Appendix B

Many faces of two-dimensional supersymmetric CP($N-1$) model

B.1 O(3) sigma model

Supersymmetric extension of the O(3) sigma model in the form discussed in this section was suggested in Refs. [7, 237]. We refer the reader to the book [238] for a pedagogical discussion of the non-supersymmetric O(3) sigma model.

One can construct supersymmetric sigma model in terms of two-dimensional $\mathcal{N}=1$ superfields as follows. Let us introduce a triplet of real superfields N^a,

$$N^a(x,\theta) = S^a(x) + \bar{\theta}\chi^a(x) + \frac{1}{2}\bar{\theta}\theta\, F^a(x), \quad a = 1,2,3, \tag{B.1}$$

where θ is a two-component Majorana (real) spinor ($\bar{\theta} = \theta\gamma^0$), χ^a is a two-component Majorana fermion field and F^a is an auxiliary boson field which will enter in the Lagrangian with no kinetic term. The superfield $N^a(x,\theta)$ is subject to the constraint

$$N^a(x,\theta)N^a(x,\theta) = 1. \tag{B.2}$$

In components this is equivalent to

$$S^aS^a = 1, \quad S^a\chi^a = 0, \quad S^aF^a = \frac{1}{2}\,\bar{\chi}^a\chi^a. \tag{B.3}$$

The action of the model takes the form

$$\begin{aligned} S &= \frac{1}{2g_0^2}\int d^2x\, d^2\theta\, \varepsilon^{\alpha\beta}\big(D_\alpha N^a\big)\big(D_\beta N^a\big) \\ &= \frac{1}{g_0^2}\int d^2x\left[\frac{1}{2}\big(\partial_\mu S^a\big)^2 + \frac{1}{2}\,\bar{\chi}^a i\gamma^\mu\partial_\mu\chi^a + \frac{1}{8}\,(\bar{\chi}\chi)^2\right] \end{aligned} \tag{B.4}$$

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where g_0^2 is the (bare) coupling constant and

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}_\alpha} - i(\gamma^\mu\theta)_\alpha\,\partial_\mu. \tag{B.5}$$

This model describes two independent (real) degrees of freedom in the bosonic and fermionic sectors. The interaction inherent to this model is due to the constraints (B.3) and the four-fermion term in (B.4). The model is O(3) symmetric, by construction. Also by construction it has $\mathcal{N} = (1, 1)$ supersymmetry (i.e. one left-handed real supercharge, and one right-handed). In fact this model has an extended $\mathcal{N} = 2$ supersymmetry (more exactly, $\mathcal{N} = (2, 2)$). The occurrence of two extra supercharges (four altogether) is automatic and is explained by the fact that the target space of the bosonic sector is S^2, which is a Kähler manifold. Minimal $\mathcal{N} = (1, 1)$ supersymmetrization of any Kählerian sigma model automatically produces $\mathcal{N} = (2, 2)$ supersymmetry. Further details can be found in the review paper [156].

B.2 CP(1) sigma model

The same model expressed in terms of unconstrained variables is usually referred to as the CP(1) model. If the unit vector S^a parametrizes the sphere, one can pass to unconstrained variables by performing the stereographic projection of the sphere onto the complex ϕ plane,

$$\phi = \frac{S^1 + iS^2}{1 + S^3}. \tag{B.6}$$

The complex field ϕ replaces two independent components of S^a. The unconstrained two-component *complex* fermion field ψ is introduced as follows:

$$\psi = \frac{\chi^1 + i\chi^2}{1 + S^3} - \frac{S^1 + iS^2}{(1 + S^3)^2}\,\chi^3. \tag{B.7}$$

The inverse transformations have the form

$$S^1 = \frac{2(\mathrm{Re}\phi)}{1 + |\phi|^2}, \quad S^2 = \frac{2(\mathrm{Im}\phi)}{1 + |\phi|^2}, \quad S^3 = \frac{1 - |\phi|^2}{1 + |\phi|^2} \tag{B.8}$$

and

$$\chi^1 = \frac{2(\mathrm{Re}\,\psi)}{1 + |\phi|^2} - \frac{2(\mathrm{Re}\,\phi)[\phi^\dagger\psi + \mathrm{H.c.}]}{(1 + |\phi|^2)^2},$$

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$$\chi^2 = \frac{2(\mathrm{Im}\,\psi)}{1+|\phi|^2} - \frac{2(\mathrm{Im}\,\phi)[\phi^\dagger\psi + \mathrm{H.c.}]}{(1+|\phi|^2)^2},$$
$$\chi^3 = -2\frac{[\phi^\dagger\psi + \mathrm{H.c.}]}{(1+|\phi|^2)^2}. \tag{B.9}$$

Substituting Eqs. (B.8) and (B.9) in the action (B.4) we get [239]

$$L_{\mathrm{CP}(1)} = G\left\{\partial_\mu\phi^\dagger\,\partial^\mu\phi + i\bar\psi\gamma^\mu\partial_\mu\psi - \frac{2i}{\chi}\,\phi^\dagger\partial_\mu\phi\,\bar\psi\gamma^\mu\psi + \frac{1}{\chi^2}\,(\bar\psi\psi)^2\right\} \tag{B.10}$$

where

$$G = \frac{2}{g_0^2\,\chi^2}, \qquad \chi = 1+|\phi|^2. \tag{B.11}$$

The above Lagrangian can be obtained in terms of $\mathcal{N}=2$ superfields which will make its $\mathcal{N}=(2,2)$ supersymmetry explicit. Namely, let us introduce a chiral superfield

$$\Phi(x_L,\theta) = \phi(x_L) + \sqrt{2}\,\varepsilon_{\alpha\beta}\,\theta^\alpha\psi^\beta(x_L) + \varepsilon_{\alpha\beta}\,\theta^\alpha\theta^\beta\,F(x_L), \tag{B.12}$$

where θ is a two-component *complex* Grassmann variable, while

$$x_L^\mu = x^\mu + i\bar\theta\gamma^\mu\theta. \tag{B.13}$$

Moreover, $\Phi^\dagger$ depends on $x_R^\mu = x^\mu - i\bar\theta\gamma^\mu\theta$ and $\bar\theta$, a conjugation of (B.12). In terms of these superfields the Lagrangian of the CP(1) model can be written as

$$L_{\mathrm{CP}(1)} = \int d^4\theta\,K(\Phi,\,\Phi^\dagger), \tag{B.14}$$

where K is the Kähler potential,

$$K = \frac{2}{g_0^2}\,\ln(1+\Phi^\dagger\Phi). \tag{B.15}$$

Needless to say, $\mathcal{N}=2$ supersymmetry is built in here. And what about the target space symmetry? The U(1) symmetry corresponding to the rotation around the third axis in the target space is realized linearly,

$$\Phi \to \Phi + i\alpha\cdot\Phi, \qquad \Phi^\dagger \to \Phi^\dagger - i\alpha\cdot\Phi^\dagger, \tag{B.16}$$

where α is a real parameter. At the same time, two other symmetry rotations are realized nonlinearly,

$$\Phi \to \beta + \beta^*\cdot\Phi^2, \qquad \Phi^\dagger \to \beta^* + \beta\cdot(\Phi^\dagger)^2, \tag{B.17}$$

with a complex parameter β.

B.3 Geometric interpretation

Equations (B.14) and (B.15) suggest a geometric interpretation (for a review see e.g. [240]) for the above formulation of the CP(1) model which, in turn, allows one to readily generalize it to the case of CP($N-1$) with arbitrary N. Indeed, let us consider $N-1$ complex superfields

$$\Phi^i(x^\mu + i\bar{\theta}\gamma^\mu\theta), \quad \Phi^{\dagger\bar{j}}(x^\mu - i\bar{\theta}\gamma^\mu\theta),$$

and the Kähler potential

$$K = \frac{2}{g_0^2} \ln\left(1 + \sum_{i,\bar{j}=1}^{N-1} \Phi^{\dagger\,\bar{j}} \delta_{\bar{j}i} \Phi^i\right). \tag{B.18}$$

(As we will see momentarily, it corresponds to the so-called round Fubini–Study metric.) The Kähler potential determines the metric of the target space according to the formula

$$G_{i\bar{j}} = \frac{\partial^2 K(\phi, \phi^\dagger)}{\partial\phi^i \partial\phi^{\dagger\,\bar{j}}}. \tag{B.19}$$

For CP($N-1$) the Riemann tensor is expressed in terms of the metric (B.19) as follows:

$$R_{i\bar{j}k\bar{m}} = -\frac{g_0^2}{2}\left(G_{i\bar{j}}G_{k\bar{m}} + G_{i\bar{m}}G_{k\bar{j}}\right), \tag{B.20}$$

while the Ricci tensor

$$R_{i\bar{j}} = \frac{g_0^2}{2}\, N\, G_{i\bar{j}}. \tag{B.21}$$

In components the Lagrangian of the CP($N-1$) model takes the form [241]

$$L = \int d^4\theta\, K = G_{i\bar{j}}\left[\partial_\mu\phi^{\dagger\,\bar{j}}\,\partial_\mu\phi^i + i\bar{\psi}^{\bar{j}}\gamma^\mu D_\mu\psi^i\right] - \frac{1}{2}\, R_{i\bar{j}k\bar{l}}\,(\bar{\psi}^{\bar{j}}\psi^i)(\bar{\psi}^{\bar{l}}\psi^k), \tag{B.22}$$

where D is the covariant derivative,

$$D_\mu\psi^i = \partial_\mu\psi^i + \Gamma^i_{kl}\left(\partial_\mu\phi^k\right)\psi^l, \tag{B.23}$$

and Γ^i_{kl} is the Christoffel symbol.

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If $N = 2$ the above expressions simplify and we get

$$G = G_{1\bar{1}} = \partial_\phi \partial_{\phi^\dagger} K\big|_{\theta=\bar{\theta}=0} = \frac{2}{g_0^2\,\chi^2},$$
$$\Gamma = \Gamma^1_{11} = -2\,\frac{\phi^\dagger}{\chi}, \quad \bar{\Gamma} = \Gamma^{\bar{1}}_{\bar{1}\bar{1}} = -2\,\frac{\phi}{\chi},$$
$$R \equiv R_{1\bar{1}} = -G^{-1} R_{1\bar{1}1\bar{1}} = \frac{2}{\chi^2}, \tag{B.24}$$

where we use the notation

$$\chi \equiv 1 + \phi\,\phi^\dagger. \tag{B.25}$$

Substituting (B.24) and (B.25) in (B.22) we arrive at the CP(1) Lagrangian (B.10).

B.4 Gauged formulation

Here we will discuss yet another formulation of $\mathcal{N} = 2$ supersymmetric sigma models with the target space

$$\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-1)\times \mathrm{U}(1)} = \mathrm{CP}(N-1), \tag{B.26}$$

which goes under the name of the gauged formulation [242]. This formulation is built on an N-plet of complex scalar fields n^i where $i = 1, 2, ..., N$. We impose the constraint

$$n_i^\dagger\, n^i = 1. \tag{B.27}$$

This leaves us with $2N - 1$ real bosonic degrees of freedom. To eliminate one extra degree of freedom we impose a local U(1) invariance $n^i(x) \to e^{i\alpha(x)} n^i(x)$. To this end we introduce a gauge field A_μ which converts the partial derivative into the covariant one,

$$\partial_\mu \to \nabla_\mu \equiv \partial_\mu - i\,A_\mu. \tag{B.28}$$

The field A_μ is auxiliary; it enters in the Lagrangian without derivatives. The kinetic term of the n fields is

$$L = \frac{2}{g_0^2}\left|\nabla_\mu n^i\right|^2. \tag{B.29}$$

The superpartner to the field n^i is an N-plet of complex two-component spinor fields ξ^i,

$$\xi^i = \begin{cases} \xi_R^i \\ \xi_L^i \end{cases} . \tag{B.30}$$

The auxiliary field A_μ has a complex scalar superpartner σ and a two-component complex spinor superpartner λ; both enter without derivatives. The full $\mathcal{N} = 2$ symmetric Lagrangian is

$$L = \frac{2}{g_0^2}\Big\{ \big|\nabla_\mu n^i\big|^2 + \bar{\xi}_i\, i\gamma^\mu \nabla_\mu\, \xi^i + 2|\sigma|^2\, |n^i|^2 \\ + \Big[i\sqrt{2}\,\sigma \xi^\dagger_{iR}\, \xi^i_L + i\sqrt{2}\, n_i^\dagger \big(\lambda_R \xi^i_L - \lambda_L \xi^i_R\big) + \text{H.c.}\Big]\Big\}. \tag{B.31}$$

The auxiliary fields can be eliminated by virtue of the equations of motion which yield the following relations:

$$\begin{aligned} & n_l^\dagger \xi^l_L = 0, \quad n_l^\dagger \xi^l_R = 0\,; \\ & A_\mu = -\frac{i}{2}\, n_l^\dagger \overset{\leftrightarrow}{\partial_\mu} n^l - \frac{1}{2}\,\bar{\xi}_l \gamma_\mu \xi^l, \\ & \sigma = \frac{i}{\sqrt{2}}\, \xi^\dagger_{lL} \xi^l_R. \end{aligned} \tag{B.32}$$

Substituting (B.32) in (B.31) we arrive at the final expression for the Lagrangian of $\mathcal{N} = 2$ sigma model with the target space (B.26),

$$L = \frac{2}{g_0^2}\,\Big\{ \big|\partial_\mu n^i\big|^2 + \frac{1}{4}\big(n_i^\dagger \overset{\leftrightarrow}{\partial_\mu} n^i\big)^2 \\ + \bar{\xi}_i\, i\gamma^\mu \left(\partial_\mu - \frac{1}{2}\, n_l^\dagger \overset{\leftrightarrow}{\partial_\mu} n^l\right) \xi^i \\ - \big(\xi^\dagger_{iR}\,\xi^i_R \cdot \xi^\dagger_{lL}\,\xi^l_L + \xi^\dagger_{iR}\,\xi^i_L \cdot \xi^\dagger_{lL}\,\xi^l_R\big)\Big\}, \tag{B.33}$$

$$n_i^\dagger\, n^i = 1, \quad n_i^\dagger\, \xi^i = 0. \tag{B.34}$$

For $N = 2$ there exists a simple local transformation converting the Lagrangian of the O(3) model discussed in Appendix B.1 into (B.33),

$$\begin{aligned} & S^a = n_i^\dagger \big(\tau^a\big)^i_k\, n^k, \\ & \chi^a = n_i^\dagger \big(\tau^a\big)^i_k\, \xi^k + \xi_i^\dagger \big(\tau^a\big)^i_k\, n^k, \end{aligned} \tag{B.35}$$

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where τ^a are the Pauli matrices. If we use the Fierz identity for the Pauli matrices,

$$(\tau^a)^i_k\,(\tau^a)^{\tilde{i}}_{\tilde{k}} = -\frac{1}{2}(\tau^a)^{\tilde{i}}_k(\tau^a)^i_{\tilde{k}} + \frac{3}{2}\delta^{\tilde{i}}_k\,\delta^i_{\tilde{k}}, \tag{B.36}$$

and substitute Eq. (B.35) in the Lagrangian (B.4) taking account of the constraints (B.3) we arrive at (B.33). The constraints (B.34) are satisfied automatically.

B.5 Heterotic CP(1)

Here we will outline derivation of the heterotic CP(1) model elaborated in Ref. [191]. We will start from the general geometric formulation presented in Appendix B.3, specify it to the CP(1) case using Eq. (B.24) and then introduce a deformation that breaks $\mathcal{N} = (2,2)$ down to $\mathcal{N} = (0,2)$. As is well known, if we limit ourselves to the set of fields present in the $\mathcal{N} = (2,2)$ sigma model, such a deformation does not exist. However, it does exist if we agree to introduce an extra *right-handed* fermion ζ_R [190].

One can obtain the deformed Lagrangian as follows. Introduce the operators

$$\mathcal{B} = \big\{\zeta_R(x^\mu + i\bar{\theta}\gamma^\mu\theta) + \sqrt{2}\theta_R\mathcal{F}\big\}\theta^\dagger_L,$$
$$\mathcal{B}^\dagger = \theta_L\big\{\zeta^\dagger_R(x^\mu - i\bar{\theta}\gamma^\mu\theta) + \sqrt{2}\theta^\dagger_R\mathcal{F}^\dagger\big\}. \tag{B.37}$$

Since θ_L and $\theta^\dagger_L$ enter in Eq. (B.37) explicitly, $\mathcal{B}$ and $\mathcal{B}^\dagger$ are *not* superfields with regards to the supertransformations with parameters ϵ_L, $\epsilon^\dagger_L$. These supertransformations are absent in the heterotic model. Only those survive which are associated with ϵ_R, $\epsilon^\dagger_R$. Note that $\mathcal{B}$ and $\mathcal{B}^\dagger$ are superfields with regards to the latter.

It is convenient to introduce a shorthand for the chiral coordinate

$$\tilde{x}^\mu = x^\mu + i\bar{\theta}\gamma^\mu\theta. \tag{B.38}$$

Then the transformation laws with the parameters ϵ_R, $\epsilon^\dagger_R$ are as follows:

$$\delta\theta_R = \epsilon_R, \quad \delta\theta^\dagger_R = \epsilon^\dagger_R, \quad \delta\tilde{x}^0 = 2i\epsilon^\dagger_R\theta_R, \quad \delta\tilde{x}^1 = 2i\epsilon^\dagger_R\theta_R. \tag{B.39}$$

With respect to such supertransformations, $\mathcal{B}$ and $\mathcal{B}^\dagger$ are superfields. Indeed,

$$\delta\zeta_R = \sqrt{2}\,\mathcal{F}\,\epsilon_R, \quad \delta\mathcal{F} = \sqrt{2}\,i\,(\partial_L\zeta_R)\epsilon^\dagger_R, \tag{B.40}$$

plus Hermitean conjugate transformations. To convert $L_{\mathrm{CP}(1)}$ into $L_{\mathrm{heterotic}}$ we add to $L_{\mathrm{CP}(1)}$ the following terms:

$$\Delta L = \int d^4\theta\big\{-2\,\mathcal{B}^\dagger\,\mathcal{B} + \big[g_0^2\,\sqrt{2}\,\gamma\,\mathcal{B}\,K + \text{H.c.}\big]\big\}, \tag{B.41}$$

where γ is generally speaking a complex constant. For simplicity we will assume γ to be real. Thus, we obviously deal here with a single deformation parameter.

First, let us check that the extra term (B.41) preserves invariance on the target space. Indeed, the invariance under the U(1) transformation of the superfields Φ, $\Phi^\dagger$,

$$\Phi \to i\delta\,\Phi, \quad \Phi^\dagger \to -i\delta\,\Phi^\dagger, \tag{B.42}$$

is obvious. Two other rotations on the sphere manifest themselves in nonlinear transformations with a complex parameter β,

$$\Phi \to \beta + \beta^*\Phi^2, \quad \Phi^\dagger \to \beta^* + \beta(\Phi^\dagger)^2. \tag{B.43}$$

Under these transformations

$$\delta K = \frac{2}{g_0^2}\,(\beta^*\,\Phi + \beta\,\Phi^\dagger). \tag{B.44}$$

It is not difficult to see that

$$\int d^4\theta\,\mathcal{B}\,\delta K = 0. \tag{B.45}$$

In other words, even before performing the component decomposition we are certain that the term (B.41) is invariant on the target space of the CP(1) model. Needless to say, it is $\mathcal{N} = (0,2)$ invariant by construction.

As usual, the $\mathcal{F}$ term enters without derivatives and can be eliminated by virtue of equations of motion,

$$\mathcal{F} = -2\,\gamma^*\,\chi^{-2}\,\psi_R^\dagger\,\psi_L, \quad \mathcal{F}^\dagger = -2\,\gamma\,\chi^{-2}\,\psi_L^\dagger\,\psi_R. \tag{B.46}$$

In addition, the F terms of the superfields Φ, $\Phi^\dagger$ also change. If before the deformation e.g. $F = (i/2)\,\Gamma\,\psi\,\gamma^0\,\psi$, after the deformation

$$F = \frac{i}{2}\,\Gamma\,\psi\,\gamma^0\,\psi - g_0^2\,\gamma\,\psi_L\,\zeta_R^\dagger, \tag{B.47}$$

plus the Hermitian conjugated expression for $F^\dagger$.

Assembling all these pieces together we get the Lagrangian of the heterotic CP(1) model,

$$L_{\text{heterotic}} = \zeta_R^\dagger \, i\partial_L \, \zeta_R + \left[\gamma \, \zeta_R \, R \left(i \, \partial_L \phi^\dagger\right)\psi_R + \text{H.c.}\right] - g_0^2 |\gamma|^2 \left(\zeta_R^\dagger \, \zeta_R\right)\left(R \, \psi_L^\dagger \psi_L\right)$$
$$+ G \left\{\partial_\mu \phi^\dagger \, \partial^\mu \phi + \frac{i}{2}\left(\psi_L^\dagger \, \overset{\leftrightarrow}{\partial}_R \, \psi_L + \psi_R^\dagger \, \overset{\leftrightarrow}{\partial}_L \, \psi_R\right) \right.$$
$$- \frac{i}{\chi}\left[\psi_L^\dagger \psi_L \left(\phi^\dagger \, \overset{\leftrightarrow}{\partial}_R \, \phi\right) + \psi_R^\dagger \, \psi_R \left(\phi^\dagger \, \overset{\leftrightarrow}{\partial}_L \, \phi\right)\right]$$
$$\left. - \frac{2(1 - g_0^2|\gamma|^2)}{\chi^2} \, \psi_L^\dagger \, \psi_L \, \psi_R^\dagger \, \psi_R \right\}, \tag{B.48}$$

where R stands for the Ricci tensor, and

$$\partial_L = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}, \qquad \partial_R = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}. \tag{B.49}$$

Generalization for arbitrary N (i.e. the $\mathcal{N} = (0, 2)$ deformed CP($N - 1$) model) is as follows:

$$L_{\text{heterotic}} = \zeta_R^\dagger \, i\partial_L \, \zeta_R + \left[\gamma \, g_0^2 \, \zeta_R \, G_{i\bar{j}} \left(i \, \partial_L \phi^{\dagger\,\bar{j}}\right)\psi_R^i + \text{H.c.}\right]$$
$$- g_0^4 \, |\gamma|^2 \left(\zeta_R^\dagger \, \zeta_R\right)\left(G_{i\bar{j}} \, \psi_L^{\dagger\,\bar{j}} \psi_L^i\right)$$
$$+ G_{i\bar{j}}\left[\partial_\mu \phi^{\dagger\,\bar{j}} \, \partial_\mu \phi^i + i\bar{\psi}^{\bar{j}} \gamma^\mu D_\mu \psi^i\right]$$
$$- \frac{g_0^2}{2}\left(G_{i\bar{j}} \psi_R^{\dagger\,\bar{j}} \, \psi_R^i\right)\left(G_{k\bar{m}} \psi_L^{\dagger\,\bar{m}} \, \psi_L^k\right)$$
$$+ \frac{g_0^2}{2}\left(1 - 2g_0^2|\gamma|^2\right)\left(G_{i\bar{j}} \psi_R^{\dagger\,\bar{j}} \, \psi_L^i\right)\left(G_{k\bar{m}} \psi_L^{\dagger\,\bar{m}} \, \psi_R^k\right). \tag{B.50}$$