and let $P \subset G$ be a subsemigroup containing the identity element 0. Here is a host of examples of isometric representations of P. Let $A \subset G$ be a non-empty set such that $P + A \subset A$. Consider the Hilbert space $\ell^2(A)$, and let $\{\delta_x : x \in A\}$ be its standard orthonormal basis. For $a \in P$, let V_a be the isometry on $\ell^2(A)$ defined by

$$V_a(\delta_x) := \delta_{a+x}.$$

Then, $V^A := \{V_a\}_{a\in P}$ is an isometric representation of P, which we call the isometric representation associated to A. Denote the C^*-algebra generated by $\{V_a : a \in P\}$ by $C^*(P, A)$. The case $A = P$ corresponds to the reduced C^*-algebra. The C^*-algebras $C^*(P, A)$ were analysed in great detail in [24] when $P = \mathbb{N}^k$.

A common feature that the isometric representations V^A share, when we vary A, is that the range projections $\{V_aV_a^* : a \in P\}$ form a commuting family. Thus, it is natural to consider the following universal C^*-algebra. Let $C_c^*(P)$ be the universal C^*-algebra generated by isometries $\{v_a : a \in P\}$ such that

1. for $a, b \in P$, $v_av_b = v_{a+b}$, and
2. for $a, b \in P$, $e_ae_b = e_be_a$. Here, $e_a := v_av_a^*$ and $e_b := v_bv_b^*$.

The subscript 'c' in $C_c^*(P)$ stands to indicate that the range projections commute.

The C^*-algebra $C_c^*(\mathbb{N}^k)$ was analysed from a groupoid perspective by Salas in [24]. Murphy, in [19] by independent methods, studied $C_c^*(\mathbb{N}^2)$ and also the universal one generated by two commuting isometries. He proved that the latter C^*-algebra is complicated, in particular not nuclear, while the former is nuclear and more manageable. The C^*-algebra $C_c^*(P)$ for a numerical semigroup P was considered in [23] and in [26]. The analog of $C_c^*(P)$ in the topological setting was investigated using groupoid methods by the last author in [25]. Both [24] and [25] borrow substantial amount of material from [18].

This paper is an attempt to understand the structure of KMS states on $C_c^*(P)$ for a natural time evolution. Let $c : G \to \mathbb{R}$ be a non-zero homomorphism. By the universal property of $C_c^*(P)$, for every $t \in \mathbb{R}$, there exists a *-homomorphism $\sigma_t : C_c^*(P) \to C_c^*(P)$ such that

$$\sigma_t(v_a) = e^{itc(a)}v_a,$$

for $a \in P$. Then, $\sigma^c := \{\sigma_t\}_{t\in\mathbb{R}}$ defines an action of $\mathbb{R}$ on $C_c^*(P)$. In this paper, we analyse the structure of KMS states for σ^c for a toy model by discussing the case when $P = \mathbb{N}^2$.

Let $c : \mathbb{Z}^2 \to \mathbb{R}$ be a non-zero homomorphism. With no loss of generality, we can assume that $c(1, 0) = 1$ and $c(0, 1) = \theta$. Let $\Omega := \{0, 1\}^{\mathbb{Z}}$ be the Cantor space equipped with the product topology. Let $\tau : \Omega \to \Omega$ be the Bernoulli shift defined by $\tau(x)_k = x_{k-1}$. Define $\chi : \Omega \to \mathbb{R}$ by

$$\chi(x) := \begin{cases} 1 & \text{if } x_{-1} = 0, \\ -\theta & \text{if } x_{-1} = 1. \end{cases}$$

Let $\rho := \{\rho_t\}_{t\in\mathbb{R}}$ be the 1-parameter group of automorphisms on $C(\Omega) \rtimes \mathbb{Z}$ defined by $\rho_t(f) = f$ for $f \in C(\Omega)$ and $\rho_t(u) = ue^{it\chi}$. Here, u is the canonical unitary of the crossed product $C(\Omega) \rtimes \mathbb{Z}$. This kind of flow, given by a continuous potential $F : X \to \mathbb{R}$, for a generic dynamical system (X, T), where X is a compact space and $T : X \to X$ is a homeomorphism, was considered in [5], and the structure of KMS states on $C(X) \rtimes \mathbb{Z}$ was analysed in depth in [5].

We establish a close link between the structure of KMS states on $C_c^*(\mathbb{N}^2)$ for σ^c and the structure of KMS states on $C(\Omega) \rtimes \mathbb{Z}$ for ρ. Strictly speaking, we only establish a bijective correspondence between the conformal measures that appear in the analysis. This brings us to the setup considered in [5]. We then apply the results of [5] to deduce the values of θ for which there is a KMS state on $C_c^*(\mathbb{N}^2)$. Taking inspiration from [5] and like in [5], we ask whether extremal KMS states on $C_c^*(\mathbb{N}^2)$ of each type t for $t \in \{I, II, III\}$ appear or not which we answer in the affirmative by exhibiting uncountably many examples of extremal KMS states of type I, II and III.

The results obtained in this paper are summarised below.

(1) If either $\theta < 0$ or $\beta < 0$, there is no β-KMS state on $C_c^*(\mathbb{N}^2)$.
(2) Suppose $\theta = 0$ and $\beta > 0$. The simplex of β-KMS states is homeomorphic to the set of probability measures on $\mathbb{T}$. In this case, the extremal β-KMS states are of type I.
(3) Suppose θ is irrational, positive and $\beta > 0$. The simplex of β-KMS states is homeomorphic to the set of probability measures m on the Cantor space Ω that are $e^{-\beta\chi}$-conformal, i.e.

$$m(\tau(B)) = \int_B e^{-\beta\chi}\, dm$$

for every Borel subset B of Ω.

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(4) Suppose $\theta > 0$ and $\beta > 0$. For each $t \in \{I, II, III\}$, there are uncountably many extremal β-KMS states of type t. Moreover, for $t \in \{II, III\}$, the extremal β-KMS states of type t are in bijective correspondence with non-atomic type t ergodic, probability measures m on Ω that are $e^{-\beta\chi}$-conformal.
(5) The simplex of tracial states of $C_c^*(\mathbb{N}^2)$ is homeomorphic to the simplex of probability measures on the torus $\mathbb{T}^2$.

We end this introduction by mentioning a result obtained in Section 4, which we believe is worth highlighting. Let X be a compact metric space, and let $\phi : X \to X$ be a homeomorphism. A probability measure m on X is said to be a conformal measure with potential F, where F is a measurable function on X, if

$$\frac{d(m \circ \phi)}{dm} = e^F.$$

Up to the authors' knowledge, the reference in the literature for the proof of the existence of an ergodic type III conformal measure with continuous potential F is the work of Katznelson ([13]). (The reader is referred to Theorem 3.2, Theorem 3.3 of Part II in [13] and is also referred to Theorem 5.2 of [5]. Also, as we will see in Section 5, Nakada's examples in [21] and in [20] provide such examples). If we demand only the measurability of the potential, then it is well known that odometers provide a rich source of such type III examples (see [11]).

Here, by making use of Arnold's dyadic adding machine, we produce an example of an ergodic type III conformal measure with continuous potential on the Cantor space $\{0, 1\}^{\mathbb{Z}}$, where the action is the usual shift. This construction is probably simpler, modulo accepting the fact that Arnold's adding machine is one of the simplest type III examples. But a drawback with our construction is that, unlike the example due to Katznelson, our dynamical system is not minimal. It is not clear to the authors whether this construction can be tweaked to produce an example, based on odometers, which is minimal.

2. A groupoid model for $C_c^*(P)$

In this section, we review the groupoid model for $C_c^*(P)$ described in [24] and in [25]. We follow the exposition given in [25]. Let G be a countable, discrete abelian group, and let P be a subsemigroup of G containing the identity element 0. We assume that P generates G, i.e. $P - P = G$.

Recall that $C_c^*(P)$ is the universal unital C^*-algebra generated by a family of isometries $\{v_a : a \in P\}$ such that

(1) for $a, b \in P$, $v_a v_b = v_{a+b}$, and
(2) for $a, b \in P$, $e_a e_b = e_b e_a$, where $e_a := v_a v_a^*$ and $e_b := v_b v_b^*$.

Let $s \in G$ be given. Choose $a, b \in P$ such that $s = a - b$. Set $w_s := v_b^* v_a$. Thanks to Proposition 3.4 of [25], w_s is well defined, and $\{w_s : s \in G\}$ is a family of partial isometries whose range projections commute. For $s \in G$, let $e_s := w_s w_s^*$.

Next, we describe a groupoid whose C^*-algebra is a quotient of $C_c^*(P)$. Let $\mathcal{P}(G)$ be the power set of G which we identify, in the usual way, with $\{0, 1\}^G$. Endow $\mathcal{P}(G)$ with the product topology inherited via this identification. Then, $\mathcal{P}(G)$ is a compact Hausdorff space and is metrisable. The map

$$\mathcal{P}(G) \times G \ni (A, s) \to A + s \in \mathcal{P}(G)$$

defines an action of G on $\mathcal{P}(G)$.

Let

$$\overline{X}_u := \{A \in \mathcal{P}(G) : 0 \in A, -P + A \subset A\}.$$

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Note that $\overline{X}_u$ is a closed subset of $\mathcal{P}(G)$ and hence a compact subset of $\mathcal{P}(G)$. Also, $\overline{X}_u$ is P-invariant, i.e. if $A \in \overline{X}_u$ and $a \in P$, then $A + a \in \overline{X}_u$. Let $\mathcal{G}$ be the reduction of the transformation groupoid $\mathcal{P}(G) \rtimes G$ onto $\overline{X}_u$, i.e.

$$\mathcal{G} := \{(A, s) \in \mathcal{P}(G) \times G : A \in \overline{X}_u, A + s \in \overline{X}_u\} = \{(A, s) : A \in \overline{X}_u, -s \in A\}.$$

The multiplication and inversion on $\mathcal{G}$ are given by

$$\begin{aligned} (A, s)(B, t) &:= (A, s + t) \quad \text{if } A + s = B, \text{ and} \\ (A, s)^{-1} &:= (A + s, -s). \end{aligned}$$

The groupoid $\mathcal{G}$ is étale and is the usual Deaconu-Renault groupoid $\overline{X}_u \rtimes P$.

For $f \in C(\overline{X}_u)$, define $\widetilde{f} \in C_c(\overline{X}_u \rtimes P)$ by

$$\widetilde{f}(A, s) := \begin{cases} f(A) & \text{if } s = 0, \\ 0 & \text{if } s \neq 0. \end{cases} \tag{2.1}$$

Observe that $C(\overline{X}_u) \ni f \to \widetilde{f} \in C_c(\overline{X}_u \rtimes P)$ is a unital *-algebra homomorphism which is injective. Via this embedding, we identify $C(\overline{X}_u)$ as a unital *-subalgebra of $C_c(\mathcal{G})$. For $f \in C(\overline{X}_u)$, we abuse notation and we denote $\widetilde{f}$ by f.

For $s \in G$, let $\overline{w}_s \in C_c(\overline{X}_u \rtimes P)$ be defined by

$$\overline{w}_s(A, t) := \begin{cases} 1 & \text{if } t = -s, \\ 0 & \text{if } t \neq -s. \end{cases} \tag{2.2}$$

For $a \in P$, set $\overline{v}_a := \overline{w}_a$. Observe the following facts. The details involving routine computations are left to the reader.

(1) For $a \in P$, $\overline{v}_a$ is an isometry and $\overline{v}_a\overline{v}_b = \overline{v}_{a+b}$.
(2) For $a, b \in P$ and $s = a - b$, $\overline{w}_s = \overline{v}_b^*\overline{v}_a$.
(3) For $s \in G$, let $\epsilon_s \in C(\overline{X}_u)$ be defined by the equation $\epsilon_s(A) = 1_A(s)$. Then, $\overline{w}_s\overline{w}_s^* = \epsilon_s$ for every $s \in G$. Consequently, $\{\overline{w}_s\overline{w}_s^* : s \in G\}$ generates $C(\overline{X}_u)$. Also, the range projections $\{\overline{v}_a\overline{v}_a^* : a \in P\}$ form a commuting family.
(4) We claim that the C^*-algebra generated by $\{\overline{v}_a : a \in P\}$ is $C^*(\mathcal{G})$. Let $f \in C_c(\mathcal{G}) = C_c(\overline{X}_u \rtimes P)$. It suffices to consider the case when f is supported on $\overline{X}_u \times \{-s\}$ for some $s \in G$. Let $h : \overline{X}_u \to \mathbb{C}$ be defined by

$$h(A) := \begin{cases} f(A, -s) & \text{if } (A, -s) \in \overline{X}_u \rtimes P, \\ 0 & \text{otherwise.} \end{cases}$$

Then, h is continuous and $h * \overline{w}_s = f$. By (3), h lies in the C^*-algebra generated by $\{\overline{w}_s : s \in G\}$. Therefore, $\{\overline{w}_s : s \in G\}$ generates $C^*(\mathcal{G})$. It follows from (2) that $\{\overline{v}_a : a \in P\}$ generates the C^*-algebra $C^*(\overline{X}_u \rtimes P)$.

Hence, there exists a unique surjective *-homomorphism $\Phi : C_c^*(P) \to C^*(\overline{X}_u \rtimes P)$ such that

$$\Phi(v_a) = \overline{v}_a. \tag{2.3}$$

Theorem 7.4 of [25] asserts that Φ is invertible, and Φ is an isomorphism. Via the map Φ, we identify $C_c^*(P)$ with $C^*(\overline{X}_u \rtimes P)$. Henceforth, we do not distinguish between w_s and $\overline{w}_s$.

For $f \in C(\overline{X}_u)$ and $s \in G$, let $R_s(f) \in C(\overline{X}_u)$ be defined by the equation

$$R_s(f)(A) := \begin{cases} f(A - s) & \text{if } A - s \in \overline{X}_u, \\ 0 & \text{otherwise.} \end{cases}$$

For $s \in G$ and $f \in C(\overline{X}_u) \subset C^*(\overline{X}_u \rtimes P)$, the covariance relation

$$w_s f w_s^* = R_s(f),$$

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is satisfied. A pleasant consequence of the above covariance relation is the fact that the linear span of $\{fw_s : f \in C(\overline{X}_u), s \in G\}$ is a unital dense *-subalgebra of $C^*(\overline{X}_u \rtimes P)$.

Let

$$\overline{Y}_u := \{A \in \mathcal{P}(G) : A \neq \emptyset, -P + A \subset A\}.$$

Observe that $\overline{Y}_u$ is a locally compact Hausdorff space and G leaves $\overline{Y}_u$ invariant. Also, $\overline{X}_u$ is a clopen set in $\overline{Y}_u$. It is clear that the groupoid $\overline{X}_u \rtimes P$ is the reduction of the transformation groupoid $\overline{Y}_u \rtimes G$ onto $\overline{X}_u$. This has the consequence that $C^*(\overline{X}_u \rtimes P)$ is isomorphic to the cut-down $p(C_0(\overline{Y}_u) \rtimes G)p$ where $p = 1_{\overline{X}_u}$. Moreover, the union $\bigcup_{a\in P}(\overline{X}_u - a) = \overline{Y}_u$. Consequently, it follows that p is a full projection in $C_0(\overline{Y}_u) \rtimes G$.

Often, we embed $C_c^*(P)$ inside the crossed product $C_0(\overline{Y}_u) \rtimes G$ and view $C_c^*(P)$ as a full corner of $C_0(\overline{Y}_u) \rtimes G$. The embedding $C_c^*(P) \to C_0(\overline{Y}_u) \rtimes G$ is given by the rules

$$C(\overline{X}_u) \ni f \to f 1_{\overline{X}_u} \in C_0(\overline{Y}_u) \text{ and } w_s \to u_s 1_{\overline{X}_u}.$$

Here, $\{u_s : s \in G\}$ are the canonical unitaries of the crossed product $C_0(\overline{Y}_u) \rtimes G$.

Tracial states on $C_c^*(P)$: We first determine the tracial states on $C_c^*(P)$. Let $C^*(G)$ be the group C^*-algebra of G, and let $\{u_s : s \in G\}$ be the canonical unitaries of $C^*(G)$. By the universal property, there exists a surjective *-homomorphism $\pi : C_c^*(P) \to C^*(G)$ such that $\pi(v_a) = u_a$. In the groupoid picture, π coincides with the restriction map $Res : C^*(\overline{X}_u \rtimes P) \to C^*(\overline{X}_u \rtimes P|_{\{G\}}) \cong C^*(G)$. Note that $\{G\}$ is an invariant closed subset of the unit space $\overline{X}_u$ of the groupoid $\mathcal{G} = \overline{X}_u \rtimes P$.

Proposition 2.1. *Every tracial state on $C_c^*(P)$ factors through $C^*(G)$. Thus, tracial states on $C_c^*(P)$ are in bijective correspondence with probability measures on $\widehat{G}$.*

Proof. Let $F := \{G\}$ and $X_u := \overline{X}_u \backslash F$. Let ω be a tracial state on $C_c^*(P)$. Let m be the measure on $\overline{X}_u$ that corresponds to the state $\omega|_{C(\overline{X}_u)}$. Let $a \in P$ be given. Note that $v_a v_a^* = 1_{\overline{X}_u + a}$. Since ω is tracial, we have

$$m(\overline{X}_u + a) = \omega(v_a v_a^*) = \omega(v_a^* v_a) = \omega(1) = 1 = m(\overline{X}_u).$$

But, $\overline{X}_u + a \subset \overline{X}_u$. Hence, $\overline{X}_u \backslash (\overline{X}_u + a)$ has measure zero. Hence, the set $X_u = \bigcup_{a\in P} \overline{X}_u \backslash (\overline{X}_u + a)$ has measure zero. Therefore, m is concentrated on $F = \{G\}$. Now the proof follows from Corollary 1.4 of [22]. □

We end this section with a few definitions that we need later.

Definition 2.1. *Let $(Y, \mathcal{B})$ be a standard Borel space on which G acts measurably. Let X be a Borel subset of Y. We say that (Y, X) is a (G, P)-space if*

(1) the set X is P-invariant, i.e. $X + P \subset X$, and
(2) the union $\bigcup_{a\in P}(X - a) = Y$.

Let (Y, X) be a (G, P)-space. We say that (Y, X) is pure if $\bigcap_{a\in P}(X + a) = \emptyset$.

Example 2.1. *The pair $(\overline{Y}_u, \overline{X}_u)$ is a (G, P)-space. Define*

$$Y_u := \overline{Y}_u \backslash \{G\};\ X_u := \overline{X}_u \backslash \{G\}.$$

Then, (Y_u, X_u) is pure.

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Definition 2.2. *Let (Y, X) be a (G, P)-space. Let $c : G \to \mathbb{R}$ be a non-zero homomorphism, and let β be a fixed real number. Suppose m is a measure on Y. We say that m is an $e^{-\beta c}$-conformal measure on the (G, P)-space (Y, X) if*

(i) $m(X) = 1$, *and*
(ii) for every Borel set $E \subset Y$ and $s \in G$,

$$m(E + s) = e^{-\beta c(s)} m(E).$$

We often abuse terminology and call an $e^{-\beta c}$-conformal measure on a (G, P)-space (Y, X) an $e^{-\beta c}$-conformal measure on Y.

Definition 2.3. *For $i = 1, 2$, let (Y_i, X_i) be a (G, P)-space, and suppose that m_i is an $e^{-\beta c}$-conformal measure on (Y_i, X_i). We say that (Y_1, X_1, m_1) and (Y_2, X_2, m_2) are metrically isomorphic, or simply isomorphic, if there exist G-invariant null sets $N_1 \subset Y_1$, $N_2 \subset Y_2$ and an invertible measurable map $S : Y_1 \backslash N_1 \to Y_2 \backslash N_2$ such that*

(i) the map S is G-equivariant, $S(X_1 \backslash N_1) = X_2 \backslash N_2$, and
(ii) for every Borel subset $E \subset Y_2 \backslash N_2$, $m_2(E) = m_1(S^{-1}(E))$.

3. KMS states and conformal measures

Let us recall the definition of a β-KMS state. Let A be a C^*-algebra, and suppose $\tau := \{\tau_t\}_{t\in\mathbb{R}}$ is a 1-parameter group of automorphisms of A. Suppose $\beta \in \mathbb{R}$. A state ω on A is a β-KMS state for τ if

$$\omega(ab) = \omega(b\tau_{i\beta}(a)),$$

for all a, b in a norm dense τ-invariant *-algebra of analytic elements in A.

Let ω be a β-KMS state on A. Then, ω is said to be *extremal* if it is an extreme point in the simplex of β-KMS states. It is well known that a β-KMS state ω is extremal if and only if the associated GNS representation π_ω is factorial, i.e. $\pi_\omega(A)''$ is a factor. We say an extremal β-KMS state ω is of type t if $\pi_\omega(A)''$ is a factor of type t.

Fix a non-zero homomorphism $c : G \to \mathbb{R}$. Recall that the 1-parameter group of automorphisms $\sigma^c := \{\sigma_t\}_{t\in\mathbb{R}}$ on $C_c^*(P)$ is defined by

$$\sigma_t(v_a) = e^{itc(a)} v_a,$$

for $a \in P$. For $t \in G$, let $D(t), R(t) \subset \overline{X}_u$ be defined by

$$D(t) := \{A \in \overline{X}_u : A + t \in \overline{X}_u\}$$
$$R(t) := \{A \in \overline{X}_u : A - t \in \overline{X}_u\}.$$

For $t \in G$, denote the map $D(t) \ni A \to A + t \in R(t)$ by T_t.

Let β be a real number. As we have already described the tracial states on $C_c^*(P)$ in Proposition 2.1, we assume for the rest of this section that β is an arbitrary, but a fixed non-zero real number. Let ω be a β-KMS state on the C^*-algebra $C_c^*(P) = C^*(\overline{X}_u \rtimes P)$ for σ^c. Restriction of ω to $C(\overline{X}_u)$ defines a probability measure m on $\overline{X}_u$. It can be checked from the covariance relation and the KMS condition that

$$\omega(f) = e^{-\beta c(t)} \omega(f \circ T_t),$$

for every $f \in C(R(t))$, or equivalently $m(E + t) = e^{-\beta c(t)} m(E)$ for every Borel subset E of $D(t)$.

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For $s, t \in G$, we say $s \leq t$ if $t - s \in P$. Then, G with the preorder $\leq$ is a directed set. Note that P is a cofinal subset of G. Moreover, if $s \leq t$, then $\overline{X}_u - s \subseteq \overline{X}_u - t$. Let $s \in G$. Define a measure m_s on $\overline{X}_u - s$ by

$$m_s(E) = e^{\beta c(s)} m(E + s).$$

Thanks to the fact that m is conformal, it is clear that $m_t|_{\overline{X}_u - s} = m_s$ for $s \leq t$. Define a σ-finite measure m_ω on $\overline{Y}_u$ by setting

$$m_\omega(E) := \lim_{s \in G} m_s(E \cap (\overline{X}_u - s)) = \lim_{s \in P} m_s(E \cap (\overline{X}_u - s)). \tag{3.4}$$

Proposition 3.1. *The measure m_ω on $\overline{Y}_u$ defined as in Equation 3.4 is an $e^{-\beta c}$-conformal measure on the (G, P)-space $(\overline{Y}_u, \overline{X}_u)$.*

Proof. It is clear that $m_\omega|_{\overline{X}_u} = m$ and so $m_\omega(\overline{X}_u) = 1$. For $t \in G$ and a Borel subset $E \subset \overline{Y}_u$,

$$\begin{aligned} m_\omega(E + t) &= \lim_s m_s((E + t) \cap (\overline{X}_u - s)) \\ &= \lim_s e^{\beta c(s)} m(((E + t) \cap (\overline{X}_u - s)) + s) \\ &= \lim_s e^{\beta c(s)} m((E \cap (\overline{X}_u - (s + t))) + s + t) \\ &= e^{-\beta c(t)} \lim_r e^{\beta c(r)} m((E \cap (\overline{X}_u - r)) + r) \ \text{(by a change of variable} r = s + t) \\ &= e^{-\beta c(t)} \lim_r m_r(E \cap (\overline{X}_u - r)) \\ &= e^{-\beta c(t)} m_\omega(E). \end{aligned}$$

Hence the proof. □

Recall that for an étale groupoid $\mathcal{G}$, the map $f \mapsto f|_{\mathcal{G}^{(0)}}$ from $C_c(\mathcal{G})$ to $C_0(\mathcal{G}^{(0)})$ defines a conditional expectation $E : C^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$. For $\mathcal{G} := \overline{X}_u \rtimes P$, the conditional expectation E has the following form

$$E(fw_s) := \begin{cases} f & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The next proposition is a direct consequence of Theorem 1.3 of [22]. Hence, we omit the proof.

Proposition 3.2. *The map $\omega \mapsto m_\omega$ from the set of β-KMS states for $\sigma = \sigma^c$ to the set of $e^{-\beta c}$-conformal measures on $\overline{Y}_u$ is surjective. More specifically, for an $e^{-\beta c}$-conformal m on $\overline{Y}_u$, the state ω_m on $C^*(\overline{X}_u \rtimes P)$ defined by*

$$\omega_m(a) = \int_{\overline{X}_u} E(a)\, dm$$

is a β-KMS state such that $m_{\omega_m} = m$.

Remark 3.1. *In general, the map $\omega \to m_\omega$ of Proposition 3.1 need not be injective. The structure of a generic KMS state is determined by Neshveyev's result (Theorem 1.3 of [22]). Let $\mathcal{G}$ be an étale groupoid, and let $\overline{c} : \mathcal{G} \to \mathbb{R}$ be a continuous homomorphism. For $t \in \mathbb{R}$, let σ_t be the automorphism of $C^*(\mathcal{G})$ defined by*

$$\sigma_t(f)(\gamma) := e^{it\overline{c}(\gamma)} f(\gamma)$$

for $f \in C_c(\mathcal{G})$. Then, $\sigma^{\overline{c}} := \{\sigma_t\}_{t \in \mathbb{R}}$ defines a 1-parameter group of automorphisms on $C^*(\mathcal{G})$. Theorem 1.3 of [22] describes the set of β-KMS states for $\sigma^{\overline{c}}$.

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In our situation, the groupoid $\mathcal{G}$ is the Deaconu-Renault groupoid $\overline{X}_u \rtimes P$, and the homomorphism $\overline{c}: \mathcal{G} \to \mathbb{R}$ is given by

$$\overline{c}(A, g) = -c(g).$$

One immediate corollary of Theorem 1.3 of [22] is that, if the homomorphism $c: G \to \mathbb{R}$ is injective, then the map $\omega \to m_\omega$ of Proposition 3.1 is a bijection.

Proposition 3.3. *Let $\beta \neq 0$, and let ω be an extremal β-KMS state for $\sigma = \sigma^c$ on the C^*-algebra $C^*(\overline{X}_u \rtimes P)$. Then, the $e^{-\beta c}$-conformal measure m_ω on $\overline{Y}_u$ defined as in Proposition 3.1 is ergodic for the G-action on $\overline{Y}_u$.*

Proof. The proof is an adaptation of the proof of Lemma 3.6 in [5] with appropriate modifications. Assume that m_ω is not ergodic. Then, there exists a G-invariant Borel subset $A \subset \overline{Y}_u$ such that neither A nor A^c has measure zero. We claim that $0 < m_\omega(A \cap \overline{X}_u) < 1$ and $0 < m_\omega(A^c \cap \overline{X}_u) < 1$.

Suppose that $m_\omega(A \cap \overline{X}_u) = 0$. By conformality, $m_\omega((A \cap \overline{X}_u) - t) = 0$ for every $t \in G$. Hence, $\bigcup_{t \in G} (A \cap \overline{X}_u) - t$ is a null set. The fact that A is G-invariant implies that

$$\bigcup_{t\in G} (A \cap \overline{X}_u) - t = A \cap \Big(\bigcup_{t \in G} (\overline{X}_u - t)\Big) = A.$$

Therefore, $m_\omega(A) = 0$ which is a contradiction. This proves that $m_\omega(A \cap \overline{X}_u) \neq 0$. Similarly, we can prove that $m_\omega(A^c \cap \overline{X}_u) \neq 0$. Consequently, $0 < m_\omega(A \cap \overline{X}_u) < 1$ and $0 < m_\omega(A^c \cap \overline{X}_u) < 1$. This proves the claim.

Let $(H_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation of ω. Extend the homomorphism $\pi_\omega|_{C(\overline{X}_u)}$ to a *- homomorphism $\overline{\pi}_\omega : L^\infty(\overline{X}_u, m_\omega) \to \pi_\omega(C^*(\overline{X}_u \rtimes P))''$ by defining

$$\overline{\pi}_\omega(h) := \lim_{n\to\infty} \pi_\omega(f_n)$$

Here, the limit is taken in the SOT sense and $\{f_n\}$ is any sequence in $C(\overline{X}_u)$ such that for every n, $|f_n| \leq ||h||_\infty$ a.e and the sequence $(f_n(x)) \to h(x)$ for almost all x.

For $s \in G$, set $W_s = \pi_\omega(w_s)$. Then, the following covariance relation is satisfied

$$W_s \overline{\pi}_\omega(f) W_s^* = \overline{\pi}_\omega(R_s(f))$$

for $s \in G$ and $f \in L^\infty(\overline{X}_u, m_\omega)$.

Next, we claim that $\overline{\pi}_\omega(1_{A\cap\overline{X}_u})$ and $\overline{\pi}_\omega(1_{A^c\cap\overline{X}_u})$ are central in $\pi_\omega(C^*(\overline{X}_u \rtimes P))''$. We will give details for $\overline{\pi}_\omega(1_{A\cap\overline{X}_u})$ and the centrality of $\overline{\pi}_\omega(1_{A^c\cap\overline{X}_u})$ follows similarly. Let $s \in P$ be given. Note that $R_{-s}(1_{A\cap\overline{X}_u}) = 1_{(A-s)\cap(\overline{X}_u - s)\cap\overline{X}_u} = 1_{A\cap\overline{X}_u}$ and $\overline{\pi}_\omega(f)$ commutes with $W_s W_s^*$ for every $f \in L^\infty(\overline{X}_u, m_\omega)$.

Calculate as follows to observe that

$$\begin{aligned}
\overline{\pi}_\omega(1_{A\cap\overline{X}_u}) W_s &= \overline{\pi}_\omega(1_{A\cap\overline{X}_u}) W_s W_s^* W_s \\
&= W_s W_s^* \overline{\pi}_\omega(1_{A\cap\overline{X}_u}) W_s \\
&= W_s \overline{\pi}_\omega(R_{-s}(1_{A\cap\overline{X}_u})) \\
&= W_s \overline{\pi}_\omega(1_{A\cap\overline{X}_u}).
\end{aligned}$$

Since $\{W_s : s \in P\}$ generates $(\pi_\omega(C^*(\overline{X}_u \rtimes P))''$, we can conclude that $\pi_\omega(1_A \cap \overline{X}_u)$ is central. This proves the claim.

Define states ω_1 and ω_2 on $C^*(\overline{X}_u \rtimes P)$ by

$$\omega_1(a) = \frac{1}{m_\omega(A \cap \overline{X}_u)} \langle \overline{\pi}_\omega(1_{A\cap\overline{X}_u})\pi_\omega(a)\Omega_\omega, \Omega_\omega\rangle,$$

$$\omega_2(a) = \frac{1}{m_\omega(A^c \cap \overline{X}_u)} \langle \overline{\pi}_\omega(1_{A^c\cap\overline{X}_u})\pi_\omega(a)\Omega_\omega, \Omega_\omega\rangle.$$

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The centrality of the projections $\overline{\pi_\omega}(1_{A\cap\overline{X}_u})$ and $\overline{\pi_\omega}(1_{A^c\cap\overline{X}_u})$ imply that ω_1 and ω_2 are β-KMS states on $C^*(\overline{X}_u \rtimes P)$. Moreover, $\omega = m_\omega(A\cap\overline{X}_u)\omega_1 + m_\omega(A^c\cap\overline{X}_u)\omega_2$ which contradicts the extremality of ω. Hence the proof. □

Next, we work out the GNS representation of the KMS state ω_m given by Proposition 3.2. Suppose m is an $e^{-\beta c}$ conformal measure on $\overline{Y}_u$, and let ω_m be the β-KMS state on the C^*-algebra $C_c^*(P) = C^*(\overline{X}_u \rtimes P)$ obtained via the conditional expectation as in Proposition 3.2. Our goal is to show that if $(H_\omega, \pi_\omega, \Omega_\omega)$ is the GNS representation of ω_m, then the von Neumann algebra $(\pi_\omega(C^*(\overline{X}_u \rtimes P)))''$ is isomorphic to the full corner $1_{\overline{X}_u}(L^\infty(\overline{Y}_u)\rtimes G)1_{\overline{X}_u}$ of $L^\infty(\overline{Y}_u)\rtimes G$.

Let $\lambda := \{\lambda_s\}_{s\in G}$ be the Koopman representation of G on $L^2(\overline{Y}_u, m)$. Recall that

$$\lambda_s\xi(A) = e^{\frac{\beta c(s)}{2}}\xi(A-s),$$

for $s\in G$ and $\xi\in L^2(\overline{Y}_u, m)$.

Let $\mathcal{K} = l^2(G, L^2(\overline{Y}_u))$ and π_0 be the representation of $C_0(\overline{Y}_u)\rtimes G$ on $\mathcal{K}$ defined by

$$\pi_0(fu_t)(\xi)(s) = f\lambda_t(\xi(s-t)).$$

Then, $(\pi_0(C_0(\overline{Y}_u)\rtimes G))'' = L^\infty(\overline{Y}_u)\rtimes G$. Let $\mathcal{H}$ be the Hilbert subspace of $\mathcal{K}$ given by

$$\mathcal{H} := \{\xi \in l^2(G, L^2(\overline{Y}_u)) : \xi(t) \in L^2(\overline{X}_u + t)\}.$$

It is clear that $L^\infty(\overline{Y}_u)\rtimes G$ leaves $\mathcal{H}$ invariant, thus giving a normal representation $\overline{\pi}$ of the crossed product $L^\infty(\overline{Y}_u)\rtimes G$ on $\mathcal{H}$. Let $\pi = \overline{\pi}\circ\pi_0$.

Notation: For $s\in G$ and $\xi\in L^2(\overline{X}_u + s)$, let $\xi\otimes\delta_s \in \mathcal{H}$ be defined by

$$\xi\otimes\delta_s(t) = \begin{cases} \xi & \text{if } t = s, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.1. *The von Neumann algebra $(\pi(C_0(\overline{Y}_u)\rtimes G))''$ is isomorphic to $L^\infty(\overline{Y}_u)\rtimes G$.*

Proof. Note that since $\pi = \overline{\pi}\circ\pi_0$ and the von Neumann algebra generated by $\pi_0(C_0(\overline{Y}_u)\rtimes G)$ is $L^\infty(\overline{Y}_u)\rtimes G$, to complete the proof, it is enough to prove that $\overline{\pi}$ is injective.

First, we claim that $\overline{\pi}$ restricted to $L^\infty(\overline{Y}_u)$ is injective. Let $f\in L^\infty(\overline{Y}_u)$ be such that $\overline{\pi}(f) = 0$. Suppose $s\in G$. Then, for every $\xi\in L^2(\overline{X}_u + s)$,

$$0 = \overline{\pi}(f)(\xi\otimes\delta_s)(s) = f\xi.$$

Thus, $f = 0$ a.e on $\overline{X}_u + s$ for every $s\in G$. Since $\overline{Y}_u = \bigcup_{s\in G}\overline{X}_u + s$, it follows that $f = 0$ a.e on $\overline{Y}_u$ proving that $\overline{\pi}$ is faithful on $L^\infty(\overline{Y}_u)$.

Let E be the usual conditional expectation from $L^\infty(\overline{Y}_u)\rtimes G \to L^\infty(\overline{Y}_u)$ given by $E(fu_s) = \delta_{s,0}f$. Here, $\delta_{s,0}$ is the Kronecker delta. Then, the conditional expectation E is faithful. Decompose $\mathcal{H}$ as $\mathcal{H} = \bigoplus_{t\in G} L^2(\overline{X}_u + t)$, and for $t\in G$, let P_t be the projection onto the subspace $L^2(\overline{X}_u + t)$. Define a map $\widetilde{E} : B(\mathcal{H}) \to B(\mathcal{H})$ by

$$\widetilde{E}(T) = \sum_{t\in G} P_tTP_t,$$

where the convergent sum is w.r.t. the strong operator topology. Clearly, $\widetilde{E}\circ\overline{\pi} = \overline{\pi}\circ E$.

Let $x\in L^\infty(\overline{Y}_u)\rtimes G$ be such that $\overline{\pi}(x) = 0$. Then, $\overline{\pi}(x^*x) = 0$. Therefore,

$$\overline{\pi}(E(x^*x)) = \widetilde{E}(\overline{\pi}(x^*x)) = 0.$$

Since $\overline{\pi}$ is faithful on $L^\infty(\overline{Y}_u)$, we have $E(x^*x) = 0$. Since E is faithful, $x = 0$. This completes the proof. □

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Recall that $C^*(\overline{X}_u \rtimes P)$ is the full corner $p(C_0(\overline{Y}_u) \rtimes G)p$ where $p = 1_{\overline{X}_u}$. Consider the Hilbert space

$$P\mathcal{H} = \{\xi \in l^2(G, L^2(\overline{Y}_u)) : \xi(t) \in L^2(\overline{X}_u \cap (\overline{X}_u + t))\}$$

where $P = \pi(p)$. Define $\widetilde{\pi} : C^*(\overline{X}_u \rtimes P) \to B(P\mathcal{H})$ by $\widetilde{\pi}(fw_t) = P\pi(\tilde{f}u_t)P$ where $\tilde{f} = f1_{\overline{X}_u}$. Define $\xi \in P\mathcal{H}$ by,

$$\xi(t) = \begin{cases} 1_{\overline{X}_u} & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.2. *With the foregoing notation, the triple $(\widetilde{\pi}, P\mathcal{H}, \xi)$ is the GNS representation of the β-KMS state ω_m. Moreover, the von Neumann algebra $\widetilde{\pi}(C^*(\overline{X}_u \rtimes P))''$ is isomorphic to the full corner $1_{\overline{X}_u}(L^\infty(\overline{Y}_u) \rtimes G)1_{\overline{X}_u}$.*

Proof. Observe that, for $f \in C(\overline{X_u})$, and $s, t \in G$,

$$\begin{aligned} \widetilde{\pi}(fw_t)\xi(s) &= P\pi(\tilde{f}u_t)P\xi(s) \\ &= \pi(1_{\overline{X}_u}\tilde{f}u_t)\xi(s) \\ &= 1_{\overline{X}_u}\tilde{f}\lambda_t(\xi(s-t)) \\ &= \delta_{s,t}1_{\overline{X}_u}f\lambda_t(\xi(0)) \\ &= e^{\frac{\beta c(s)}{2}}\delta_{s,t}1_{\overline{X}_u}f1_{\overline{X}_u+s} \end{aligned}$$

From the above formula, it is clear that $\langle\widetilde{\pi}(a)\xi, \xi\rangle = \omega_m(a)$ for every $a \in C^*(\overline{X}_u \rtimes P)$. Also, $span\{\widetilde{\pi}(fw_t)\xi : f \in C(\overline{X}_u), t \in G\} = span\{\eta \otimes \delta_s : \eta \in C(\overline{X}_u \cap (\overline{X}_u + s)), s \in G\}$, and the latter set is dense in $P\mathcal{H}$ proving that ξ is cyclic for $\widetilde{\pi}$. Thus, $(\widetilde{\pi}, P\mathcal{H}, \xi)$ is the GNS representation of the β-KMS state ω_m.

From the definition of $\widetilde{\pi}$, we have

$$(\widetilde{\pi}(C^*(\overline{X}_u \rtimes P)))'' = (P\pi(C_0(\overline{Y}_u) \rtimes G)P)'' = P(\pi(C_0(\overline{Y}_u) \rtimes G))''P.$$

The conclusion is now clear from Lemma 3.1. □

We have the following main theorem of this section establishing the factor types of extremal β-KMS states ω_m.

Theorem 3.1. *Suppose m is an $e^{-\beta c}$-conformal measure on $\overline{Y}_u$. Let $\omega := \omega_m$ be the β-KMS state as in Proposition 3.2. Denote the GNS representation of ω by π_ω. Suppose $t \in \{I_\infty, II_\infty, III\}$. Then,*

(1) $(\pi_\omega(C^*(\overline{X}_u \rtimes P)))''$ is isomorphic to the full corner $1_{\overline{X}_u}(L^\infty(\overline{Y}_u) \rtimes G)1_{\overline{X}_u}$.
(2) ω_m is extremal $\iff L^\infty(\overline{Y}_u) \rtimes G$ is a factor $\iff m$ is ergodic and the G-action is essentially free. Moreover, ω_m is of type t if and only if m is of type t.

Proof. The first equivalence in (2) follows from Lemma 3.2 and the following facts. It is well known that

(i) a β-KMS state ω is extremal if and only if its GNS representation π_ω is factorial, and
(ii) a full corner pMp is a factor of type t if and only if M is a factor of type t.

Other conclusions are standard. □

A few remarks are in order.

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Remark 3.2. *Let β be a non-zero real number.*

(1) *Let ω be a β-KMS state. Let $m := m_\omega$ be the $e^{-\beta c}$-conformal measure on $\overline{Y}_u$ associated with ω as in Proposition 3.1. Then, m is concentrated on $Y_u := \overline{Y}_u \backslash \{G\}$. Note that, for every $s \in G$,*

$$m(\{G\}) = m(\{G\} + s) = e^{-\beta c(s)} m(\{G\}).$$

Since $\{G\}$ is of finite measure (as $\{G\} \subset \overline{X}_u$) and βc is a non-zero homomorphism, we have $m(\{G\}) = 0$.

(2) *Suppose ω is a β-KMS state on $A := C_c^*(P)$. Assume that ω is extremal, and denote the GNS representation of A associated with ω by π_ω. Denote the associated cyclic vector by Ω_ω. Then, the factor $M := \pi_\omega(A)''$ is neither of type I_n for n finite nor of type II_1. To see this, let m be the $e^{-\beta c}$-conformal measure on $\overline{Y}_u$ associated with ω. Pick $a \in P$ such that $c(a) \neq 0$. Note that $\pi_\omega(e_a) = \pi_\omega(1_{\overline{X}_u + a})$. Here $e_a = v_a v_a^*$. Then,*

$$\langle \pi_\omega(1 - e_a)\Omega_\omega | \Omega_\omega \rangle = 1 - m(\overline{X}_u + a) = 1 - e^{-\beta c(a)} \neq 0.$$

This implies that $\pi_\omega(1 - e_a) \neq 0$. Thus, $\pi_\omega(e_a)$ is a proper subprojection of 1 which is Murray von Neumann equivalent to 1 in M. Hence the conclusion.

In the next proposition, we work out the extremal β-KMS states whose associated measure is supported on an orbit.

Proposition 3.4.

(1) *Let m be an $e^{-\beta c}$-conformal measure on Y_u supported on an orbit $Orb(A)$ for some A. Denote the of A by H, i.e.*

$$H := \{s \in G : A + s = A\}.$$

Let χ be a character of H. Define a state $\omega_{\chi,m}$ on $C_c^*(P)$ by

$$\omega_{\chi,m}(fw_s) := \begin{cases} 0 & \text{if } s \notin H \\ \overline{\chi(s)} \displaystyle\int_{X_u} f(A)dm(A) & \text{if } s \in H. \end{cases}$$

Then, $\omega_{\chi,m}$ is an extremal β-KMS state on $C_c^*(P)$. Moreover, $\omega_{\chi,m}$ is of type I.

(2) *Let ω be an extremal β-KMS state on $C_c^*(P)$, and let m be the $e^{-\beta c}$-conformal measure on Y_u associated with ω. Suppose that m is atomic and concentrated on $Orb(A)$ for some $A \in Y_u$. Let H be the stabiliser of A. Then, ω is of the form $\omega_{\chi,m}$ for some character χ of H.*

Proof. The only thing that requires proof is that $\omega_{\chi,m}$ is of type I. Other assertions follow from Corollary 1.4 of [22]. Let m be an $e^{-\beta c}$-conformal measure on Y_u supported on an orbit $Orb(A)$ for some A and denote the of A by H. Let χ be a character of H, and let $\omega := \omega_{\chi,m}$. Let π_ω be the GNS representation of $\omega_{\chi,m}$. To prove that $\omega_{\chi,m}$ is extremal of type I, it suffices to show that the von Neumann algebra $\pi_\omega(C^*(\overline{X}_u \rtimes P))''$ is a factor of type I. The proof is similar to the proof of Theorem 3.1.

By the conformality of m, for a Borel set $E \subset Orb(A)$ with $m(E) > 0$ and $t \in H$, we have

$$m(E) = m(E + t) = e^{-\beta c(t)} m(E),$$

implying that $c(t) = 0$. Thus, $H \subset c^{-1}(0)$. Let $\lambda = \{\lambda_t\}_{t \in G}$ be the Koopman representation of G on $L^2(\overline{Y}_u)$. Note that since $Y_u = Orb(A)$ upto a null set, $L^2(\overline{Y}_u) = L^2(Orb(A))$ and we can conclude that $\lambda_t = \lambda_s$, whenever $t - s \in H$. Extend the character χ of H to a character of G which we denote again by χ.

Let $\mathcal{K}$ be the Hilbert space defined by $\mathcal{K} = l^2(G/H, L^2(\overline{Y}_u))$, and let π_0 be the representation of $C_0(\overline{Y}_u) \rtimes G$ on $\mathcal{K}$ given by

$$\pi_0(fu_t)(\xi)(\overline{s}) = \overline{\chi(t)} f \lambda_t(\xi(\overline{s - t})).$$

Clearly, $\pi_0(C_0(\overline{Y}_u) \rtimes G)'' = L^\infty(\overline{Y}_u) \rtimes G/H$. Let $\mathcal{H}$ be the Hilbert subspace of $\mathcal{K}$ given by

$$\mathcal{H} := \{\xi \in l^2(G/H, L^2(\overline{Y}_u)) : \xi(\bar{t}) \in L^2(\overline{X}_u + t)\}.$$

We have $\bar{s} = \bar{t}$ implies $\overline{X}_u + s = \overline{X}_u + t$ upto a null set, and hence, the definition of $\mathcal{H}$ makes sense. Since $L^\infty(\overline{Y}_u) \rtimes G/H$ leaves $\mathcal{H}$ invariant, we get a representation $\overline{\pi} : L^\infty(\overline{Y}_u) \rtimes G/H \to B(\mathcal{H})$. Define $\pi = \overline{\pi} \circ \pi_0$. Consider the Hilbert space $P\mathcal{H}$ where $P = \pi(1_{\overline{X}_u})$. Let $\xi \in P\mathcal{H}$ be given by,

$$\xi(\bar{t}) = \begin{cases} 1_{\overline{X}_u} & \text{if } \bar{t} = 0, \\ 0 & \text{otherwise .} \end{cases}$$

Let $\widetilde{\pi} : C^*(\overline{X}_u \rtimes P) \to B(P\mathcal{H})$ be defined by $\widetilde{\pi}(fw_t) = P\pi(\tilde{f}u_t)P$ where $\tilde{f} = f 1_{\overline{X}_u}$.

Arguing as before, we can conclude the following.

1. The map $\overline{\pi}$ is injective, and hence, the von Neumann algebra $\pi(C_0(\overline{Y}_u) \rtimes G)''$ is isomorphic to $L^\infty(\overline{Y}_u) \rtimes G/H$.
2. The triple $(\widetilde{\pi}, P\mathcal{H}, \xi)$ is the GNS representation of $\omega_{\chi,m}$.
3. The von Neumann algebra generated by $\widetilde{\pi}(C^*(\overline{X}_u \rtimes P))$ is isomorphic to the full corner $1_{\overline{X}_u}(L^\infty(\overline{Y}_u) \rtimes G/H)1_{\overline{X}_u}$.

Thus, $\pi_\omega(C^*(\overline{X}_u \rtimes P))''$ is isomorphic to the full corner $1_{\overline{X}_u}(L^\infty(\overline{Y}_u) \rtimes G/H)1_{\overline{X}_u}$ which is clearly a factor of type I (as the measure m is atomic). This completes the proof. □

We end this section with a proposition that allows us to construct conformal measures on the (G, P)-space (Y_u, X_u), and consequently, KMS states on $C^*_c(P)$ of the desired type. Let (Y, X) be a pure (G, P)-space. For $y \in Y$, let

$$Q_y := \{s \in G : y - s \in X\}.$$

First, let us check that for $y \in Y$, $Q_y \in Y_u$. Fix $y \in Y$.

(1) Since, $Y = \bigcup_{a \in P}(X - a)$, it follows that there exists $a \in P$ such that $y \in X - a$. Then, $-a \in Q_y$. Thus, Q_y is non-empty.

(2) As the intersection $\bigcap_{a \in P}(X + a) = \emptyset$, it follows that there exists $a \in P$ such that $y \notin X + a$. Then, $a \notin Q_y$ for such an element a. Hence, Q_y is a proper subset of G.

(3) The fact that $X + P \subset X$ implies that $-P + Q_y \subset Q_y$.

Proposition 3.5. *Let (Y, X) be a pure (G, P)-space, and let m be an $e^{-\beta c}$-conformal measure on Y. Denote the map*

$$Y \ni y \to Q_y \in Y_u,$$

by T. Assume that T is 1-1. Then, we have the following.

(1) The map T is G-equivariant, measurable, and $T^{-1}(X_u) = X$.

(2) The push-forward measure $T_*m := m \circ T^{-1}$ is an $e^{-\beta c}$-conformal measure on the pure (G, P)-space (Y_u, X_u). Moreover, (Y, X, m) and (Y_u, X_u, T_*m) are metrically isomorphic.

Consequently, for $t \in \{I, II, III\}$, (Y, m, G) is essentially free, ergodic and is of type t if and only if (Y_u, T_*m, G) is essentially free, ergodic and is of type t.

Proof. Let $y \in Y$ and $s \in G$ be given. Note that for $t \in G$, $t \in Q_{y+s}$ iff $y + s - t \in X$ iff $y - (t - s) \in X$ iff $t - s \in Q_y$ iff $t \in Q_y + s$. Thus, $Q_{y+s} = Q_y + s$. This shows that T is G-equivariant. To show that T is measurable, it suffices to show that for every $s \in G$, the map

$$Y \ni y \to 1_{Q_y}(s) \in \{0, 1\}$$

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is measurable. Fix $s \in G$. Clearly, for every $y \in Y$,

$$1_{Q_y}(s) = 1_{X+s}(y).$$

Since $X + s$ is a measurable subset of Y, it follows that the map

$$Y \ni y \to 1_{Q_y}(s) \in \{0, 1\},$$

is measurable. Hence, T is a measurable map. Note that for $y \in Y$, $y \in T^{-1}(X_u)$ if and only if $0 \in Q_y$ if and only if $y \in X$. Thus, $T^{-1}(X_u) = X$. This completes the proof of (1).

Since $T^{-1}(X_u) = X$, and $m(X) = 1$, it follows that $T_*m(X_u) = 1$. Let E be a Borel subset of Y_u, and let $s \in G$ be given. Thanks to the equivariance of T, we have $T^{-1}(E + s) = T^{-1}(E) + s$. Since m is $e^{-\beta c}$-conformal,

$$T_*m(E + s) = m(T^{-1}(E + s)) = m(T^{-1}(E) + s) = e^{-\beta c(s)}m(T^{-1}(E)) = e^{-\beta c(s)}T_*m(E).$$

Therefore, T_*m is an $e^{-\beta c}$-conformal measure on (Y_u, X_u).

Set $Y'_u := T(Y)$, and $X'_u := Y'_u \cap X_u$. Since T is one-one, and Y and Y_u are standard Borel spaces, Y'_u is a Borel subset of Y_u, and the map $T : Y \to Y'_u$ is a Borel isomorphism. Also, $T(X) = X'_u$. Observe that T_*m is supported on Y'_u. Thus, T is an isomorphism between (Y, X, m) and (Y'_u, X'_u, T_*m). Since Y'_u is a co-null G-invariant subset of Y_u, (Y'_u, X'_u, T_*m) and (Y_u, X_u, T_*m) are metrically isomorphic. Hence, (Y, X, m) and (Y_u, X_u, T_*m) are metrically isomorphic. This completes the proof of (2).

The final assertion is clear as metric isomorphism preserves essential freeness, ergodicity and type. □

4. The case $P = \mathbb{N}^2$

In this section, we discuss the structure of KMS states on $C^*_c(P)$ for σ^c when $P = \mathbb{N}^2$. Let us fix notation. Set $e_1 := (1, 0)$ and $e_2 := (0, 1)$. Define $v_1 := e_1$ and $v_2 := e_1 + e_2$. Let $c : \mathbb{Z}^2 \to \mathbb{R}$ be a non-zero homomorphism. We normalise and assume that $c(e_1) = 1$ and $c(e_2) = \theta$. This does not lead to any loss of generality. In what follows, the homomorphism c will be fixed.

First, we obtain a reasonable parametrisation of Y_u. Let $\Omega := \{0, 1\}^{\mathbb{Z}}$ be the Cantor space. Consider the Bernoulli shift $\tau : \Omega \to \Omega$ defined by

$$\tau(x)_k := x_{k-1}.$$

Define a $\mathbb{Z}^2$-action on $\Omega \times \mathbb{Z}$ by the following formulae.

$$\begin{aligned}\big(x, t\big) + v_1 &= \big(\tau(x), t + x_{-1}\big), \text{ and}\\ \big(x, t\big) + v_2 &= \big(x, t + 1\big).\end{aligned}$$

For $(x, t) \in \Omega \times \mathbb{Z}$, define $a(x, t) := a = (a_m)_{m\in\mathbb{Z}}$ as follows.

$$a(x, t)_m := a_m = \begin{cases} t - (x_0 + x_1 + \cdots + x_{m-1}) & \text{if } m > 0,\\ t & \text{if } m = 0,\\ t + (x_{-1} + x_{-2} + \cdots + x_m) & \text{if } m < 0.\end{cases} \tag{4.5}$$

Note that $a_m - a_{m+1} = x_m$. Let $A(x, t)$ be defined by

$$A(x, t) := \{mv_1 + nv_2 : n \le a_m\}.$$

It is not difficult to verify that $-e_1 + A(x, t) \subset A(x, t)$ and $-e_2 + A(x, t) \subset A(x, t)$. In other words, $A(x, t) \in Y_u$ for every $(x, t) \in \Omega \times \mathbb{Z}$.

Lemma 4.1. *Let $(x^{(k)}, t^{(k)})$ be a sequence in $\Omega \times \mathbb{Z}$, and let $(x, t) \in \Omega \times \mathbb{Z}$. For $m \in \mathbb{Z}$ and for $k \in \mathbb{N}$, set $a^{(k)}_m := a(x^{(k)}, t^{(k)})_m$ and $a_m := a(x, t)_m$. Let $A_k := A(x^{(k)}, t^{(k)})$, and $A := A(x, t)$. Then, the following are equivalent.*

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(1) As $k \to \infty$, $(x^{(k)}, t^{(k)}) \to (x, t)$.
(2) For every m, $a_m^{(k)} \to a_m$ as $k \to \infty$.
(3) The sequence $(A_k)_k \to A$.

Proof. It is clear from equation (4.5) that (1) and (2) are equivalent. Suppose that (2) holds. Let $m, n \in \mathbb{Z}$. Suppose $1_A(mv_1 + nv_2) = 1$. Then, $n \leq a_m$. Since $a_m^{(k)} \to a_m$, it follows that eventually $a_m^{(k)} = a_m$. Thus, for large k, $n \leq a_m^{(k)}$, i.e. for large k, $1_{A_k}(mv_1 + nv_2) = 1$. If $1_A(mv_1 + nv_2) = 0$. By the same argument, we can conclude that for large k, $1_{A_k}(mv_1 + nv_2) = 0$. Thus, $1_{A_k}(mv_1 + nv_2) \to 1_A(mv_1 + nv_2)$ for every $m, n \in \mathbb{Z}$. Hence, $(A_k)_k \to A$. This completes the proof of the implication (2) $\implies$ (3).

Assume that (3) holds. Let $m \in \mathbb{Z}$ be given. Note that $1_A(mv_1 + a_m v_2) = 1$. Since $(A_k)_k \to A$, it follows that for large k, $1_{A_k}(mv_1 + a_m v_2) = 1$. In other words, for large k, $a_m \leq a_m^{(k)}$. Also, note that $1_A(mv_1 + (a_m + 1)v_2) = 0$. Since $(A_k)_k \to A$, it follows that for large k, $1_{A_k}(mv_1 + (a_m + 1)v_2) = 0$, i.e. for large k, $a_m + 1 > a_m^{(k)}$. Thus, eventually $a_m \leq a_m^{(k)} < a_m + 1$, i.e. eventually $a_m^{(k)} = a_m$. Hence, for every $m \in \mathbb{Z}$, $a_m^{(k)} \to a_m$ as $k \to \infty$. This completes the proof of the implication (3) $\implies$ (2). □

Proposition 4.1. *With the foregoing notation, the map*

$$\Omega \times \mathbb{Z} \ni (x, t) \mapsto A(x, t) \in Y_u$$

is a $\mathbb{Z}^2$-equivariant homeomorphism.

Proof. First, we check that the prescribed map is $\mathbb{Z}^2$-equivariant. Let $(x, t) \in \Omega \times \mathbb{Z}$ be given. By definition, for $m, n \in \mathbb{Z}$, $mv_1 + nv_2 \in A(x, t) + v_1$ if and only if $n \leq a(x, t)_{m-1}$. Thus,

$$A(x, t) + v_1 = \{mv_1 + nv_2 : n \leq a(x, t)_{m-1}\}.$$

From equation (4.5), it is clear that $a(\tau(x), t + x_{-1})_m = a(x, t)_{m-1}$. Thus, $A(x, t) + v_1 = A(\tau(x), t + x_{-1})$.

For $m, n \in \mathbb{Z}$, $mv_1 + nv_2 \in A(x, t) + v_2$ if and only if $n \leq a(x, t)_m + 1$. Thus,

$$A(x, t) + v_2 = \{mv_1 + nv_2 : n \leq a(x, t)_m + 1\}.$$

From equation (4.5), it is clear that $a(x, t)_m + 1 = a(x, t + 1)_m$. Therefore, $A(x, t) + v_2 = A(x, t + 1)$. As a consequence, $A(x, t) + v_1 = A(\tau(x), t + x_{-1})$ and $A(x, t) + v_2 = A(x, t + 1)$. Since $\{v_1, v_2\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}^2$, it follows that the map

$$\Omega \times \mathbb{Z} \ni (x, t) \mapsto A(x, t) \in Y_u$$

is $\mathbb{Z}^2$-equivariant.

Let $(x, t), (y, s) \in \Omega \times \mathbb{Z}$ be given. Suppose $A(x, t) = A(y, s)$. Then,

$$\{mv_1 + nv_2 : n \leq a(x, t)_m\} = \{mv_1 + nv_2 : n \leq a(y, s)_m\}.$$

The above equality clearly implies that $a(x, t)_m = a(y, s)_m$ for every m. It follows from equation (4.5) that $x = y$ and $t = s$. Thus, the map $\Omega \times \mathbb{Z} \ni (x, t) \mapsto A(x, t) \in Y_u$ is 1-1.

We now show that it is onto. Let $A \in Y_u$ be given. By translating, if necessary, we can assume that $0 \in A$ which implies that $-\mathbb{N}^2 \subset A$.

Let $m \in \mathbb{Z}$ be given. We claim that the set $\{k \in \mathbb{Z} : mv_1 + kv_2 \in A\}$ is non-empty and bounded above. Choose $k_0 < 0$ such that $m + k_0 < 0$. Then, we have

$$mv_1 + k_0 v_2 = (m + k_0)e_1 + k_0 e_2 \in -\mathbb{N}^2 \subseteq A.$$

Hence, the set $\{k \in \mathbb{Z} : mv_1 + kv_2 \in A\}$ is non-empty. Suppose $\{k \in \mathbb{Z} : mv_1 + kv_2 \in A\}$ is not bounded above. Then, for every $k \in \mathbb{N}$, there exists a natural number $n_k \geq k$ such that $mv_1 + n_k v_2 \in A$. Since $A - \mathbb{N}^2 \subset A$, we have $mv_1 + (n_k - \ell)v_2 \in A$ for every $\ell \in \mathbb{N}$ and for every $k \in \mathbb{N}$. This means that, for every $n \in \mathbb{Z}$, $mv_1 + nv_2 \in A$. Again for any $k, \ell \in \mathbb{N}$ and $n \in \mathbb{Z}$, we have

$$(m + n - k)e_1 + (n - \ell)e_2 = mv_1 + nv_2 - ke_1 - \ell e_2 \in A - \mathbb{N}^2 \subset A.$$

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This implies that $A=\mathbb{Z}^2$ which contradicts the fact that $A \in Y_u$. Hence, $\{k \in \mathbb{Z} : mv_1 + kv_2 \in A\}$ is bounded above. The proof of the claim is now complete.

For $m \in \mathbb{Z}$, define

$$a_m := \max\{k \in \mathbb{Z} : mv_1 + kv_2 \in A\}.$$

Then, a_m is an integer. Let $A_a = \{mv_1 + nv_2 : n \leq a_m\}$. We claim that $A = A_a$ and for every $m \in \mathbb{Z}$, $0 \leq a_m - a_{m+1} \leq 1$.

Clearly, $A \subset A_a$. Suppose $mv_1 + nv_2 \in A_a$. By the definition of a_m, $mv_1 + a_m v_2 \in A$. Note that

$$mv_1 + nv_2 = mv_1 + a_m v_2 + (n - a_m)(e_1 + e_2) \in A - \mathbb{N}^2 \subseteq A.$$

Hence, $A_a \subseteq A$. Therefore, $A = A_a$.

Let $mv_1 + nv_2 \in A_a$ be given. Since $A_a - \mathbb{N}^2 \subset A_a$, we have $mv_1 + nv_2 - ke_1 - \ell e_2 \in A_a$, for every $k, l \geq 0$. Therefore, whenever $mv_1 + nv_2 \in A_a$, we have $a_{m-k+\ell} \geq n - \ell$. In particular, when $n = a_m, k = 1$ and $\ell = 0$, we have $a_{m-1} \geq a_m$, and when $n = a_m, k = 0$ and $\ell = 1$, we have $a_{m+1} \geq a_m - 1$. Thus, $0 \leq a_m - a_{m+1} \leq 1$ for every $m \in \mathbb{Z}$. The proof of the claim is now over.

Define $x \in \Omega$ by setting $x_m := a_m - a_{m+1}$ and let $t := a_0$. Clearly, for every m, $a_m = a(x,t)_m$. Therefore, $A_a = A(x,t)$. Consequently, $A = A(x,t)$. This proves the surjectivity of the map

$$\Omega \times \mathbb{Z} \ni (x,t) \to A(x,t) \in Y_u.$$

The fact that the map $\Omega \times \mathbb{Z} \ni (x,t) \to A(x,t) \in Y_u$ is a homeomorphism follows from Lemma 4.1. This completes the proof. □

Remark 4.1. *For $A \in Y_u$, let G_A be the stabiliser of A, i.e.*

$$G_A := \{(m,n) \in \mathbb{Z}^2 : A + (m,n) = A\}.$$

Let $(x,t) \in \Omega \times \mathbb{Z}$, and let $A := A(x,t) \in Y_u$ be the set defined as in Proposition 4.1. Then, $G_A \neq 0$ if and only if x is a periodic point, i.e. there exists $p > 0$ such that $x_{m+p} = x_m$ for all $m \in \mathbb{Z}$. Moreover, there are only countably many periodic points in Ω. This has the consequence that the groupoid $\mathcal{G} := \overline{X}_u \rtimes \mathbb{N}^2$ (and also the transformation groupoid $\overline{Y}_u \rtimes \mathbb{Z}^2$) has only countably many points in its unit space whose stabiliser is non-trivial.

We identify Y_u with $\Omega \times \mathbb{Z}$ via the map prescribed in Proposition 4.1. After an abuse of notation, we write $Y_u = \Omega \times \mathbb{Z}$. Then, $X_u = \Omega \times \mathbb{N}$. Let m be a probability measure on Ω. Define a measure $\overline{m}$ on $\Omega \times \mathbb{Z}$ by setting

$$\overline{m}(E \times \{n\}) = (1 - e^{-\beta(1+\theta)})e^{-\beta(1+\theta)n} m(E)$$

for a measurable subset $E \subset \Omega$.

Define $\chi : \Omega \to \mathbb{R}$ by

$$\chi(x) := \begin{cases} 1 & \text{if } x_{-1} = 0, \\ -\theta & \text{if } x_{-1} = 1. \end{cases}$$

Let β be a real number, and let m be a probability measure on Ω. The probability measure m is said to be $e^{-\beta\chi}$-conformal for the Bernoulli shift τ if for every Borel subset $E \subset \Omega$,

$$m(\tau(E)) = \int_E e^{-\beta\chi} dm.$$

Proposition 4.2. *Suppose $\beta(\theta + 1) > 0$. Then, the map $m \to \overline{m}$ defines a bijection between the set of $e^{-\beta\chi}$-conformal measures on Ω and the set of $e^{-\beta c}$-conformal measures on the $(\mathbb{Z}^2, \mathbb{N}^2)$-space $(Y_u, X_u) = (\Omega \times \mathbb{Z}, \Omega \times \mathbb{N})$.*

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Proof. The proof is not difficult. We have included some details for completeness. Suppose that m is an $e^{-\beta\chi}$-conformal measure on Ω. Clearly, $\overline{m}(X_u)=1$. Let $F\subset\Omega$ be a Borel subset and let $n\in\mathbb{Z}$ be given. It suffices to show that

$$\overline{m}((F\times\{n\})+v_1)=e^{-\beta}\overline{m}(F\times\{n\}),\ \text{ and}$$
$$\overline{m}((F\times\{n\})+v_2)=e^{-\beta(1+\theta)}\overline{m}(F\times\{n\}).$$

Define $F_0:=\{x\in F:x_{-1}=0\}$ and $F_1:=\{x\in F:x_{-1}=1\}$. Calculate as follows to observe that

$$\begin{aligned}\frac{1}{1-e^{-\beta(1+\theta)}}\overline{m}((F\times\{n\})+v_1)&=\frac{1}{1-e^{-\beta(1+\theta)}}\Big(\overline{m}(\tau(F_0)\times\{n\})+\overline{m}(\tau(F_1)\times\{n+1\})\Big)\\&=e^{-\beta n(1+\theta)}m(\tau(F_0))+e^{-\beta(n+1)(1+\theta)}m(\tau(F_1))\\&=e^{-\beta n(1+\theta)}e^{-\beta}m(F_0)+e^{-\beta(n+1)(1+\theta)}e^{\beta\theta}m(F_1)\\&=e^{-\beta n(1+\theta)}e^{-\beta}\big(m(F_0)+m(F_1)\big)\\&=e^{-\beta}\frac{1}{1-e^{-\beta(1+\theta)}}\overline{m}(F\times\{n\}).\end{aligned}$$

Similarly, we can prove that $\overline{m}((F\times\{n\})+v_2)=e^{-\beta(1+\theta)}\overline{m}(F\times\{n\})$. Hence, $\overline{m}$ is an $e^{-\beta c}$-conformal measure on $(\Omega\times\mathbb{Z},\Omega\times\mathbb{N})$.

Conversely, suppose μ is an $e^{-\beta c}$-conformal measure on (Y_u,X_u). Define a measure m on Ω by

$$m(E)=\frac{1}{1-e^{-\beta(1+\theta)}}\mu(E\times\{0\}),$$

for a Borel subset $E\subset\Omega$. Note that for a Borel set $E\subset\Omega$, $E\times\{n\}=(E\times\{0\})+nv_2$. Using the conformality condition on μ and the fact that $E\times\{n\}=(E\times\{0\})+nv_2$, it is routine to see that $\overline{m}=\mu$.

Let $F\subset\Omega$ be measurable. Set $F_0:=\{x\in F:x_{-1}=0\}$ and $F_1:=\{x\in F:x_{-1}=1\}$. Thanks, to the conformality condition on μ and the calculation done earlier, it follows that the equality

$$\mu((F\times\{0\})+v_1)=e^{-\beta}\mu(F\times\{0\}),$$

is equivalent to the equality

$$m(\tau(F_0))+e^{-\beta(1+\theta)}m(\tau(F_1))=e^{-\beta}\big(m(F_0)+m(F_1)\big)=e^{-\beta}m(F).$$

In other words, for every Borel subset $F\subset\Omega$,

$$m(F)=\int_{\tau(F)}e^{\beta(\chi\circ\tau^{-1})}dm.$$

Hence, $\frac{d(m\circ\tau)}{dm}=e^{-\beta\chi}$, i.e. m is $e^{-\beta\chi}$-conformal. The fact that the map $m\to\overline{m}$ is a bijection is clear. □

Remark 4.2. *Note that for the existence of an $e^{-\beta c}$-conformal measure on (Y_u,X_u), it is necessary that $\beta(1+\theta)>0$. This is because, if $\overline{m}$ is an $e^{-\beta c}$-conformal measure on Y_u, then*

$$1=\overline{m}(X_u)=\sum_{n=0}^{\infty}\overline{m}(\Omega\times\{n\})=\sum_{n=0}^{\infty}\overline{m}((\Omega\times\{0\})+nv_2)=\sum_{n=0}^{\infty}e^{-\beta(1+\theta)n}\overline{m}(\Omega\times\{0\}).$$

Hence, the condition $\beta(1+\theta)>0$ is necessary.

The bijection $m\to\overline{m}$ of Proposition 4.2 preserves essential freeness, ergodicity and type. Thus, (Ω,τ,m) is essentially free if and only if $(Y_u,\mathbb{Z}^2,\overline{m})$ is essentially free. Similarly, (Ω,τ,m) is ergodic and is of type t if and only if $(Y_u,\mathbb{Z}^2,\overline{m})$ is ergodic and is of type t. Here, $t\in\{I,II,III\}$.

We are now in a position to determine the values of β and θ for which there is a β-KMS state on $C_c^*(\mathbb{N}^2)$ for $\sigma^c:=\{\sigma_t\}_{t\in\mathbb{R}}$. Recall that the flow $\sigma^c:=\{\sigma_t\}_{t\in\mathbb{R}}$ on $C_c^*(\mathbb{N}^2)$ is defined by

$$\sigma_t(v_{(m,n)})=e^{i(m+n\theta)t}v_{(m,n)}.$$

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Proposition 4.3. *Suppose β is a non-zero real number. The following are equivalent.*

(1) There is a β-KMS state on $C_c^*(\mathbb{N}^2)$ for σ^c.
(2) There is an $e^{-\beta c}$-conformal measure on (Y_u, X_u).
(3) There is an $e^{-\beta\chi}$-conformal measure on Ω and $\beta(1+\theta) > 0$.
(4) $\beta > 0$ and $\theta \geq 0$.

Proof. The equivalence between (1), (2) and (3) follow from Propositions 3.2, 4.2 and Remark 4.2. We prove the equivalence between (3) and (4). Assume that (3) holds. Let m be an $e^{-\beta\chi}$-conformal measure on Ω. Suppose $\theta < 0$. Then, the potential $\beta\chi$ is strictly negative or strictly positive. Suppose $\beta\chi$ is strictly positive. Then, there exists a real number $a > 0$ such that $\beta\chi > a$. Note that

$$1 = m(\tau(\Omega)) = \int_\Omega e^{-\beta\chi} dm \leq e^{-a} < 1.$$

This is a contradiction. A similar contradiction will be met if $\beta\chi$ is strictly negative. Therefore, $\theta \geq 0$. (A direct application of Theorem 6.2 of [5] could also have been made instead). The condition $\beta(1+\theta) > 0$ implies that $\beta > 0$. This completes the proof of (3) $\Longrightarrow$ (4).

Assume now that $\beta > 0$ and $\theta \geq 0$. Let us fix some notation. For $n \in \mathbb{Z}$, define $S_n(\chi)$ by setting

$$S_n(\chi)(x) := \begin{cases} \sum_{j=0}^{n-1} \chi \circ \tau^j(x) & \text{if } n \geq 1, \\ 0 & \text{if } n = 0 \\ -\sum_{j=1}^{|n|} \chi \circ \tau^{-j}(x) & \text{if } n \leq -1. \end{cases}$$

Note that if $n \leq -1$, then $S_n(\chi)(x) = -|n| + (1+\theta)(x_0 + x_1 + \cdots + x_{|n|-1})$, and if $n \geq 1$, we have $S_n(\chi)(x) = n - (1+\theta)(x_{-1} + x_{-2} + \cdots + x_{-n})$.

We apply Lemma 4.3 of [5] which states the following. Let $x \in \Omega$ be given. If x is periodic of period $p > 0$, then there is an $e^{-\beta\chi}$-conformal measure on Ω concentrated on the orbit of x iff $S_p(\chi)(x) = 0$. If x is not periodic, then there is an $e^{-\beta\chi}$-conformal measure concentrated on the orbit of x if and only if $\sum_{n\in\mathbb{Z}} e^{-\beta S_n(\chi)(x)} < \infty$.

Case 1: Suppose $\theta = 0$. Let $x \in \Omega$ be such that $x_m = 1$ for every $m \in \mathbb{Z}$. Clearly, x is periodic of period 1, and $S_1(\chi)(x) = 0$. Therefore, there is an $e^{-\beta\chi}$-conformal measure supported on the orbit of x.

Case 2: Suppose $\theta > 0$. We exhibit uncountably many non-periodic points x for which the series $\sum_{n\in\mathbb{Z}} e^{-\beta S_n(\chi)(x)}$ converges. Let ℓ be a positive integer such that $\ell > 1+\theta$. Suppose $S \subset \ell\mathbb{N}$. Denote the indicator function of S by 1_S. Define $x \in \Omega$ by setting

$$x_m := \begin{cases} 1 & \text{if } m \geq 0 \\ 1_S(-m) & \text{if } m < 0 \end{cases}$$

Note that $S_n(\chi)(x) = |n|\theta$ if $n \leq -1$. Therefore, the series $\sum_{n\leq -1}^{\infty} e^{-\beta S_n(\chi)(x)}$ is convergent.

For $n \geq 1$, observe that since $x_{-1} + x_{-2} + \cdots + x_{-n} = |S \cap \{1, 2, \cdots, n\}| \leq \frac{n}{\ell}$,

$$S_n(\chi)(x) = n - (1+\theta)(x_{-1} + x_{-2} + \cdots + x_{-n}) \geq n(1 - \frac{1+\theta}{\ell}).$$

Thus, $e^{-\beta S_n(\chi)(x)} \leq e^{-\beta n(1-\frac{1+\theta}{\ell})}$. Consequently, the series $\sum_{n\geq 1}^{\infty} e^{-\beta S_n(\chi)(x)}$ converges. Thanks to Lemma 4.9 of [5], there is an $e^{-\beta\chi}$-conformal measure supported on the orbit of x. The proof is complete. □

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We discuss the structure of KMS states in more detail now. Assume hereafter that β is an arbitrary positive real number.

The case $\theta = 0$: Assume that $\theta = 0$. Denote the element in $\{0, 1\}^{\mathbb{Z}}$ whose every entry is 1 by $\underline{1}$. Suppose that m is an $e^{-\beta\chi}$-conformal measure on Ω. We claim that m is concentrated on $\underline{1}$. We claim that $m(\{x : x_{-1} = 0\}) = 0$. Suppose $m(\{x : x_{-1} = 0\}) > 0$. Then, by conformality, we have

$$\begin{aligned} 1 &= m(\tau(\Omega)) \\ &= \int e^{-\beta\chi}\,dm \\ &= e^{-\beta} m(\{x : x_{-1} = 0\}) + m(\{x : x_{-1} = 1\}) \\ &< m(\{x : x_{-1} = 0\}) + m(\{x : x_{-1} = 1\}) = 1 \end{aligned}$$

which is a contradiction.

Therefore, for almost all x, $x_{-1} = 1$. Since m is quasi-invariant for the Bernoulli shift τ, we have, for every k, $x_k = 1$ for almost all x. Hence, $x = \underline{1}$ for almost all x. This proves the claim.

Let ω be a β-KMS state on $C_c^*(\mathbb{N}^2)$. Thanks to Propositions 4.1 and 4.2 and by what we have proved now, its associated $e^{-\beta c}$-conformal measure m_ω is supported on $Orb(A(\underline{1}, 0))$. Note that

$$A := A(\underline{1}, 0) = \{(m, n) : m \leq 0\}$$

whose stabiliser $G_A = \{0\} \times \mathbb{Z} = c^{-1}(\{0\})$. Appealing to Corollary 1.4 of [22], we see that there exists a state ϕ on $C^*(G_A)$, or equivalently a probability measure μ on $\mathbb{T}$ such that

$$\omega(fw_{(m,n)}) = \delta_{m,0}\Big(\int z^n d\mu(z)\Big)(1 - e^{-\beta}) \sum_{k=0}^{\infty} e^{-\beta k} f(A(\underline{1}, k)). \tag{4.6}$$

Conversely, given a probability measure μ on $\mathbb{T}$, if we define ω as in equation (4.6), then ω will be a β-KMS state. Thus, when $\theta = 0$, the simplex of β-KMS states for σ^c is homeomorphic to the simplex of probability measures on the circle.

Hereafter, we assume that $\beta > 0$ and $\theta > 0$. We summarise the structure of extremal β-KMS states on $C_c^*(\mathbb{N}^2)$, that result from our discussions so far, as follows.

(1) Suppose ω is an extremal β-KMS state on $C_c^*(\mathbb{N}^2)$. Denote the associated conformal measure on Y_u by $\overline{m}$, and let m be the corresponding $e^{-\beta\chi}$-conformal measure on Ω obtained via the map in Proposition 4.2. Then, thanks to Proposition 3.3, m is ergodic. Suppose that m is atomic and m is concentrated on $Orbit(x)$ for some $x \in \{0, 1\}^{\mathbb{Z}}$. Set $A := A(x, 0)$. Thanks to Proposition 3.4, if the stabiliser $G_A = \{0\}$, then $\omega = \omega_{\overline{m}}$. If $G_A \neq \{0\}$, then ω is as in Proposition 3.4. In both cases, ω is of type I. If m is non-atomic, then $\omega = \omega_{\overline{m}}$ where $\omega_{\overline{m}}$ is the state corresponding to $\overline{m}$ obtained via the conditional expectation. This follows by applying Corollary 1.2 of [22] and by Remark 4.1. In this case, ω is of type II or III depending upon whether m (or equivalently $\overline{m}$) is of type II or of type III.

(2) Conversely, suppose m is an ergodic $e^{-\beta\chi}$-conformal measure on Ω. Let $\overline{m}$ be the corresponding $e^{-\beta c}$-conformal measure on Y_u given by Proposition 4.2. If m is non-atomic, then $\omega_{\overline{m}}$ is an extremal β-KMS state and its type is the same as that of m. To see that $\omega_{\overline{m}}$ is extremal, it suffices to show that the GNS representation is factorial. Thanks to Theorem 3.1, it suffices to prove that the $\mathbb{Z}^2$-action on Y_u is essentially free.

Let $(m, n) \neq (0, 0)$ be given. Then, by Remark 4.1, the set

$$\{A \in Y_u : A + (m, n) = A\},$$

is countable. As $\overline{m}$ is non-atomic, it follows that $\{A \in Y_u : A + (m, n) = A\}$, which is countable, is a set of measure zero. Thus, the $\mathbb{Z}^2$-action on Y_u is essentially free.

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Suppose $\overline{m}$ is atomic and $\overline{m}$ is concentrated on $Orbit(A)$ for some $A \in Y_u$. From Proposition 3.4, if the stabiliser $G_A = \{0\}$, then $\omega_{\overline{m}}$ is an extremal β-KMS state of type I. If $G_A \neq \{0\}$, then $\omega_{\chi,m}$, as defined in Proposition 3.4, is an extremal β-KMS state of type I.

(3) We have exhibited, in the proof of Proposition 4.3, uncountably many atomic probability measures on Ω that are $e^{-\beta\chi}$-conformal. Thus, there are uncountably many type I β-KMS states on $C_c^*(\mathbb{N}^2)$ for σ^c.

We end this section by exhibiting a type II KMS state when θ is irrational and a type III KMS state when $\theta = 1$. We prove this by appealing to Proposition 3.5. Uncountably, many such examples for every $\theta > 0$ and for every $\beta > 0$ will be constructed in Section 5.

A type II example: Assume that $\theta \in (0, \infty)$ is irrational. Suppose $\beta > 0$. Let $Y := \mathbb{R}$ and $X := [0, \infty)$. Define a $\mathbb{Z}^2$-action on Y by

$$t + e_1 := t + 1; \text{ and } t + e_2 := t + \theta.$$

Clearly, $X + \mathbb{N}^2 \subset X$. Also, (Y, X) is a pure $(\mathbb{Z}^2, \mathbb{N}^2)$-space. Since θ is irrational, the $\mathbb{Z}^2$-action on Y is free. Let m be the measure on $\mathbb{R}$ such that $dm = \beta e^{-\beta t}dt$. Then, $m(X) = 1$ and m is $e^{-\beta c}$-conformal. As $\mathbb{Z} + \mathbb{Z}\theta$ is a dense subgroup of $\mathbb{R}$, the $\mathbb{Z}^2$-action on Y is ergodic. Clearly, it is of type II as m is absolutely continuous w.r.t. the Lebesgue measure.

Observe that, for $t \in \mathbb{R}$,

$$Q_t := \{(m, n) \in \mathbb{Z}^2 : t - (me_1 + ne_2) \in X\} = \{(m, n) \in \mathbb{Z}^2 : t \geq m + n\theta\}.$$

If $t_1 < t_2$, the density of $\mathbb{Z} + \mathbb{Z}\theta$ in $\mathbb{R}$ implies that there exists $(m, n) \in \mathbb{Z}^2$ such that $t_1 < m + n\theta < t_2$. Then, $(m, n) \in Q_{t_2}$ but $(m, n) \notin Q_{t_1}$. Therefore, the map

$$Y \ni t \to Q_t \in Y_u$$

is injective.

Applying Proposition 3.5, we get an extremal β-KMS state of type II when θ is irrational.

A type III example: Assume that $\theta = 1$. We make use of Arnold's dyadic adding machine to produce a type III example in this situation. Let us recall the basics on adding machine from [1]. Let $\beta > 0$ be fixed. Let $p \in (0, \frac{1}{2})$ be such that $\frac{1-p}{p} = e^{\beta}$.

Let

$$\mathcal{D} := \prod_{n=1}^{\infty}\{0, 1\} := \{(x_1, x_2, \cdots,) : x_n \in \{0, 1\}\}$$

be the group of dyadic integers. Let $\mu := \otimes_{k=1}^{\infty}\mu_k$ be the product measure where the measure μ_k on $\{0, 1\}$ is given by

$$\mu_k(\{0\}) = 1 - p; \mu_k(\{1\}) = p.$$

We remove from $\mathcal{D}$ the null set of eventually constant sequences, and we denote the resulting set again by $\mathcal{D}$.

Denote the map on $\mathcal{D}$ that corresponds to addition by $\underline{1}$ by τ. Recall that

$$\tau(1, 1, 1, \cdots, 1, 0, *, *, \cdots) = (0, 0, 0, \cdots, 0, 1, *, *, \cdots).$$

Define $\phi : \mathcal{D} \to \mathbb{Z}$ by $\phi(x) := \min\{n \geq 1 : x_n = 0\} - 2$. Then,

$$\frac{d(\mu \circ \tau)}{d\mu} = e^{\beta\phi}.$$

Let $Y := \mathcal{D} \times \mathbb{Z}$ and $X := \mathcal{D} \times \{0, 1, 2, \cdots\}$. Let $\overline{\mu}$ be the measure on Y given by

$$d\overline{\mu} := (1 - e^{-\beta})e^{-\beta n}d\mu dn$$

where dn is the counting measure on $\mathbb{Z}$.

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Define a $\mathbb{Z}^2$-action on Y by

$$(x, t) + e_1 := (\tau(x), \phi(x) + t + 1); \text{ and } (x, t) + e_2 := (x, t + 1).$$

Since $\phi \geq -1$, it follows that $X + \mathbb{N}^2 \subset X$. Also, it is routine to verify that (Y, X) is a pure $(\mathbb{Z}^2, \mathbb{N}^2)$-space and $\overline{\mu}$ is an $e^{-\beta c}$-conformal measure.

Lemma 4.2. *Let $x, y \in \mathcal{D}$ be such that $x \neq y$. Then, there exists an integer m such that $\phi(\tau^m(x)) \neq \phi(\tau^m(y))$.*

Proof. Let k be the least integer for which $x_k \neq y_k$. We can, without loss of generality, assume that $x_k = 0$ and $y_k = 1$. Let $m := 2^{k-1} - 1 - (x_1 + 2x_2 + \cdots + 2^{k-2}x_{k-1})$. A calculation with 'binary arithmetic' shows that

$$\tau^m(x) = (\underbrace{1, 1, \cdots, 1}_{k-1}, 0, *, *, *, \cdots)$$
$$\tau^m(y) = (\underbrace{1, 1, 1, \cdots 1}_{k-1}, 1, *, *, *, \cdots).$$

Thus, $\phi(\tau^m(x)) = k - 2 < \phi(\tau^m(y))$. The proof is complete. □

Proposition 4.4. *With the foregoing notation, we have the following.*

(1) The $\mathbb{Z}^2$-action on Y is essentially free and ergodic.
(2) The measure $\overline{\mu}$ is of type III.
(3) The map

$$Y \ni (x, t) \to Q_{(x,t)} \in Y_u,$$

is injective.

Proof. The first two statements are straightforward consequences of Proposition 1.2.8 and Theorem 1.2.9 of [1]. We include a brief explanation.

Note that the e_2-action on Y is by translation by 1 on the second coordinate. This forces that any $\mathbb{Z}^2$-invariant subset of Y must be of the form $E \times \mathbb{Z}$ for some subset E of $\mathcal{D}$. The fact that $E \times \mathbb{Z}$ is invariant under the action of e_1 implies that $\tau(E) = E$. By Proposition 1.2.8 of [1], it follows that E is either null or co-null. Hence, the $\mathbb{Z}^2$-action is ergodic. Using the fact that the action of τ on $\mathcal{D}$ is essentially free, it is routine to check that the $\mathbb{Z}^2$-action is essentially free.

Suppose $\overline{\mu}$ is not of type III. Let $\overline{\nu}$ be a σ-finite measure that is absolutely continuous w.r.t. $\overline{\mu}$ such that $\overline{\nu}$ is $\mathbb{Z}^2$-invariant. The invariance of $\overline{\nu}$ under e_2 implies that $d\overline{\nu} := d\nu dn$ for some σ-finite measure ν on $\mathcal{D}$ which is absolutely w.r.t. μ. The invariance of $\overline{\nu}$ under e_1 implies that ν is τ-invariant which is a contradiction to Theorem 1.2.9 of [1]. The proof of (1) and (2) is complete.

For $m \in \mathbb{Z}$ and $x \in \mathcal{D}$, let $c(m, x) \in \mathbb{Z}$ be defined by

$$c(m, x) := \begin{cases} \sum_{k=0}^{m-1} \phi(\tau^k(x)) & \text{if } m \geq 1, \\ 0 & \text{if } m = 0, \\ -\sum_{k=1}^{|m|} \phi(\tau^{-k}x) & \text{if } m \leq -1. \end{cases} \tag{4.7}$$

Let $(x, s), (y, t) \in Y$ be such that $Q_{(x,s)} = Q_{(y,t)}$. Notice that

$$Q_{(x,s)} = \{(m, n) \in \mathbb{Z}^2 : (x, s) - (me_1 + ne_2) \in X\} = \{(m, n) \in \mathbb{Z}^2 : n \leq -m + c(-m, x) + s\}.$$

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The equality $Q_{(x,s)} = Q_{(y,t)}$ implies in particular that for every $m \in \mathbb{Z}$,

$$-m + c(-m, x) + s = -m + c(-m, y) + t.$$

Substituting, $m = 0$ in the above equality, we deduce that $s = t$. The fact that $c(-m, x) = c(-m, y)$ for every $m \in \mathbb{Z}$ implies that $\phi(\tau^m x) = \phi(\tau^m y)$ for each $m \in \mathbb{Z}$. By Lemma 4.2, we have $x = y$. This proves (3) and the proof is complete. □

Applying Proposition 3.5, we see that when $\theta = 1$, there is an extremal β-KMS state of type III.

5. Examples of type II and type III

In this section, we construct uncountably many extremal β-KMS states of type II and of type III for every $\beta > 0$ and for every $\theta > 0$. It suffices to exhibit uncountably many non-atomic ergodic $e^{-\beta c}$-conformal measures on the $(\mathbb{Z}^2, \mathbb{N}^2)$-space (Y_u, X_u) of the desired type. We do this by appealing to Proposition 3.5. For the rest of this section, we assume that β is an arbitrary positive real number. Recall that $e_1 = (1, 0)$, $e_2 = (0, 1)$, $v_1 = e_1$ and $v_2 = e_1 + e_2$.

Type II examples for an irrational θ: Assume that $\theta > 0$ is irrational. Choose $\alpha \in (0, 1)$ such that $1, \alpha$ and θ are rationally independent. For $\delta \in (0, \theta]$, let C^δ be the cone in $\mathbb{R}^2$ generated by $(0, 1)$ and (α, δ), i.e.

$$C^\delta := \{x(0, 1) + y(\alpha, \delta) : x, y \geq 0\}.$$

Let $\pi : \mathbb{R}^2 \to \frac{\mathbb{R}^2}{\mathbb{Z}\times\{0\}} = \mathbb{T} \times \mathbb{R}$ be the quotient map. We denote $\pi(x, y)$ by $\overline{(x, y)}$. Denote the subgroup generated by $\overline{(\alpha, \theta)}$ and $\overline{(0, 1)}$ by Γ. Let $\phi : \mathbb{Z}^2 \to \Gamma$ be the homomorphism defined by

$$\phi(e_1) = \overline{(0, 1)}; \ \phi(e_2) = \overline{(\alpha, \theta)}.$$

Note that ϕ is an isomorphism as α and θ are irrational.

Set $Y^\delta := \frac{\mathbb{R}^2}{\mathbb{Z}\times\{0\}} = \mathbb{T} \times \mathbb{R}$ and $X^\delta := \pi(C^\delta)$. It is not difficult to see that X^δ is closed. Define a $\mathbb{Z}^2$-action on Y^δ by

$$\overline{(x, y)} + e_1 := \overline{(x, y)} + \phi(e_1) = \overline{(x, y + 1)}$$
$$\overline{(x, y)} + e_2 := \overline{(x, y)} + \phi(e_2) = \overline{(x + \alpha, y + \theta)}.$$

Since, $\delta \in (0, \theta]$, it follows that $(\alpha, \theta) \in C^\delta$. The fact that C^δ is a cone implies that X^δ is invariant under $\mathbb{N}^2$, i.e. $X^\delta + \mathbb{N}^2 \subset X^\delta$. Thus, (Y^δ, X^δ) is a $(\mathbb{Z}^2, \mathbb{N}^2)$-space. We leave it to the reader to verify that it is pure.

Define a measure μ on Y^δ by $d\mu := e^{-\beta y}dxdy$. Here, dx is the Haar measure on $\mathbb{T}$ and dy is the Lebesgue measure on $\mathbb{R}$. Note that $X^\delta \subset \mathbb{T} \times [0, \infty)$ and the latter set has finite μ-measure. Thus, $\mu(X^\delta) < \infty$. Define a measure m on Y^δ by $dm := \frac{1}{\mu(X^\delta)} d\mu$. It is clear that m is an $e^{-\beta c}$-conformal measure.

Remark 5.1. *We need the following 'pictorially' obvious facts. We omit the rigorous proofs as they are elementary.*

(1) The interior $Int(X^\delta)$ is dense in X^δ.
(2) If $X^\delta + \overline{(x, y)} = X^\delta$ for some $\overline{(x, y)} \in \mathbb{T} \times \mathbb{R}$, then $\overline{(x, y)} = \overline{(0, 0)}$.
(3) Let $\delta_1, \delta_2 \in (0, \theta]$. Suppose there exists $\overline{(x, y)} \in \mathbb{T} \times \mathbb{R}$ for which $X^{\delta_1} + \overline{(x, y)} = X^{\delta_2}$. Then, $\delta_1 = \delta_2$.

Proposition 5.1. *Keep the foregoing notation.*

(1) The $\mathbb{Z}^2$-action on Y^δ is essentially free and ergodic.
(2) The measure m is of type II.

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(3) The map, denoted T^δ,

$$Y^\delta \ni \overline{(x,y)} \to Q_{\overline{(x,y)}} \in Y_u$$

is injective.

For $\delta \in (0,\theta]$, let $\mu^\delta := T^\delta_* m$ be the push-forward measure on Y_u. If $\delta_1 \neq \delta_2$, then $\mu^{\delta_1} \neq \mu^{\delta_2}$. In particular, there are uncountably many extremal β-KMS states of type II for every irrational θ.

Proof. Note that the measure m is absolutely continuous with respect to $dxdy$, which is the Haar measure on the cylinder $\mathbb{T}\times\mathbb{R}$, and $dxdy$ is clearly $\mathbb{Z}^2$-invariant. As $1, \alpha$ and θ are rationally independent, the subgroup Γ is dense in $\mathbb{T}\times\mathbb{R}$. Hence, the action of $\mathbb{Z}^2$ on Y^δ, which is by translations by Γ via the isomorphism $\phi:\mathbb{Z}^2\to\Gamma$, is ergodic. As the map ϕ is an isomorphism, it is also clear that the $\mathbb{Z}^2$-action on Y^δ is essentially free. The proof of (1) and (2) are now complete.

Let $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$ be such that $Q_{\overline{(x_1,y_1)}} = Q_{\overline{(x_2,y_2)}}$. Note that for $i=1,2$,

$$\begin{aligned} Q_{\overline{(x_i,y_i)}} &:= \{(m,n) : \overline{(x_i,y_i)} - m\overline{(\alpha,\theta)} - n\overline{(0,1)} \in X^\delta\} \\ &= \{(m,n) : m\overline{(\alpha,\theta)} + n\overline{(0,1)} \in -X^\delta + \overline{(x_i,y_i)}\}. \end{aligned}$$

The above equation, together with the equality $Q_{\overline{(x_1,y_1)}} = Q_{\overline{(x_2,y_2)}}$ implies that

$$(-X^\delta + \overline{(x_1,y_1)})\cap\Gamma = (-X^\delta + \overline{(x_2,y_2)})\cap\Gamma.$$

Taking closure in the above equality, and using the fact that $\overline{Int(X^\delta)} = X^\delta$ and Γ is dense in Y^δ, we deduce that $-X^\delta + \overline{(x_1,y_1)} = -X^\delta + \overline{(x_2,y_2)}$. By (2) of Remark 5.1, we have $\overline{(x_1,y_1)} = \overline{(x_2,y_2)}$. This proves (3).

Let $\delta_1,\delta_2\in(0,\theta]$. Suppose $\mu^{\delta_1}=\mu^{\delta_2}=\mu$. Note that, for $i=1,2$, μ^{δ_i} is concentrated on the image of T^{δ_i}. Thus, up to a set of μ-measure zero, $T^{\delta_1}(Y^{\delta_1}) = T^{\delta_2}(Y^{\delta_1})$. Choose $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$ such that $T^{\delta_1}(\overline{(x_1,y_1)}) = T^{\delta_2}(\overline{(x_2,y_2)})$. This implies that

$$(-X^{\delta_1} + \overline{(x_1,y_1)})\cap\Gamma = (-X^{\delta_2} + \overline{(x_2,y_2)})\cap\Gamma.$$

Taking closure, we deduce

$$-X^{\delta_1} + \overline{(x_1,y_1)} = -X^{\delta_2} + \overline{(x_2,y_2)}.$$

By (3) of Remark 5.1, we have $\delta_1=\delta_2$. This completes the proof. □

Type II examples for $\theta = 1$: Assume that $\theta=1$. Recall that the homomorphism $c:\mathbb{Z}^2\to\mathbb{R}$ is defined by $c(e_1)=1$ and $c(e_2)=1$. For an irrational $\alpha\in(0,1)$, let R_α be the rotation by angle α on $\mathbb{T} := [0,1)$. In additive notation, $R_\alpha(x) = x+\alpha \mod 1$. For $\eta\in(0,1)$, let $h_\eta := 1_{[0,\eta)}$ and let $\phi_{\eta,\alpha} := h_\eta\circ R_\alpha - h_\eta$.

Lemma 5.1. *Let $\alpha\in(0,1)$ be irrational and let $\eta\in(0,1)$ be given.*

(1) Suppose $x,y\in\mathbb{T}$ are distinct points. Then, there exists $m\in\mathbb{Z}$ such that $h_\eta(R^m_\alpha(x)) \neq h_\eta(R^m_\alpha(y))$.
(2) Suppose $\eta<\min\{\alpha,1-\alpha\}$ and let $x,y\in\mathbb{T}$ be distinct points. Then, there exists $m\in\mathbb{Z}$ such that $\phi_{\eta,\alpha}(R^m_\alpha(x)) \neq \phi_{\eta,\alpha}(R^m_\alpha(y))$.

Proof. We denote the map $h_\eta:[0,1)\to\mathbb{R}$ simply by h. Denote the periodic extension of h to $\mathbb{R}$ by $\widetilde{h}$. Suppose, for $x,y\in\mathbb{T}$, $h(R^m_\alpha(x)) = h(R^m_\alpha(y))$ for every $m\in\mathbb{Z}$. Then,

$$\widetilde{h}(x+m\alpha+n) = \widetilde{h}(y+m\alpha+n)$$

for all $m,n\in\mathbb{Z}$. This means that for all $m,n\in\mathbb{Z}$, $(T_x\widetilde{h})(m\alpha+n) = (T_y\widetilde{h})(m\alpha+n)$, where $T_x\widetilde{h}, T_y\widetilde{h}$ denote the translations of $\widetilde{h}$ by x,y respectively.

Since the set $D=\{m\alpha+n : m,n\in\mathbb{Z}\}$ is dense in $\mathbb{R}$, $T_x\widetilde{h}$, $T_y\widetilde{h}$ are right continuous and they agree on D, we can conclude that they agree on $\mathbb{R}$. Then, $T_z\widetilde{h}=\widetilde{h}$ on $\mathbb{R}$, where $z=x-y$. This clearly implies that z is an integer. This proves (1).

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Let $\eta < \min\{\alpha, 1-\alpha\}$ be given. Denote $\phi_{\eta,\alpha}$ by ϕ. Note that for $x \in \mathbb{T}$,

$$\phi(R_\alpha^{-1}x) := \begin{cases} 1 & \text{if } x \in [0, \eta), \\ 0 & \text{if } x \in [\eta, \alpha), \\ -1 & \text{if } x \in [\alpha, \alpha+\eta), \\ 0 & \text{if } x \in [\alpha+\eta, 1). \end{cases} \tag{5.8}$$

Let $\chi : \{1, 0, -1\} \to \{0, 1\}$ be defined by $\chi(1) = 1$ and $\chi(j) = 0$ if $j \in \{0, -1\}$. Then,

$$\chi(\phi(R_\alpha^{-1}x)) = h_\eta(x).$$

The proof of (2) follows from the above equality and (1). This completes the proof. □

Let $\alpha \in (0, 1)$ be irrational, and let $\eta \in (0, 1)$ be such that $\eta < \min\{\alpha, 1-\alpha\}$. Denote h_η by h and $\phi_{\eta,\alpha}$ by ϕ. Let $Y_{\alpha,\eta} := \mathbb{T} \times \mathbb{Z}$, and let $X_{\alpha,\eta} := \mathbb{T} \times \mathbb{N}$. Define a $\mathbb{Z}^2$-action on $Y_{\alpha,\eta}$ by

$$(x, t) + e_1 := (x + \alpha, \phi(x) + t + 1); \text{ and } (x, t) + e_2 := (x, t + 1).$$

Since $\phi \geq -1$, it follows that $X_{\alpha,\eta} + \mathbb{N}^2 \subset X_{\alpha,\eta}$. Also, it is routine to verify that $(Y_{\alpha,\eta}, X_{\alpha,\eta})$ is a pure $(\mathbb{Z}^2, \mathbb{N}^2)$-space.

Let β be an arbitrary positive real number. Define a probability measure $\nu_{\alpha,\eta}$ on $\mathbb{T}$ by

$$\nu_{\alpha,\eta}(E) := \frac{1}{||e^{\beta h}||_1} \int_E e^{\beta h(x)} dx.$$

In the above formula, dx is the Haar measure on the circle. Clearly,

$$\frac{d(\nu_{\alpha,\eta} \circ R_\alpha)}{d\nu_{\alpha,\eta}} = e^{\beta\phi}.$$

Define a measure $\mu_{\alpha,\eta}$ on $Y_{\alpha,\eta}$ by

$$\mu_{\alpha,\eta}(E \times \{n\}) = (1 - e^{-\beta})e^{-\beta n}\nu_{\alpha,\eta}(E).$$

It is easily verifiable that $\mu_{\alpha,\eta}$ is an $e^{-\beta c}$-conformal measure on the $(\mathbb{Z}^2, \mathbb{N}^2)$-space $(Y_{\alpha,\eta}, X_{\alpha,\eta})$.

Proposition 5.2. *With the foregoing notation, we have the following.*

1. *The $\mathbb{Z}^2$-action on $Y_{\alpha,\eta}$ is essentially free and ergodic.*
2. *The measure $\mu_{\alpha,\eta}$ is of type II.*
3. *The map*

$$Y_{\alpha,\eta} \ni (x, t) \to Q_{(x,t)} \in Y_u$$

is injective.

4. *Suppose $\alpha_1, \alpha_2 \in (0, \frac{1}{2})$ are distinct irrationals, and let $\eta_1, \eta_2 \in (0, 1)$ be such that for $i = 1, 2$, $\eta_i < \min\{\alpha_i, 1-\alpha_i\}$. Then, the $(\mathbb{Z}^2, \mathbb{N}^2)$-spaces $(Y_{\alpha_1,\eta_1}, X_{\alpha_1,\eta_1}, \mu_{\alpha_1,\eta_1})$ and $(Y_{\alpha_2,\eta_2}, X_{\alpha_2,\eta_2}, \mu_{\alpha_2,\eta_2})$ are not isomorphic. In particular, there are uncountably many type II extremal β-KMS states for $\theta = 1$.*

Proof. Note that $\mu_{\alpha,\eta}$ is absolutely continuous w.r.t. $dxdn$ which is a $\mathbb{Z}^2$-invariant measure. The proofs of (1) and (3) are very similar to the proof of Proposition 4.4. The proof that the map

$$Y_{\alpha,\eta} \ni (x, t) \to Q_{(x,t)} \in Y_u$$

is injective is similar to Proposition 4.4, where instead of Lemma 4.2, we use Part (2) of Lemma 5.1.

We now prove (4). Let $\alpha_1, \alpha_2 \in (0, \frac{1}{2})$ be irrationals, and let $\eta_1, \eta_2 \in (0, 1)$ be such that for $i = 1, 2$, $\eta_i < \min\{\alpha_i, 1-\alpha_i\}$. Suppose $(Y_{\alpha_1,\eta_1}, X_{\alpha_1,\eta_1}, \mu_{\alpha_1,\eta_1})$ and $(Y_{\alpha_2,\eta_2}, X_{\alpha_2,\eta_2}, \mu_{\alpha_2,\eta_2})$ are isomorphic. Then, there exist $\mathbb{Z}^2$-invariant null sets $N_{\alpha_i} \subset Y_{\alpha_i,\eta_i}$ for $i = 1, 2$ and an invertible measurable map $S : Y_{\alpha_1,\eta_1} \backslash N_{\alpha_1} \to Y_{\alpha_2,\eta_2} \backslash N_{\alpha_2}$ such that

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(i) the map S is $\mathbb{Z}^2$-equivariant, $S(X_{\alpha_1,\eta_1}\backslash N_{\alpha_1}) = X_{\alpha_2,\eta_2}\backslash N_{\alpha_2}$, and
(ii) for every Borel subset $E \subset Y_{\alpha_2,\eta_2}\backslash N_{\alpha_2}$, $\mu_{\alpha_2,\eta_2}(E) = \mu_{\alpha_1,\eta_1}(S^{-1}(E))$.

Since μ_{α_i,η_i} is absolutely continuous w.r.t. $dxdn$, it follows that S preserves the measure $dxdn$. Let $S(x, t) = (S_1(x, t), S_2(x, t))$, where S_1 and S_2 are the coordinate functions. Note that e_2 acts by translation by 1 on the second coordinate. Hence, for $i = 1, 2$, N_{α_i} must be of the form $N_{\alpha_i} = U_{\alpha_i} \times \mathbb{Z}$ for some R_{α_i}-invariant subset $U_{\alpha_i} \subseteq \mathbb{T}$. Since the map S is $\mathbb{Z}^2$-equivariant, from the action of e_2, we notice that,

$$S_1(x, t) = S_1(x, t+1) \quad \text{and} \quad S_2(x, t+1) = S_2(x, t) + 1. \tag{5.9}$$

In particular, the function $(x, t) \to S_1(x, t)$ does not depend on t. As S maps $X_{\alpha_1,\eta_1}\backslash N_{\alpha_1}$ onto $X_{\alpha_2,\eta_2}\backslash N_{\alpha_2}$ and $X_{\alpha_i,\eta_i}\backslash(X_{\alpha_i,\eta_i} + e_2) = \mathbb{T} \times \{0\}$, it follows that $S_2(x, 0) = 0$ for $x \in \mathbb{T}\backslash U_{\alpha_1}$. Equation (5.9) implies that $S_2(x, t) = t$ for $(x, t) \in \mathbb{T}\backslash U_{\alpha_1} \times \mathbb{Z}$.

Now, from the action of e_1, and by equation (5.9), we can conclude that

$$R_{\alpha_2} S_1(x, 0) = S_1(R_{\alpha_1}x, \phi_{\eta_1,\alpha_1}(x) + 1) = S_1(R_{\alpha_1}x, 0).$$

Define a map $\widetilde{S} : \mathbb{T}\backslash U_{\alpha_1} \to \mathbb{T}\backslash U_{\alpha_2}$ by $\widetilde{S}(x) = S_1(x, 0)$. Then, $\widetilde{S}$ is an isomorphism between $(\mathbb{T}, dx, R_{\alpha_1})$ and $(\mathbb{T}, dx, R_{\alpha_2})$. This implies that $\alpha_1 = \alpha_2$. This completes the proof. □

Type III examples: The uncountably many type III examples that we construct are directly based on the works of Nakada ([21] and [20]) on cylinder flows. We follow the exposition given in [2] and recall a few facts from [2] concerning the ergodicity of cylinder flows. Let $\beta > 0$ be fixed. For an irrational $\alpha \in (0, 1)$, let R_α be the rotation on $\mathbb{T}$ by angle α. We use additive notation. Thus, $\mathbb{T} = [0, 1)$ and $R_\alpha(x) = x + \alpha \mod 1$. For $\gamma > 0$, let $F_\gamma : [0, 1) \to \mathbb{C}$ be defined by

$$F_\gamma(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{\gamma}{\gamma+1}), \\ -\gamma & \text{if } x \in [\frac{\gamma}{\gamma+1}, 1). \end{cases} \tag{5.10}$$

Remark 5.2 (Nakada). *Thanks to Proposition 1.1, Theorem 1.4 and Theorem 1.6 of [2], we have the following.*

(a) There exists a unique non-atomic probability measure $m := m_{\alpha,\beta,\gamma}$ *on* $\mathbb{T}$ *such that* $\frac{d(m\circ R_\alpha)}{dm} = e^{-\beta F_\gamma}$*. Moreover,* (R_α, m) *is ergodic.*
(b) The measure $m_{\alpha,\beta,1}$ *is of type III.*
(c) Suppose α *has bounded partial quotients and* $\frac{\gamma}{\gamma+1} \notin \mathbb{Q} + \mathbb{Q}\alpha$*. Then,* $m_{\alpha,\beta,\gamma}$ *is of type III.*

We also need the description of the spectrum of R_α *given in [20] (Page 476, Paragraph 1). Let* $\beta, \gamma > 0$ *be given. Define*

$$Spec(R_\alpha) := \{\lambda \in \mathbb{T} : \text{There exists } \xi \in L^\infty(\mathbb{T}, m_{\alpha,\beta,\gamma}) \text{ such that } \xi \circ R_\alpha = \lambda\xi\}.$$

If α *has bounded partial quotients, then* $Spec(R_\alpha) = \{e^{2\pi i n\alpha} : n \in \mathbb{Z}\}$*. In particular, if* $\alpha_1, \alpha_2 \in (0, \frac{1}{2})$ *are distinct irrationals having bounded partial quotients, then the dynamical systems* $(\mathbb{T}, m_{\alpha_1,\beta,\gamma}, R_{\alpha_1})$ *and* $(\mathbb{T}, m_{\alpha_2,\beta,\gamma}, R_{\alpha_2})$ *are not metrically isomorphic.*

The case $\theta = 1$: Let $\alpha \in (0, 1)$ be an irrational, and let R_α denote the rotation on $\mathbb{T}$ by angle α. Take $\gamma = 1$ in equation (5.10), and consider F_1 given by,

$$F_1(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}), \\ -1 & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$$

Then by (*a*) and (*b*) of Remark 5.2, there exists a unique non-atomic type III probability measure $m := m_{\alpha,\beta,1}$ such that $\frac{d(m\circ R_\alpha)}{dm} = e^{-\beta F_1}$.

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Recall that the homomorphism $c: G \to \mathbb{R}$ is given by $c(v_1) = 1$ and $c(v_2) = 2$. Define $Y_\alpha = \mathbb{T} \times \mathbb{Z}$, and set $X_\alpha = \mathbb{T} \times \{0, 1, 2, \cdots\}$. Let μ_α be the measure on Y_α defined by,

$$\mu_\alpha(E \times \{n\}) = (1 - e^{-2\beta})e^{-2n\beta}m(E),$$

for a measurable subset $E \subset \mathbb{T}$. Clearly, $\mu_\alpha(X_\alpha) = 1$. Define a $\mathbb{Z}^2$-action on Y_α by

$$(x, t) + v_1 := (R_\alpha(x), 1_{[\frac{1}{2},1)}(x) + t); \text{ and } (x, t) + v_2 := (x, t + 1).$$

It is easy to see that (Y_α, X_α) is a pure $(\mathbb{Z}^2, \mathbb{N}^2)$-space and μ_α is an $e^{-\beta c}$-conformal measure on Y_α.

Proposition 5.3. *With the foregoing notation, we have the following.*

1. *The $\mathbb{Z}^2$-action on Y_α is essentially free and ergodic.*
2. *The measure μ_α is of type III.*
3. *The map*

$$Y_\alpha \ni (x, t) \to Q_{(x,t)} \in Y_u$$

is injective.

4. *If $\alpha_1 \neq \alpha_2$ are distinct irrationals in $(0, \frac{1}{2})$ having bounded partial quotients, then the $(\mathbb{Z}^2, \mathbb{N}^2)$-spaces $(Y_{\alpha_1}, X_{\alpha_1}, \mu_{\alpha_1})$ and $(Y_{\alpha_2}, X_{\alpha_2}, \mu_{\alpha_2})$ are not isomorphic. In particular, there are uncountably many type III extremal β-KMS states for $\theta = 1$.*

Proof. With Remark 5.2 in hand, the proofs of (1), (2) and (3) are exactly similar to the proof of Proposition 4.4, and hence, we omit the proof. For example, the proof for the injectivity of the map

$$Y_\alpha \ni (x, t) \to Q_{(x,t)} \in Y_u$$

is similar to that of Proposition 4.4, where we work in the coordinate system determined by $\{v_1, v_2\}$, the function ϕ is replaced with $1_{[\frac{1}{2},1)}$ and in place of Lemma 4.2, we appeal to Lemma 5.1 with $\eta = \frac{1}{2}$. With the description of the spectrum alluded to in Remark 5.2, the proof of (4) is very similar to the proof of Part (4) of Lemma 5.2. We leave the details to the reader. □

The case of an irrational θ**:** The construction of type III examples for an irrational θ is similar to the case $\theta = 1$. Recall that the homomorphism $c: G \to \mathbb{R}$ is given by $c(e_1) = 1$ and $c(e_2) = \theta$. Take $\gamma = \theta$ in 5.10, and let F_θ be the function defined by

$$F_\theta(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{\theta}{\theta+1}), \\ -\theta & \text{if } x \in [\frac{\theta}{\theta+1}, 1). \end{cases}$$

Choose $\alpha \in [0, 1)\backslash\mathbb{Q}$ such that α has bounded partial quotients and $\frac{\theta}{\theta+1} \notin \mathbb{Q} + \mathbb{Q}\alpha$. Then, by (*a*) and (*c*) of Remark 5.2, there exists a unique non-atomic type III probability measure $m := m_{\alpha,\beta,\theta}$ on $\mathbb{T}$ such that $\frac{d(m\circ R_\alpha)}{dm} = e^{-\beta F_\theta}$.

Define $Y_\alpha = \mathbb{T} \times \mathbb{Z}$, and set $X_\alpha = \mathbb{T} \times \{0, 1, 2, \cdots\}$. Let μ_α be the measure on Y_α given by,

$$\mu_\alpha(E \times \{n\}) = (1 - e^{-\beta(1+\theta)})e^{-\beta n(\theta+1)}m(E)$$

for a measurable subset $E \subset \mathbb{T}$. Clearly, $\mu_\alpha(X_\alpha) = 1$. Define a $\mathbb{Z}^2$-action on Y_α by

$$(x, t) + v_1 := (R_\alpha(x), 1_{[\frac{\theta}{\theta+1},1)}(x) + t); \text{ and } (x, t) + v_2 := (x, t + 1).$$

It is easy to see that (Y_α, X_α) is a pure $(\mathbb{Z}^2, \mathbb{N}^2)$-space and μ_α is an $e^{-\beta c}$-conformal measure on Y_α.

If we make use of Remark 5.2, the proof of the following proposition is similar to that of Proposition 5.3, and hence, the proof is omitted.

Proposition 5.4. *With the foregoing notation, we have the following.*

1. *The $\mathbb{Z}^2$-action on Y_α is essentially free and ergodic.*
2. *The measure μ_α is of type III.*

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3. *The map*

$$Y_\alpha \ni (x,t) \to Q_{(x,t)} \in Y_u$$

is injective.

4. *Suppose $\alpha_1 \neq \alpha_2$ are distinct irrationals in $(0, \frac{1}{2})$ having bounded partial quotients such that $\frac{\theta}{1+\theta} \notin \mathbb{Q} + \mathbb{Q}\alpha_i$ for $i = 1, 2$. Then, the $(\mathbb{Z}^2, \mathbb{N}^2)$-spaces $(Y_{\alpha_1}, X_{\alpha_1}, \mu_{\alpha_1})$ and $(Y_{\alpha_2}, X_{\alpha_2}, \mu_{\alpha_2})$ are not isomorphic. In particular, there are uncountably many type III extremal β-KMS states when θ is irrational.*

The type II and type III examples for other rational values of θ can be constructed from the base case $\theta = 1$ as follows. Let $\theta := \frac{p}{q}$ be a positive rational. Assume that $\gcd(p, q) = 1$. Choose a matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in SL_2(\mathbb{Z})$ such that $x, y, z, w \geq 0$ and

$$x + z = q; y + w = p.$$

To see that this is possible, observe that if $q = 1$, then the matrix $\begin{bmatrix} 1 & p-1 \\ 0 & 1 \end{bmatrix}$ will do. Suppose $q \geq 2$. Choose $x \in \{1, 2, \cdots, q-1\}$ such that $xp \equiv 1 \mod q$. Write $xp = yq + 1$ with $y \in \mathbb{Z}$. Set $z := q - x$ and $w := p - y$. Then, $x, y, z, w \geq 0$, $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in SL_2(\mathbb{Z})$ and

$$x + z = q; y + w = p.$$

Let us fix notation. Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ be the isomorphism that corresponds to the matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Note that $\phi(\mathbb{N}^2) \subset \mathbb{N}^2$. We denote the homomorphism $\mathbb{Z}^2 \to \mathbb{R}$ that sends $e_1 \to 1$ and $e_2 \to \theta$ by c_θ. If $\theta = 1$, we denote c_θ simply by c. It is clear that $\frac{1}{q}(c \circ \phi) = c_\theta$.

Fix $\beta > 0$ and let $t \in \{II, III\}$. Let (Y, X) be a pure $(\mathbb{Z}^2, \mathbb{N}^2)$-space together with an $e^{-\frac{\beta c}{q}}$-conformal measure m on Y such that

(A1) the map

$$Y \ni y \to Q_y := \{s \in G : y - s \in X\} \in Y_u$$

is injective, and

(A2) the dynamical system (Y, m, G) is essentially free, ergodic and is of type t.

Define

$$Y^\phi := Y; \; X^\phi := X; \; m^\phi := m.$$

Define a new $\mathbb{Z}^2$-action on Y^ϕ as follows:

$$y \oplus (m, n) := y + \phi(m, n).$$

With this new action, (Y^ϕ, X^ϕ, m^ϕ) is a pure $(\mathbb{Z}^2, \mathbb{N}^2)$-space and m^ϕ is $e^{-\beta c_\theta}$-conformal. Since ϕ is an automorphism of $\mathbb{Z}^2$, it follows that $(Y^\phi, m^\phi, \mathbb{Z}^2)$ inherits the essential freeness, ergodicity and the type from that of $(Y, m, \mathbb{Z}^2)$. Note that for $y \in Y^\phi$,

$$Q_y^\phi := \{(m, n) \in \mathbb{Z}^2 : y \ominus (m, n) \in X^\phi\} = \phi^{-1}(Q_y).$$

Since the map $Y \ni y \to Q_y \in Y_u$ is injective, it follows that $Y^\phi \ni y \to Q_y^\phi \in Y_u$ is injective. Thus, the $(\mathbb{Z}^2, \mathbb{N}^2)$-space (Y^ϕ, X^ϕ) together with the $e^{-\beta c_\theta}$-conformal measure satisfies $(A1)$ and $(A2)$.

Moreover, since ϕ is an automorphism, it is clear that the above construction, i.e.

$$(Y, X, m) \rightarrow (Y^{\phi}, X^{\phi}, m^{\phi})$$

maps metrically non-isomorphic spaces to metrically non-isomorphic spaces.

As we have already proved the existence of a continuum of metrically non-isomorphic $(\mathbb{Z}^2, \mathbb{N}^2)$-spaces satisfying $(A1)$ and $(A2)$ for $\theta = 1$ and for every $\beta > 0$, we can conclude that that there is a continuum of metrically non-isomorphic $(\mathbb{Z}^2, \mathbb{N}^2)$-spaces that satisfy $(A1)$ and $(A2)$ for every rational $\theta > 0$ and for every $\beta > 0$. We can apply Proposition 3.5 to conclude that for a rational $\theta > 0$ and for every $\beta > 0$, there are uncountably many β-KMS states of type II and of type III.

We have now proved the following theorem.

Theorem 5.1. *Suppose* $\theta > 0$ *Let* $\sigma := \{\sigma_t\}_{t\in\mathbb{R}}$ *be the* 1*-parameter group of automorphisms on* $C_c^*(\mathbb{N}^2)$ *given by*

$$\sigma_t(v_{(m,n)}) = e^{i(m+n\theta)t} v_{(m,n)}.$$

Then, for every $\beta > 0$*, there is a continuum of extremal* β*-KMS states on* $C_c^*(\mathbb{N}^2)$ *for* σ *of both type II and type III.*

We end our paper by posing a problem. Constructing type III product measures for the Bernoulli shift on two symbols have always been a subject of interest. The first such example was due to Hamachi ([12]). We refer the reader to [3] and the references therein for more recent developments. In the recent years, the focus is on constructing product measures of various Krieger types. Given this interest, the following question, we believe, is worth investigating.

Question: Let $\theta > 0$ be irrational, and suppose $\beta > 0$. Let $\chi : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$ be the potential defined by

$$\chi(x) = \begin{cases} 1 & \text{if } x_0 = 0, \\ -\theta & \text{if } x_0 = 1. \end{cases}$$

(1) Does there exist an ergodic probability measure for the Bernoulli shift of type II_∞ that is $e^{-\beta\chi}$-conformal?

(2) Let $\lambda \in [0, 1)$ be given. Does there exist an ergodic probability measure of type III_λ that is $e^{-\beta\chi}$-conformal?

Nakada's work shows that type III_1 is possible. Note that the type II examples that we have constructed in this paper are of type II_1.

Declaration of competing interest. The authors declare none.

References

[1] J. Aaronson, An introduction to infinite ergodic theory, *Mathematical Surveys and Monographs*, vol. **50** (American Mathematical Society, Providence, RI, 1997).

[2] J. Aaronson, H. Nakada, O. Sarig and R. Solomyak, Invariant measures and asymptotics for some skew products, *Israel J. Math* **128** (2002), 93–134.

[3] M. Björklund, Z. Kosloff, and S. Vaes, Ergodicity and type of nonsingular Bernoulli actions, *Invent. Math.* **224**(2) (2021), 573–625.

[4] N. Brownlowe, N. S. Larsen, J. Ramagge and N. Stammeier, C*-algebras of right LCM monoids and their equilibrium states, *Trans. Amer. Math. Soc.* **373**(7) (2020), 5235–5273.

[5] Johannes Christensen and Klaus Thomsen, KMS states on the crossed product C*-algebra of a homeomorphism,*Ergodic Theory Dynam. Syst.* **42**(4) (2022), 1373–1414.

https://doi.org/10.1017/S0017089523000071 Published online by Cambridge University Press

[6] J. Cuntz, C*-algebras associated with the $ax+b$-semigroup over $\mathbb{N}$, K-theory and noncommutative geometry, *EMS Ser. Congr. Rep.* (2008), 201–215. Available at https://mathscinet.ams.org/mathscinet/search/publications.html?fmt=bibtex&pg1=MR&s1=2513338

[7] J. Cuntz, C. Deninger and M. Laca, C*-algebras of Toeplitz type associated with algebraic number fields, *Math. Ann* **355**(4) (2013), 1383–1423.

[8] J. Cuntz, S. Echterhoff and X. Li, On the K-theory of crossed products by automorphic semigroup actions, *Q J. Math.* **64**(3) (2013), 747–784.

[9] J. Cuntz and X. Li, The regular C∗-algebra of an integral domain, Quanta of maths, *Clay math. proc*, vol. **11** (Amer. Math. Soc., Providence, RI, 2010), 149–170.

[10] J. Cuntz and X. Li, C*-algebras associated with integral domains and crossed products by actions on adele spaces, *J. Noncommut. Geom.* **5**(1) (2011), 1–37.

[11] A. I. Danilenko and C. E. Silva, Ergodic theory: non-singular transformations, *Mathematics of complexity and dynamical systems*, Vols. **1-3** (Springer, Newyork, 2012), 329–356.

[12] T. Hamachi, On a Bernoulli shift with nonidentical factor measures, *Ergodic Theory Dynam. Syst.* **1**(3) (1981), 273–283, **1**982.

[13] Y. Katznelson, Sigma-finite invariant measures for smooth mappings of the circle, *J. Anal. Math.* **31** (1997), 1–18.

[14] M. Laca and S. Neshveyev, Type III_1 equilibrium states of the Toeplitz algebra of the affine semigroup over the natural numbers, *J. Funct. Anal.* **261**(1) (2011), 169–187.

[15] M. Laca and I. Raeburn, Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers, *Adv. Math.* **225**(2) (2010), 643–688.

[16] X. Li, Semigroup C*-algebras and amenability of semigroups, *J. Funct. Anal.* **262**(10) (2012), 4302–4340.

[17] X. Li, Nuclearity of semigroup C*-algebras and the connection to amenability, *Adv. Math.* **244** (2013), 626–662.

[18] P. S. Muhly and J. N. Renault, C*-algebras of multivariable Wiener-Hopf operators, *Trans. Amer. Math. Soc.* **274**(1) (1982), 1–44.

[19] G. J. Murphy, C*-algebras generated by commuting isometries, *Rocky Mountain J. Math.* **26**(1) (1996), 237–267.

[20] H. Nakada, Piecewise linear homeomorphisms of type III and the ergodicity of cylinder flows, *Keio Math. Sem. Rep.*7 (1982), 29–40. Available at https://mathscinet.ams.org/mathscinet/search/publications.html?fmt=bibtex&pg1=MR&s1=667804

[21] H. Nakada, On a classification of PL-homeomorphisms of a circle, Probability theory and mathematical statistics (Tbilisi, 1982), *Lecture Notes in Math*, vol. **1021** (Springer, Berlin, 1983), 474–480.

[22] S. Neshveyev, KMS states on the C*-algebras of non-principal groupoids, *J. Operator Theory* **70**(2) (2013), 513–530.

[23] I. Raeburn and S. T. Vittadello, The isometric representation theory of a perforated semigroup, *J. Operator Theory* **62**(2) (2009), 357–370.

[24] H. N. Salas, Semigroups of isometries with commuting range projections, *J. Operator Theory* **14**(2) (1985), 311–346.

[25] S. Sundar, C*-algebras associated to topological Ore semigroups, *Münster J. Math.* **9**(1) (2016), 155–185.

[26] S. T. Vittadello, The isometric representation theory of numerical semigroups, *Integral Equations Operator Theory* **64**(4) (2009), 573–597.

https://doi.org/10.1017/S0017089523000071 Published online by Cambridge University Press