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IMPROVING ON BOLD PLAY WHEN THE GAMBLER IS RESTRICTED

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Abstract

Suppose that a gambler starts with a fortune in (0, 1) and wishes to attain a fortune of 1 by making a sequence of bets. Assume that whenever the gambler stakes an amount s, the gambler's fortune increases by s with probability w and decreases by s with probability $1 - w$, where $w < \frac{1}{2}$. Dubins and Savage showed that the optimal strategy, which they called 'bold play', is always to bet $\min\{f, 1 - f\}$, where f is the gambler's current fortune. Here we consider the problem in which the gambler may stake no more than ℓ at one time. We show that the bold strategy of always betting $\min\{\ell, f, 1 - f\}$ is not optimal if ℓ is irrational, extending a result of Heath, Pruitt, and Sudderth.

Keywords: Bold play; red-and-black; gambling; supermartingale

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1. Introduction and background

Suppose that a gambler starts with a fortune in (0, 1) and wishes to attain a fortune of 1 by making a sequence of bets. If the gambler's current fortune is f then the gambler may stake any amount less than or equal to f. The gambler wins the amount of the stake with probability w and loses the stake with probability $1 - w$. Following [8], we refer to this game as red-and-black. Clearly the gambler should never stake more than $1 - f$, which is enough to ensure that the gambler will reach the goal if the bet is won. The strategy in which the gambler always stakes $\min\{f, 1 - f\}$ is called bold play.

In [8], Dubins and Savage developed a general theory for gambling problems. For red-and-black, they showed that if $0 < w < \frac{1}{2}$, which means that the game is subfair, then bold play is the optimal strategy, in the sense that it maximizes the probability that the gambler will eventually reach the goal. Their proof is also given in [2, Chapter 7] and [10, Chapter 24]. See [1] for some computations comparing the probability that a gambler will reach the goal using bold play to the probability that a gambler will reach the goal using other strategies.

This result has been extended in several ways. Dubins and Savage [8] also considered primitive casinos, in which the gambler loses the stake s with probability $1 - w$ and wins $s(1 - r)/r$ with probability w, where $0 < r < 1$. Note that the game in which $r = \frac{1}{2}$ is red-and-black. They showed that bold play is optimal when the game is subfair, which in this case means $w < r$. Chen [5] considered red-and-black with inflation, in which the goal is not to reach 1 but to reach $(1+\alpha)^n$ after n bets, for some n. He showed that bold play is optimal when $w \leq \frac{1}{2}$. A different extension is to incorporate a discount factor, so that the gambler receives

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a utility of β^n, where $0 < \beta \leq 1$, from reaching 1 on the nth bet. Klugman [12] showed that bold play is optimal for subfair red-and-black with a discount factor. However, for some subfair primitive casinos, there exist discount factors for which bold play is not optimal (see [4] and [6]). See also [17] for a discussion of the optimality of bold play in some two-person games.

Several authors have considered discrete versions of this problem in which the gambler's initial fortune and the amount of each bet must be integers and the gambler's goal is to attain a fortune of n. An extensive discussion of discrete gambling problems such as this can be found in [13]. Bold play remains optimal when $w < \frac{1}{2}$. Ross [15] showed that the timid strategy of staking exactly 1 each time is optimal in the superfair case when $w > \frac{1}{2}$. See also [16] for an analysis of the superfair case when the minimum bet is 2. Dubins [7] showed, however, that if the win probability is less than $\frac{1}{2}$ but is allowed to depend on the gambler's fortune, then bold play need not be optimal.

Another direction of work concerns gambling problems in which there is a limit to how much the gambler may bet. The simplest problem of this type involves red-and-black in which the gambler may bet no more than $\ell \in (0, \frac{1}{2})$ at one time. In this case, we define bold play as the strategy in which the gambler, whose current fortune is f, always stakes $\min\{\ell, f, 1 - f\}$. Wilkins [18] showed that if $w < \frac{1}{2}$ and $\ell = 1/n$, for some positive integer $n \geq 3$, then bold play maximizes the chance that the gambler will reach the goal. Chen [3] showed that bold play remains optimal when there is a discount factor in addition to a limit of $1/n$ on the stake. In [14], the optimality of bold play in continuous-time gambling problems was established under rather general restrictions on the gambler.

However, Heath *et al.* [11] obtained an important negative result for discrete-time red-and-black. They showed that if the gambler can stake at most ℓ, and if $1/(n + 1) < \ell < 1/n$ for some $n \geq 3$ or if ℓ is irrational and $\frac{1}{3} < \ell < \frac{1}{2}$, then there exists an $\varepsilon > 0$ such that if $0 < w < \varepsilon$ then bold play is not optimal. To see heuristically why this is true, suppose that $\frac{1}{4} < \ell < \frac{1}{3}$ and the gambler's initial fortune is $f = \frac{1}{2} - \delta$, where δ is small. If the gambler plays boldly and loses the first bet, then the gambler's fortune after one bet will be $\frac{1}{2} - \ell - \delta$. The gambler's fortune can at most double to $1 - 2\ell - 2\delta$ after the second bet and, therefore, can be at most $1 - 2\delta$ after two more wins. However, if the gambler first stakes $\ell - \delta$ and plays boldly thereafter then, even with an initial loss, the gambler can reach the goal by winning the next three bets. Consequently, for sufficiently small δ, first betting $\ell - \delta$ makes the gambler more likely to achieve the goal after winning three or fewer bets. As $w \downarrow 0$, the probability that the gambler can win four bets before going bankrupt becomes very small relative to the probability that the gambler wins three bets. Therefore, first betting $\ell - \delta$ is a better strategy than bold play for sufficiently small w.

The purpose of the present paper is to extend this result by showing that when ℓ is irrational, bold play fails to be optimal for all $w < \frac{1}{2}$, not just for very small w. The case of rational ℓ remains open except when $\ell = 1/n$ for some $n \geq 3$. Note that when ℓ is rational and $\frac{1}{3} < \ell < \frac{1}{2}$, it is not even known whether or not bold play can be improved upon for very small w.

To state our result more precisely, define the function $s\colon [0, 1] \to [0, 1]$ by $s(f) = \min\{\ell, f, 1 - f\}$. We think of $s(f)$ as the bold stake for a gambler whose fortune is f. Denote by X_k the gambler's fortune after k bets, when the gambler plays boldly. Note that $(X_k)_{k=0}^{\infty}$ is a Markov chain whose transition probabilities are given by

$$\mathrm{P}(X_{k+1} = f + s(f) \mid X_k = f) = w, \tag{1}$$

$$\mathrm{P}(X_{k+1} = f - s(f) \mid X_k = f) = 1 - w. \tag{2}$$

Define $Q(f) = \mathrm{P}(X_k = 1 \text{ for some } k \mid X_0 = f)$, which is the probability that a gambler who starts with a fortune of f will eventually reach the goal. Our main result is given in the following theorem.

Theorem 1. *Suppose that $w < \frac{1}{2}$ and ℓ is irrational. Then there exist $f \in (0, 1)$ and $\varepsilon \in (0, s(f))$ such that*

$$wQ(f + s(f) - \varepsilon) + (1 - w)Q(f - s(f) + \varepsilon) > Q(f). \tag{3}$$

If a gambler begins with a fortune of f and stakes $s(f) - \varepsilon$ then the gambler's fortune after one bet will be $f + s(f) - \varepsilon$ with probability w and $f - s(f) + \varepsilon$ with probability $1 - w$. Consequently, the left-hand side of (3) is the probability that the gambler will eventually reach the goal using the strategy of first staking $s(f) - \varepsilon$ and playing boldly thereafter, while the right-hand side of (3) is the probability that the gambler will reach the goal using (only) bold play. Therefore, (3) implies that the strategy of first staking $s(f) - \varepsilon$ and then playing boldly is superior to bold play and, hence, bold play is not optimal.

2. Proof of Theorem 1

In this section, we will prove Theorem 1. The key to the proof will be the following proposition. Here, and throughout the rest of the paper, all logarithms are assumed to be base 2; that is, we write $\log n$ instead of $\log_2 n$.

Proposition 1. *Let $S = \{f \colon \mathrm{P}(X_k = 1 - \ell \text{ for some } k \mid X_0 = f) > 0\}$. That is, S is the set of all f such that a gambler who starts with a fortune of f and plays boldly could have a fortune of exactly $1 - \ell$ after a finite number of bets.*

(i) *Suppose that $f \in S$. Then there exists a constant $C > 0$ such that if $0 < \varepsilon < \ell$ then $Q(f) - Q(f - \varepsilon) \geq C(1 - w)^{-\log \varepsilon}$.*

(ii) *Suppose that $f \notin S$. For all $C > 0$, there exists a $\delta > 0$ such that if $0 < \varepsilon < \delta$ then $Q(f) - Q(f - \varepsilon) \leq C(1 - w)^{-\log \varepsilon}$.*

(iii) *If ℓ is irrational then there exists an $f \in (\ell, 1 - \ell)$ such that $f - \ell \in S$ and $f + \ell \notin S$.*

Proposition 1 implies that $Q(f) - Q(f - \varepsilon)$ is larger when $f \in S$ than when $f \notin S$. In other words, the difference between having a fortune of f and having a fortune of $f - \varepsilon$ matters more to the gambler when $f \in S$ than it does when $f \notin S$. Proposition 1(iii) states that when ℓ is irrational, we can find an f such that $f - s(f) \in S$ and $f + s(f) \notin S$. We will show that if a gambler starts with a fortune slightly below f then it is better to make slightly less than the bold stake, so that the fortune will not fall below $f - s(f)$ if the bet is lost. This will imply Theorem 1.

An important tool for the proof of Proposition 1 is a coupling construction in which we follow two gamblers simultaneously. We present this construction in Section 2.1. We prove Proposition 1(i), 1(ii), and 1(iii) in Sections 2.2, 2.3, and 2.4 respectively. Then, in Section 2.5, we show how Theorem 1 follows from Proposition 1.

2.1. A coupling construction

Throughout this and the next two subsections, we consider two Markov chains $(X_k)_{k=0}^{\infty}$ and $(Y_k)_{k=0}^{\infty}$. We define $X_0 = f_1$ and $Y_0 = f_2$, where $f_1 \geq f_2$. Both chains evolve with the transition probabilities given by (1) and (2). Consequently, we can think of X_k as the fortune, after k bets, of a gambler whose initial fortune is f_1, while Y_k is the fortune, after k bets, of a gambler whose initial fortune is f_2.

We assume that these sequences are coupled, so that both gamblers win and lose the same bets. To construct this coupling, we work with the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ defined as follows. Let $\Omega = \{0, 1\}^\infty$, and denote sequences in Ω by $\omega = (\omega_1, \omega_2, \dots)$, so that $\omega \to \omega_i$ is the ith coordinate function. Let $\mathcal{F}_0$ be the trivial σ-field and, for positive integers k, let $\mathcal{F}_k$ be the σ-field generated by the first k coordinate functions. Let $\mathcal{F} = \sigma(\mathcal{F}_1, \mathcal{F}_2, \dots)$ be the product σ-field. Let P be the product probability measure with the property that $\mathrm{P}(\omega_i = 1) = w$ and $\mathrm{P}(\omega_i = 0) = 1 - w$ for all i. We then say that the two gamblers win the ith bet if $\omega_i = 1$ and lose the ith bet if $\omega_i = 0$. In particular, for $k \geq 0$, we define $X_{k+1}(\omega) = X_k(\omega) + s(X_k(\omega))$ and $Y_{k+1}(\omega) = Y_k(\omega) + s(Y_k(\omega))$ if $\omega_{k+1} = 1$, and $X_{k+1}(\omega) = X_k(\omega) - s(X_k(\omega))$ and $Y_{k+1}(\omega) = Y_k(\omega) - s(Y_k(\omega))$ if $\omega_{k+1} = 0$.

We now make some remarks pertaining to this construction.

Remark 1. Since $s(f) = \min\{\ell, f, 1 - f\}$, we see that if $f \leq g$ then $|s(g) - s(f)| \leq g - f$. Therefore, $f + s(f) \leq g + s(g)$ and $f - s(f) \leq g - s(g)$. It then follows by induction and the construction of the sequences $(X_k)_{k=0}^\infty$ and $(Y_k)_{k=0}^\infty$ that $X_k \geq Y_k$ for all k. Likewise, the fact that $|s(g) - s(f)| \leq g - f$ implies that $X_k - Y_k \leq 2^k(f_1 - f_2)$, for all k.

Remark 2. The fact that $X_k \geq Y_k$ for all k means that if $Y_k = 1$ then $X_k = 1$. Since $Q(f_1) = \mathrm{P}(X_k = 1 \text{ for some } k)$ and $Q(f_2) = \mathrm{P}(Y_k = 1 \text{ for some } k)$, it follows that $Q(f_1) \geq Q(f_2)$. That is, the function $f \mapsto Q(f)$ is nondecreasing.

Remark 3. Note that

$$\begin{aligned} \mathrm{E}[X_{k+1} \mid \mathcal{F}_k] &= w(X_k + s(X_k)) + (1 - w)(X_k - s(X_k)) \\ &= X_k + (2w - 1)s(X_k) \\ &\leq X_k, \end{aligned}$$

where the last inequality holds because $w < \frac{1}{2}$. Therefore, $(X_k)_{k=0}^\infty$ is a supermartingale with respect to $(\mathcal{F}_k)_{k=0}^\infty$. By the same argument, $(Y_k)_{k=0}^\infty$ is a supermartingale with respect to $(\mathcal{F}_k)_{k=0}^\infty$. By the martingale convergence theorem (see [9, Chapter 4]), there exist random variables L_1 and L_2 such that $X_k \to L_1$ almost surely (a.s.) and $Y_k \to L_2$ a.s. as $k \to \infty$. If $0 < \varepsilon < \ell$ then $s(f) > \varepsilon$ for $f \in [\varepsilon, 1 - \varepsilon]$. It follows that L_1 and L_2 must be $\{0, 1\}$-valued random variables. Furthermore, it is easy to see that, for sufficiently large k, $X_k = 1$ and $Y_k = 1$ on $\{L_1 = 1\}$ and $\{L_2 = 1\}$, respectively. Thus, $Q(f_1) = \mathrm{P}(L_1 = 1)$ and $Q(f_2) = \mathrm{P}(L_2 = 1)$, from which it follows that $Q(f_1) - Q(f_2) = \mathrm{P}(L_1 = 1 \text{ and } L_2 = 0)$.

2.2. Proof of Proposition 1(i)

We begin with the following lemma, in which we compute the gambler's probability of reaching the goal starting from a sequence of fortunes approaching 1.

Lemma 1. *For all $n \geq 0$, we have $Q(1 - 2^{-n}\ell) = 1 - (1 - w)^n(1 - Q(1 - \ell))$.*

Proof. The statement is obvious when $n = 0$. Suppose that the result holds for some $n \geq 0$. Since $s(1 - 2^{-(n+1)}\ell) = 2^{-(n+1)}\ell$, a gambler whose fortune is $1 - 2^{-(n+1)}\ell$ will, after the next bet, have a fortune of 1 with probability w and a fortune of $1 - 2^{-n}\ell$ with probability $1 - w$.

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Thus, by the Markov property,

$$\begin{aligned}Q(1-2^{-(n+1)}\ell) &= w + (1-w)Q(1-2^{-n}\ell)\\ &= w + (1-w)(1-(1-w)^n(1-Q(1-\ell)))\\ &= 1-(1-w)^{n+1}(1-Q(1-\ell)).\end{aligned}$$

The lemma now follows by induction on n.

Proof of Proposition 1(i). Let $f_1 = f$ and $f_2 = f - \varepsilon$, where $0 < \varepsilon < \ell$. Since $f \in S$, there exists a positive integer k such that if B denotes the event that $X_k = 1-\ell$ and $X_{k+1} = 1$, then $\mathrm{P}(B) > 0$. Note that, for $0 \le j < k$, we have $X_{j+1} - Y_{j+1} \ge X_j - Y_j$ unless either $X_j > 1-\ell$ and $X_{j+1} = 1$ or $Y_j < \ell$ and $Y_{j+1} = 0$. Therefore, if B occurs then $Y_k \le 1-\ell-\varepsilon$ and, thus, $Y_{k+1} \le 1-\varepsilon$. Combining this observation with Remarks 2–3, we obtain

$$\begin{aligned}Q(f) - Q(f-\varepsilon) &= \mathrm{P}(L_1 = 1 \text{ and } L_2 = 0)\\ &\ge \mathrm{P}(B)\,\mathrm{P}(L_2 = 0 \mid B)\\ &\ge \mathrm{P}(B)(1-Q(1-\varepsilon)).\end{aligned}$$

Choose a nonnegative integer n such that $2^{-(n+1)}\ell < \varepsilon \le 2^{-n}\ell$, which implies that $n \le \log\ell - \log\varepsilon$. By Lemma 1,

$$\begin{aligned}Q(1-\varepsilon) &\le Q(1-2^{-(n+1)}\ell)\\ &= 1-(1-w)^{n+1}(1-Q(1-\ell))\\ &\le 1-(1-w)^{1+\log\ell-\log\varepsilon}(1-Q(1-\ell)).\end{aligned}$$

Thus, $Q(f) - Q(f-\varepsilon) \ge C(1-w)^{-\log\varepsilon}$, where $C = \mathrm{P}(B)(1-w)^{1+\log\ell}(1-Q(1-\ell))$.

2.3. Proof of Proposition 1(ii)

Our next step is to prove Proposition 1(ii), which gives an upper bound for $Q(f) - Q(f-\varepsilon)$ when $f \notin S$. We will compare the sequences $(X_k)_{k=0}^{\infty}$ and $(Y_k)_{k=0}^{\infty}$ when $f_1 = f$ and $f_2 = f - \varepsilon$. Although $(X_k - Y_k)_{k=0}^{\infty}$ is not a supermartingale, we will be able to construct a supermartingale by considering the differences between the gamblers' fortunes at a sequence of stopping times. It will then follow that the gamblers' fortunes remain similar enough for us to obtain the desired upper bound on $Q(f) - Q(f-\varepsilon)$ when $f \notin S$.

Given f and f^* such that $0 \le f^* \le f \le 1$, we define

$$h(f, f^*) = \begin{cases} 1 & \text{if } f = f^*,\\ \dfrac{s(f) - s(f^*)}{f - f^*} & \text{otherwise.}\end{cases}$$

Note that $-1 \le h(f, f^*) \le 1$ for all f and f^*. If $\ell \le f^* \le f \le 1-\ell$ then $s(f) = s(f^*) = \ell$, which means that $h(f, f^*) = 0$. If $f^* \ge \ell$ and $f \ge 1-\ell$ then $h(f, f^*) \le 0$, while if $f^* \le \ell$ and $f \le 1-\ell$ then $h(f, f^*) \ge 0$. Also, recall that $\omega_k = 1$ if the gamblers win the kth bet, and that $\omega_k = 0$ if the gamblers lose the kth bet. We have

$$X_{k+1}(\omega) - Y_{k+1}(\omega) = \begin{cases} (1 + h(X_k(\omega), Y_k(\omega)))(X_k(\omega) - Y_k(\omega)) & \text{if } \omega_{k+1} = 1,\\ (1 - h(X_k(\omega), Y_k(\omega)))(X_k(\omega) - Y_k(\omega)) & \text{if } \omega_{k+1} = 0.\end{cases} \tag{4}$$

Define

$$W_k = (1-w)^{-\log(X_k - Y_k)} = (X_k - Y_k)^{-\log(1-w)}. \tag{5}$$

By (4), we have

$$\begin{aligned} \mathrm{E}[W_{k+1} \mid \mathcal{F}_k] &= w(1+h(X_k, Y_k))^{-\log(1-w)} W_k + (1-w)(1-h(X_k, Y_k))^{-\log(1-w)} W_k \\ &= g(h(X_k, Y_k)) W_k, \end{aligned} \tag{6}$$

where

$$g(x) = w(1+x)^{-\log(1-w)} + (1-w)(1-x)^{-\log(1-w)},$$

for $-1 \le x \le 1$. Note that

$$g'(x) = -\log(1-w)(w(1+x)^{-\log(1-w)-1} - (1-w)(1-x)^{-\log(1-w)-1}).$$

Suppose that $0 < x < 1$. Since $0 < -\log(1-w) < 1$, we have $(1+x)^{-\log(1-w)-1} < 1$ and $(1-x)^{-\log(1-w)-1} > 1$. Therefore,

$$g'(x) \le -\log(1-w)(w - (1-w)) < 0.$$

Since $g(0) = 1$, it follows that $0 < g(x) < 1$, for $x \in (0, 1]$.

We now introduce four lemmas that will help us to define a supermartingale.

Lemma 2. *Suppose that* $1 - \ell \le f_2 \le f_1 \le 1$. *Then* $\mathrm{E}[W_1] = W_0$.

Proof. We have $s(f_1) = 1 - f_1$ and $s(f_2) = 1 - f_2$. Therefore, $h(f_1, f_2) = -1$. Since $g(-1) = 1$, it follows from (6) that $\mathrm{E}[W_1] = W_0$.

Lemma 3. *Suppose that* $f_2 \le f_1 < 1-\ell$. *Define a stopping time* R *as follows. If* $h(f_1, f_2) \ge \frac{1}{2}$ *then let* $R = 0$. *If* $h(f_1, f_2) < \frac{1}{2}$ *then let* $R(\omega) = \inf\{j : \omega_j = 1$ *or* $h(X_j(\omega), Y_j(\omega)) \ge \frac{1}{2}\}$. *Let* $L = \lfloor 1 + (1-2\ell)/\ell \rfloor$, *where* $\lfloor\cdot\rfloor$ *is the integer-part function. Then* $R \le L$ *and* $\mathrm{E}[W_R] \le W_0$.

Proof. Proceeding by contradiction, suppose that $R(\omega) > L$ for some ω. Then the gamblers must lose the first L bets. However, by the definition of L, any gambler who starts with a fortune of at most $1-\ell$ and then loses L consecutive bets has a fortune of at most ℓ. Therefore, there exists a $j \le L$ such that $0 < X_j \le \ell$. Since $X_j \ge Y_j$, it follows that $Y_j \le \ell$ and, thus, $s(X_j) = X_j$ and $s(Y_j) = Y_j$. However, this means that $h(X_j, Y_j) = 1$ and, thus, $R \le j$, i.e. a contradiction. Hence, $R \le L$.

For $j < R$, we have $Y_j \le X_j < 1 - \ell$ and, therefore, $0 \le h(X_j, Y_j) \le 1$. Since $g(x) \le 1$ for $x \in [0, 1]$, we have, with the aid of (6),

$$\begin{aligned} \mathrm{E}[W_{(j+1)\wedge R} \mid \mathcal{F}_j] &= W_{j\wedge R}\, \mathbf{1}_{\{R \le j\}} + \mathrm{E}[W_{j+1} \mid \mathcal{F}_j]\, \mathbf{1}_{\{R > j\}} \\ &= W_{j\wedge R}\, \mathbf{1}_{\{R \le j\}} + g(h(X_j, Y_j)) W_j\, \mathbf{1}_{\{R > j\}} \\ &\le W_{j\wedge R}, \end{aligned}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function and $x \wedge y$ denotes $\min\{x, y\}$. Therefore, $(W_{j\wedge R})_{j=0}^{\infty}$ is a supermartingale with respect to $(\mathcal{F}_j)_{j=0}^{\infty}$. Note that $0 \le W_{j\wedge R} \le 1$ for all j, so the optional stopping theorem (see [9, Chapter 4]) gives $\mathrm{E}[W_R] \le W_0$.

Lemma 4. *Suppose that $f_2 \le f_1 < 1 - \ell$. Let*

$$T(\omega) = \inf\{j \ge 1 \colon \omega_j = 1 \text{ or } h(X_{j-1}(\omega), Y_{j-1}(\omega)) \ge \tfrac{1}{2}\}.$$

Let $L = \lfloor 1 + (1 - 2\ell)/\ell \rfloor$ as in Lemma 3. Then $T \le L + 1$ and $\mathrm{E}[W_T] \le \alpha W_0$, where

$$\alpha = 1 - (1 - g(\tfrac{1}{2}))(1 - w)^{2L}.$$

Proof. Define the stopping time R as in Lemma 3. Then $T = R$ if and only if the gamblers win the Rth bet; otherwise $T = R + 1$. Clearly $T \le L + 1$ by Lemma 3. Let A be the event that the gamblers win the Rth bet. Then

$$\mathrm{E}[W_T] = \mathrm{E}[\mathrm{E}[W_T \mid \mathcal{F}_R]] = \mathrm{E}[W_R \mathbf{1}_A + \mathrm{E}[W_{R+1} \mid \mathcal{F}_R] \mathbf{1}_{A^c}].$$

By the strong Markov property and (6), $\mathrm{E}[W_{R+1} \mid \mathcal{F}_R] = g(h(X_R, Y_R)) W_R$. If the gamblers lose the Rth bet then $h(X_R, Y_R) \ge \frac{1}{2}$. Therefore, since g is decreasing on [0, 1],

$$\mathrm{E}[W_{R+1} \mid \mathcal{F}_R] \mathbf{1}_{A^c} \le g(\tfrac{1}{2}) W_R \mathbf{1}_{A^c}.$$

Thus,

$$\mathrm{E}[W_T] \le \mathrm{E}[W_R \mathbf{1}_A + g(\tfrac{1}{2}) W_R \mathbf{1}_{A^c}] = \mathrm{E}[W_R - (1 - g(\tfrac{1}{2})) W_R \mathbf{1}_{A^c}]. \tag{7}$$

If A^c occurs then the gamblers lose the first R bets, and $h(X_j, Y_j) < \frac{1}{2}$ for all $j < R$. If $h(X_j, Y_j) < \frac{1}{2}$ and the gamblers lose the $(j+1)$th bet, then $X_{j+1} - Y_{j+1} \ge (X_j - Y_j)/2$. Thus, on A^c we have $X_R - Y_R \ge 2^{-R}(f_1 - f_2) \ge 2^{-L}(f_1 - f_2)$. Therefore,

$$\begin{aligned}
\mathrm{E}[W_R \mathbf{1}_{A^c}] &\ge \mathrm{E}[(2^{-L}(f_1 - f_2))^{-\log(1-w)} \mathbf{1}_{A^c}] \\
&= 2^{L \log(1-w)} W_0 \, \mathrm{P}(A^c) \\
&= \mathrm{P}(A^c)(1 - w)^L W_0.
\end{aligned} \tag{8}$$

Since A^c occurs when the gamblers lose the first L bets, we have $\mathrm{P}(A^c) \ge (1 - w)^L$. Thus, since $\mathrm{E}[W_R] \le W_0$ by Lemma 3, combining (7) and (8) gives

$$\begin{aligned}
\mathrm{E}[W_T] &\le \mathrm{E}[W_R] - (1 - g(\tfrac{1}{2}))(1 - w)^{2L} W_0 \\
&\le (1 - (1 - g(\tfrac{1}{2}))(1 - w)^{2L}) W_0 \\
&= \alpha W_0,
\end{aligned}$$

which completes the proof.

Lemma 5. *Suppose that $f_2 < 1 - \ell$ and $1 - \ell \le f_1 < 1 - \ell/2$. Define the stopping time T by*

$$T(\omega) = \begin{cases} 1 & \text{if } \omega_1 = 1, \\ \inf\{j \ge 2 \colon \omega_j = 1 \text{ or } h(X_{j-1}(\omega), Y_{j-1}(\omega)) \ge \frac{1}{2}\} & \text{otherwise.} \end{cases}$$

Let $N(\omega) = 1 - \alpha$ if $\omega_1 = 1$ and let $N(\omega) = 1$ if $\omega_1 = 0$. Then $T \le L + 2$ and $\mathrm{E}[N W_T] \le W_0$.

Proof. Let A be the event that the gamblers win the first bet, which means that $\omega_1 = 1$. We have

$$\mathrm{E}[N W_T] = \mathrm{E}[\mathrm{E}[N W_T \mid \mathcal{F}_1]] = \mathrm{E}[(1 - \alpha) W_1 \mathbf{1}_A + \mathrm{E}[W_T \mid \mathcal{F}_1] \mathbf{1}_{A^c}].$$

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If the gamblers lose the first bet then $X_1 = f_1 - s(f_1) = 2f_1 - 1 < 1 - \ell$. Therefore, by Lemma 4 and the Markov property, we have $T \le L + 2$ and $\mathrm{E}[W_T \mid \mathcal{F}_1]\,\mathbf{1}_{A^c} \le \alpha W_1\,\mathbf{1}_{A^c}$. Thus,

$$\begin{aligned}
\mathrm{E}[NW_T] &\le \mathrm{E}[(1-\alpha)W_1\,\mathbf{1}_A + \alpha W_1\,\mathbf{1}_{A^c}] \\
&= w(1-\alpha)[(1+h(f_1, f_2))(f_1 - f_2)]^{-\log(1-w)} \\
&\quad + (1-w)\alpha[(1-h(f_1, f_2))(f_1 - f_2)]^{-\log(1-w)} \\
&= [w(1-\alpha)(1+h(f_1, f_2))^{-\log(1-w)} + (1-w)\alpha(1-h(f_1, f_2))^{-\log(1-w)}]W_0 \\
&\le [w(1-\alpha)2^{-\log(1-w)} + (1-w)\alpha 2^{-\log(1-w)}]W_0 \\
&= \left(\frac{w}{1-w}(1-\alpha) + \alpha\right)W_0 \\
&\le W_0,
\end{aligned}$$

which completes the proof.

By combining Lemmas 2, 4, and 5, we can obtain Proposition 2, below, in which we construct the supermartingale needed to prove Proposition 1(ii). We first inductively define a sequence of stopping times $(T_k)_{k=0}^{\infty}$. Let $T_0 = 0$. Given T_k, we define T_{k+1} according to the following rules.

1. If $Y_{T_k}(\omega) \ge 1 - \ell$ then let $T_{k+1}(\omega) = T_k(\omega) + 1$.

2. If $X_{T_k}(\omega) < 1 - \ell$ then let

$$T_{k+1}(\omega) = \inf\{j \ge T_k(\omega) + 1 \colon \omega_j = 1 \text{ or } h(X_{j-1}(\omega), Y_{j-1}(\omega)) \ge \tfrac{1}{2}\}.$$

3. Suppose that $Y_{T_k}(\omega) < 1 - \ell$ and $1 - \ell \le X_{T_k}(\omega) < 1 - \ell/2$. If $\omega_{T_k(\omega)+1} = 1$, meaning the gamblers win the $(T_k + 1)$th bet, then let $T_{k+1}(\omega) = T_k(\omega) + 1$. Otherwise, let $T_{k+1}(\omega) = \inf\{j \ge T_k(\omega) + 2 \colon \omega_j = 1 \text{ or } h(X_{j-1}(\omega), Y_{j-1}(\omega)) \ge \tfrac{1}{2}\}$.

4. If $Y_{T_k} < 1 - \ell$ and $X_{T_k} \ge 1 - \ell/2$ then let $T_{k+1} = T_k$.

Proposition 2. *Define the sequence of stopping times* $(T_k)_{k=0}^{\infty}$ *as above. For* $k \ge 0$, *let*

$$B_k = \sharp\{j \in \{0, 1, \dots, k-1\} \colon X_{T_j} < 1 - \ell\},$$

where $\sharp S$ *denotes the cardinality of the set* S. *Let* $N_k = 1 - \alpha$ *in the event that, for some* $j \le T_k$, *we have* $Y_{j-1} < 1 - \ell \le X_{j-1}$ *and the gamblers win the* j*th bet. Otherwise, let* $N_k = 1$. *Define* $Z_k = \alpha^{-B_k} N_k W_{T_k}$. *Then* $(Z_k)_{k=0}^{\infty}$ *is a supermartingale with respect to the filtration* $(\mathcal{F}_{T_k})_{k=0}^{\infty}$. *Furthermore,* $T_{k+1} \le T_k + L + 2$ *for all* $k \ge 0$.

Proof. Let $A_{1,k}$ be the event that $Y_{T_k} \ge 1 - \ell$ and let $A_{2,k}$ be the event that $X_{T_k} < 1 - \ell$. Let $A_{3,k}$ be the event that $Y_{T_k} < 1 - \ell$ and $1 - \ell \le X_{T_k} < 1 - \ell/2$. Let $A_{4,k}$ be the event that $Y_{T_k} < 1 - \ell$ and $X_{T_k} \ge 1 - \ell/2$. Note that, for all k, exactly one of these four events occurs. We consider the four cases separately.

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First, suppose that $A_{1,k}$ occurs. Then $X_{T_k} \geq 1 - \ell$, so $B_{k+1} = B_k$. Also, note that $T_{k+1} = T_k + 1$ and $Y_{T_k} \geq 1 - \ell$, so $N_{k+1} = N_k$. Therefore, by Lemma 2 and the strong Markov property,

$$\begin{aligned} \mathrm{E}[Z_{k+1} \mathbf{1}_{A_{1,k}} \mid \mathcal{F}_{T_k}] &= \alpha^{-B_k} N_k \, \mathrm{E}[W_{T_{k+1}} \mid \mathcal{F}_{T_k}] \mathbf{1}_{A_{1,k}} \\ &= \alpha^{-B_k} N_k \, \mathrm{E}[W_{T_k+1} \mid \mathcal{F}_{T_k}] \mathbf{1}_{A_{1,k}} \\ &= \alpha^{-B_k} N_k W_{T_k} \mathbf{1}_{A_{1,k}} \\ &= Z_k \mathbf{1}_{A_{1,k}} . \end{aligned} \tag{9}$$

Next, suppose that $A_{2,k}$ occurs. Then $X_{T_k} < 1 - \ell$, so $B_{k+1} = B_k + 1$. The gamblers lose bets $T_k + 1, \dots, T_{k+1} - 1$, so $X_j < 1 - \ell$ for $T_k \leq j \leq T_{k+1} - 1$. Therefore, $N_{k+1} = N_k$. By Lemma 4 and the strong Markov property,

$$\begin{aligned} \mathrm{E}[Z_{k+1} \mathbf{1}_{A_{2,k}} \mid \mathcal{F}_{T_k}] &= \alpha^{-(B_k+1)} N_k \, \mathrm{E}[W_{T_{k+1}} \mid \mathcal{F}_{T_k}] \mathbf{1}_{A_{2,k}} \\ &\leq \alpha^{-(B_k+1)} N_k (\alpha W_{T_k}) \mathbf{1}_{A_{2,k}} \\ &= \alpha^{-B_k} N_k W_{T_k} \mathbf{1}_{A_{2,k}} \\ &= Z_k \mathbf{1}_{\{A_{2,k}\}} . \end{aligned} \tag{10}$$

Suppose that $A_{3,k}$ occurs. Then $X_{T_k} \geq 1 - \ell$, so $B_{k+1} = B_k$. Since $X_{T_k} < 1$, we have $N_k = 1$. If the gamblers win the $(T_k + 1)$th bet then $N_{k+1} = 1 - \alpha$. Otherwise, $X_{T_k+1} = X_{T_k} - s(X_{T_k}) = 2X_{T_k} - 1 < 1 - \ell$ and the gamblers lose bets $T_k + 2, \dots, T_{k+1} - 1$, so $N_{k+1} = 1$. By Lemma 5 and the strong Markov property,

$$\begin{aligned} \mathrm{E}[Z_{k+1} \mathbf{1}_{A_{3,k}} \mid \mathcal{F}_{T_k}] &= \alpha^{-B_k} \, \mathrm{E}[N_{k+1} W_{T_{k+1}} \mid \mathcal{F}_{T_k}] \mathbf{1}_{A_{3,k}} \\ &\leq \alpha^{-B_k} W_{T_k} \mathbf{1}_{A_{3,k}} \\ &= Z_k \mathbf{1}_{A_{3,k}} . \end{aligned} \tag{11}$$

Finally, if $A_{4,k}$ occurs then $T_{k+1} = T_k$. Therefore, $N_{k+1} = N_k$ and $W_{T_{k+1}} = W_{T_k}$. Since $X_{T_k} > 1 - \ell$, we also have $B_{k+1} = B_k$ and, thus, $Z_{k+1} = Z_k$. Therefore, $\mathrm{E}[Z_{k+1} \mathbf{1}_{A_{4,k}} \mid \mathcal{F}_{T_k}] = Z_k \mathbf{1}_{A_{4,k}}$. This fact, combined with (9)–(11), gives $\mathrm{E}[Z_{k+1} \mid \mathcal{F}_{T_k}] \leq Z_k$. Hence, $(Z_k)_{k=0}^{\infty}$ is a supermartingale with respect to the filtration $(\mathcal{F}_{T_k})_{k=0}^{\infty}$.

To complete the proof, note that clearly $T_{k+1} \leq T_k + L + 2$ if $A_{1,k}$ or $A_{4,k}$ occurs. The strong Markov property, combined with Lemmas 4 and 5, implies that $T_{k+1} \leq T_k + L + 2$ if $A_{2,k}$ or $A_{3,k}$ occurs.

We now use Proposition 2 to establish an upper bound on $Q(f) - Q(f - \varepsilon)$ when $f \notin S$. We will need one more lemma.

Lemma 6. *Fix $f \notin S$ and let N be a positive integer. Then there exist a positive integer M and a positive real number δ such that if $f_1 = f$ and $f_2 = f - \varepsilon$, where $0 < \varepsilon < \delta$, then the following statements hold.*

(i) *If $X_k \geq 1 - \ell$ and $Y_k < 1 - \ell$ then $k > M(L + 2)$, where $L = \lfloor 1 + (1 - 2\ell)/\ell \rfloor$.*

(ii) *Let $D_k = \sharp\{j \in \{0, 1, \dots, k - 1\} \colon X_{T_j} < 1 - \ell \text{ or } X_{T_j} = 1\}$. Then $D_M \geq N$.*

Proof. Let R'_k be the set of all possible values of X_k, and let $R_k = \bigcup_{j=0}^{k} R'_j$. Note that R_k is a finite set because there are only 2^k possible outcomes for the first k bets. For all $g \in [0, 1)$, let $v(g)$ be the number of consecutive bets that a gambler whose fortune is g must lose for the

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fortune to drop below $1-\ell$. That is, $v(g)=0$ when $0 \le g < 1-\ell$ and, for positive integers k, $v(g)=k$ when $1-2^{-k+1}\ell \le g < 1-2^{-k}\ell$. Let $V_k = \max\{v(g),\ g \in R_k\}$. Let $M_0 = 0$. Let $M_{i+1} = M_i + V_{M_i(L+2)} + 1,\ i \ge 0$, and let $M = M_N$. Choose $\theta > 0$ small enough that $R_{M(L+2)} \cap (1-\ell, 1-\ell+\theta) = \varnothing$. Let $\delta = 2^{-M(L+2)}\theta$. We will show that the two conditions of Lemma 6 are satisfied for these choices of M and δ.

Suppose that $k \le M(L+2)$. If $X_k \ge 1-\ell$ then $X_k \ge 1-\ell+\theta$, since $f \notin S$ and $R_k' \cap (1-\ell, 1-\ell+\theta) = \varnothing$. By Remark 1, we have $X_k - Y_k \le 2^k\varepsilon < 2^{M(L+2)}\delta = \theta$. Therefore, $Y_k \ge 1-\ell$. This proves the first part of the lemma.

To prove the second part we claim that, for $i = 0, 1, \dots, N-1$,

$$D_{M_i + V_{M_i(L+2)}+1} \ge D_{M_i} + 1. \tag{12}$$

To see how (12) implies the second part of the lemma, first note that $D_{M_0} = D_0 = 0$. Suppose that $D_{M_i} \ge i$ for some $i \ge 0$. Then $D_{M_{i+1}} = D_{M_i + V_{M_i(L+2)}+1} \ge D_{M_i} + 1 \ge i+1$ by (12). Hence, by induction, (12) implies that $D_M \ge N$. Thus, we need only to prove (12). First, suppose that either $X_{T_{M_i}} < 1-\ell$ or $X_{T_{M_i}} = 1$. Then $D_{M_i+1} = D_{M_i} + 1$. Since $(D_i)_{i=0}^{\infty}$ is a nonincreasing sequence and $V_{M_i(L+2)} \ge 0$, we have (12).

Thus, it remains only to prove (12) when $1-\ell \le X_{T_{M_i}} < 1$. Write v for $v(X_{T_{M_i}})$. Note that $v \le V_{T_{M_i}}$, and $T_{M_i} \le M_i(L+2)$ by Proposition 2. Therefore,

$$\begin{aligned} T_{M_i} + v &\le T_{M_i} + V_{T_{M_i}} \\ &\le M_i(L+2) + V_{M_i(L+2)} \\ &\le (M_i + V_{M_i(L+2)})(L+2) \\ &\le M(L+2). \end{aligned} \tag{13}$$

We now consider two cases. First, suppose that the gamblers lose the bets $T_{M_i}+1, \dots, T_{M_i}+v$. Then $X_j \ge 1-\ell$ for $T_{M_i} \le j \le T_{M_i+v-1}$ and $X_{T_{M_i}+v} < 1-\ell$. Also, by (13) and the first part of the lemma, we have $Y_j \ge 1-\ell$ for $T_{M_i} \le j \le T_{M_i+v-1}$. Therefore, by the definition of the sequence $(T_j)_{j=0}^{\infty}$, we have $T_{M_i+k} = T_{M_i} + k$ for $1 \le k \le v$. It follows that $X_{T_{M_i+v}} < 1-\ell$, which means that $D_{M_i+v+1} = D_{M_i}+1$. Thus, $D_{M_i+V_{M_i(L+2)}+1} \ge D_{M_i+v+1} = D_{M_i}+1$, which is (12). Finally, we consider the case in which, for some $j \in \{1, \dots, v\}$, the gamblers lose the bets $T_{M_i}+1, \dots, T_{M_i}+j-1$ but win the bet $T_{M_i}+j$. Then, $X_{T_{M_i}+j} = X_{T_{M_i+j}} = 1$ and $D_{M_i+j+1} = D_{M_i}+1$. Hence, $D_{M_i+V_{M_i(L+2)}+1} \ge D_{M_i}+1$, which is (12).

Proof of Proposition 1(ii). Fix $C > 0$ and $f \notin S$. Since $0 < \alpha < 1$, there exists a positive integer N such that

$$\alpha^{-N}(1-\alpha)(1-w)^{-\log(\ell/2)} \ge C^{-1}.$$

Define M and δ as in Lemma 6 and fix $\varepsilon \in (0, \delta)$. Define $(T_k)_{k=0}^{\infty}$ and $(Z_k)_{k=0}^{\infty}$ as in Proposition 2, with $f_1 = f$ and $f_2 = f - \varepsilon$.

By Remark 3, there exist random variables L_1 and L_2 such that $X_k \to L_1$ a.s. and $Y_k \to L_2$ a.s. as $k \to \infty$, and $Q(f) - Q(f-\varepsilon) = \mathrm{P}(L_1 = 1 \text{ and } L_2 = 0)$. Let A be the event that $L_1 = 1$ and $L_2 = 0$. Then there is an integer-valued random variable K such that, on the event A, we have $X_{T_K} \ge 1-\ell/2$ and $Y_{T_K} < 1-\ell$. By Lemma 6(i), in the event A, we have $T_K > M(L+2)$ and, thus, $K \ge M$. It also follows from Lemma 6(i) that if $X_{T_j} = 1$ for $j \le M$ then $Y_{T_j} = 1$ and, therefore, $L_2 = 1$. Consequently, in the event A, we can see from the definitions of $(B_i)_{i=0}^{\infty}$ and $(D_i)_{i=0}^{\infty}$ that $B_M = D_M$ and, thus, using Lemma 6(ii), $B_K \ge B_M = D_M \ge N$.

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Since $(Z_k)_{k=0}^{\infty}$ is a nonnegative supermartingale, it follows from the martingale convergence theorem (see [9, Corollary 4.2.11]) that there exists a random variable Z such that $Z_k \to Z$ a.s. and $\mathrm{E}[Z] \le \mathrm{E}[Z_0]$. On the event A, if $j > K$ then $T_j = T_K$ and, thus, $Z_j = Z_K = Z$. Hence, using (5),

$$\begin{aligned} Z\,\mathbf{1}_A &= Z_K\,\mathbf{1}_A \\ &= \alpha^{-B_K} N_K (1-w)^{-\log(X_{T_K} - Y_{T_K})}\,\mathbf{1}_A \\ &\ge \alpha^{-N}(1-\alpha)(1-w)^{-\log(l/2)}\,\mathbf{1}_A \\ &\ge C^{-1}\,\mathbf{1}_A\,. \end{aligned}$$

It follows that $\mathrm{E}[Z] \ge C^{-1}\,\mathrm{P}(A)$. Thus, $Q(f) - Q(f-\varepsilon) = \mathrm{P}(A) \le C\,\mathrm{E}[Z] \le C\,\mathrm{E}[Z_0] = C(1-w)^{-\log \varepsilon}$, as claimed.

2.4. Proof of Proposition 1(iii)

Let $D^1 = S \cap [0, l]$ and $D^2 = S \cap [1-\ell, 1]$. Define a sequence of stopping times $(\tau_k)_{k=0}^{\infty}$ by $\tau_0 = 0$ and $\tau_{k+1} = \inf\{n > \tau_k \colon X_n \in [0,\ell] \cup [1-\ell, 1]\}$, for all $k \ge 0$. Then define

$$D_k = \{f \colon \mathrm{P}(X_{\tau_j} = 1 - l \text{ for some } j \le k \mid X_0 = f) > 0\}.$$

Let $D_k^1 = D_k \cap [0,\ell]$ and $D_k^2 = D_k \cap [1-\ell, \ell]$. Note that $D^1 = \bigcup_{k=0}^{\infty} D_k^1$ and $D^2 = \bigcup_{k=0}^{\infty} D_k^2$. We have $D_0^1 = \varnothing$ and $D_0^2 = \{1-\ell\}$. For $k \ge 1$,

$$\begin{aligned} D_k^1 &= \{f \in [0,\ell] \colon \mathrm{P}(X_{\tau_1} \in D_{k-1} \mid X_0 = f) > 0\} \cup D_{k-1}^1, \\ D_k^2 &= \{f \in [1-\ell, 1] \colon \mathrm{P}(X_{\tau_1} \in D_{k-1} \mid X_0 = f) > 0\} \cup D_{k-1}^2. \end{aligned}$$

Suppose that $X_0 = f$. If $f \in (\ell, 1-\ell)$ then $s(f) = \ell$ for all $k < \tau_1$. Therefore, $X_{\tau_1} = f + n\ell$ for some $n \in \mathbb{Z}$. If, instead, $f \in [0,\ell]$ then $s(f) = f$, in which case either $X_1 = X_{\tau_1} = 0$ or $X_1 = 2f$. If $X_1 = 2f$ then $X_{\tau_1} = 2f + n\ell$ for some $n \in \mathbb{Z}$, where $n = 0$ if $2f \in [0, l] \cup [1-\ell, 1]$. Likewise, suppose that $f \in [1-\ell, \ell]$. Then $s(f) = 1 - f$, so either $X_1 = X_{\tau_1} = 1$ or $X_1 = 2f - 1$. If $X_1 = 2f - 1$ then $X_{\tau_1}(f) = 2f - 1 + n\ell$ for some $n \in \mathbb{Z}$, where $n = 0$ if $2f - 1 \in [0,\ell] \cup [1-\ell, 1]$.

We claim that if $f \in D^1 \cup D^2$ then there exist integers a, b, and c such that $f = 2^{-c}(a + b\ell)$. Furthermore, we claim that if $f \ne 1-\ell$ then we can choose a, b, and c such that $c \ge 1$, $a \ge 1$, a or b is odd, and $a \ge 2$ if $f \in D^2$. We will prove these claims by induction on k. Note that $D_0 = \{1-\ell\}$ so, for $f \in D_0$, we can take $a = 1$, $b = -1$, and $c = 0$. Now, suppose that our claims hold when $f \in D_{k-1}$, where $k \ge 1$. To show that our claims hold when $f \in D_k$, we consider two cases.

First, suppose that $f \in D_k^1 \setminus D_{k-1}$. Then $\mathrm{P}(X_{\tau_1} = g \mid X_0 = f) > 0$ for some $g \in D_{k-1}$. Since $0 \notin S$, we must have $2f + n\ell = g$ or, equivalently, $f = (g - n\ell)/2$ for some $n \in \mathbb{Z}$ and $g \in D_{k-1}$. If $g = 1 - \ell$ then $f = (1 - (n+1)\ell)/2$, so $f = 2^{-c}(a + b\ell)$, where $a = 1$, $b = -(n+1)$, and $c = 1$. If $g \ne 1 - \ell$ then $g = 2^{-c}(a + b\ell)$, where $c \ge 1$, $a \ge 1$, and a or b is odd. Then $f = 2^{-(c+1)}(a + b\ell - 2^c n\ell) = 2^{-(c+1)}(a + (b - 2^c n)\ell)$. Note that $c + 1 \ge 1$, $a \ge 1$, and $b - 2^c n$ is odd if b is odd, so a or $b - 2^c n$ is odd.

Next, suppose that $f \in D_k^2 \setminus D_{k-1}$. Then $\mathrm{P}(X_{\tau_1} = g \mid X_0 = f) > 0$ for some $g \in D_{k-1}$. Since $1 \notin S$, we have $2f - 1 + n\ell = g$ or, equivalently, $f = (1 + g - n\ell)/2$ for some $n \in \mathbb{Z}$ and $g \in D_{k-1}$. If $g = 1 - \ell$ then $f = (2 - (n+1)\ell)/2$. If $n + 1$ were even then $f = 1 - m\ell$ would hold for some positive integer m; since $D^2 \subseteq [1-\ell, 1]$, we would have $f \in \{1-\ell, 1\}$, which is

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a contradiction because $1 \notin D_k^2$ and $1-\ell \in D_{k-1}$. Therefore, $n+1$ is odd, so $f = 2^{-c}(a+b\ell)$, where $c = 1$, $a = 2$, and b is odd. If, instead, $g \neq 1-\ell$ then $g = 2^{-c}(a+b\ell)$, where $c \geq 1$, $a \geq 1$, and a or b is odd. Then $f = 2^{-(c+1)}(2^c+a+b\ell-2^c n\ell) = 2^{-(c+1)}[(2^c+a)+(b-2^c n)\ell]$. Note that $c+1 \geq 1$, $2^c+a \geq 2$, and either 2^c+a or $b-2^c n$ is odd because 2^c and $2^c n$ are even and either a or b is odd. It now follows, by induction, that our claims hold for all $f \in D^1 \cup D^2$.

Since $\ell < \frac{1}{2}$, we can choose a positive integer m such that $1 - m\ell \in (\ell, 2\ell]$. We can then choose positive integers d and n such that $2^{-d}(1-m\ell) < 1-2\ell$ and $2^{-d}(1-\ell) + n\ell \in (1-2\ell, 1-\ell)$. Let $f = 2^{-d}(1-m\ell) + n\ell$. Note that $f \in (\ell, 1-\ell)$ and that $f - \ell = 2^{-d}(1-m\ell) + (n-1)\ell$. Suppose that a gambler who starts with a fortune of $f-\ell$ loses the first $n-1$ bets, then wins the next $d+m-1$ bets. After the $n-1$ losses, the gambler's fortune will be $2^{-d}(1-m\ell)$. Then, after d wins, the fortune will be $1-m\ell$. After $m-1$ additional wins, the gambler's fortune will be $1-\ell$. Consequently,

$$\mathrm{P}(X_{n+m+d-2} = 1-\ell \mid X_0 = f-\ell) > 0,$$

which means that $f-\ell \in S$. We now show by contradiction that $f+\ell \notin S$, which will complete the proof. Suppose that $f+\ell \in S$. Since $f+\ell > 1-\ell$, there exist integers a, b, and c such that $a \geq 2$, $c \geq 1$, a or b is odd, and $f+\ell = 2^{-c}(a+b\ell)$. We also have $f+\ell = 2^{-d}(1+(2^d(n+1)-m)\ell)$. Therefore, $2^d(a+b\ell) = 2^c(1+(2^d(n+1)-m)\ell)$ and, so, $2^d a - 2^c = (2^c(2^d(n+1)-m) - 2^d b)\ell$. Since ℓ is irrational, we must have $2^d a - 2^c = 2^c(2^d(n+1)-m) - 2^d b = 0$. Thus, $2^d a = 2^c$ and, since $a \geq 2$, it follows that a is even and $c > d$. Therefore, b is odd and $b = 2^{c-d}(2^d(n+1)-m)$, which is a contradiction.

2.5. Obtaining Theorem 1 from Proposition 1

Suppose that ℓ is irrational. By Proposition 1(iii), there exists an $f_0 \in (\ell, 1-\ell)$ such that $f_0 - \ell \in S$ and $f_0 + \ell \notin S$. Let $f = f_0 - \varepsilon$, where $0 < \varepsilon < \ell$ and ε is small enough that $f \in (\ell, 1-\ell)$. We will show that, for sufficiently small ε, we have

$$wQ(f+\ell-\varepsilon) + (1-w)Q(f-\ell+\varepsilon) > Q(f), \tag{14}$$

which implies Theorem 1 because $s(f) = \ell$. Note that

$$wQ(f+\ell-\varepsilon) + (1-w)Q(f-\ell+\varepsilon) = wQ(f_0+\ell-2\varepsilon) + (1-w)Q(f_0-\ell). \tag{15}$$

Since $f \mapsto Q(f)$ is nondecreasing, we have

$$Q(f) = wQ(f+\ell) + (1-w)Q(f-\ell) \leq wQ(f_0+\ell) + (1-w)Q(f_0-\ell-\varepsilon). \tag{16}$$

Since $f_0 - \ell \in S$, it follows from Proposition 1(i) that there exists a constant $C > 0$ such that

$$Q(f_0-\ell) - Q(f_0-\ell-\varepsilon) \geq C(1-w)^{-\log\varepsilon}. \tag{17}$$

Let $C_0 = C(1-w)$. Since $f_0 + \ell \notin S$, Proposition 1(ii) implies that, for sufficiently small ε, we have

$$Q(f_0+\ell) - Q(f_0+\ell-2\varepsilon) \leq C_0(1-w)^{-\log 2\varepsilon} = C(1-w)^{-\log\varepsilon}. \tag{18}$$

Let $B = C(1-w)^{-\log\varepsilon}$. Equations (15)–(18) imply that

$$wQ(f+\ell-\varepsilon) + (1-w)Q(f-\ell+\varepsilon) - Q(f) \geq -wB + (1-w)B = (1-2w)B > 0,$$

for sufficiently small ε, which gives (14).

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