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THE DERIVATION ALGEBRA OF $M_4^8(C)$

BY

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1. **The Main Theorem.** Let C by a Cayley-Dickson algebra over an algebraically closed field F of characteristic 0. A multiplication table for a basis of this 8-dimensional alternative algebra can be found in [3], page 137, where we take $\alpha=\beta=\gamma=-1$. C has an involution $x\mapsto\bar{x}$, and a matrix $X=(x_{ij})$ with entries in C is called hermitian if $X=(\bar{x}_{ji})$. The space $M_n^8(C)$ of all $n\times n$ hermitian matrices with entries in C becomes a commutative algebra under the product $X\circ Y=(XY+YX)/2$, where juxtaposition denotes the standard matrix product. It is not hard to see that all these algebras are simple, but for $n>3$, they otherwise remain a mystery. They are not Jordan algebras [4] as is $M_3^8(C)$, and hence, because the identity of each is a sum of n pairwise orthogonal idempotents, they are not even power-associative by Theorem 1 of [1]. Since a search for identities these algebras might satisfy seems a difficult problem, it seems more fruitful instead to investigate their derivation algebras, $\mathscr{D}(M_n^8(C))$. The reason for this is that $\mathscr{D}(M_3^8(C))$ is known to be F_4, one of the exceptional simple Lie algebras ([2]; pp. 142–145). Our study has revealed that $\mathscr{D}(M_2^8(C))$ is B_4, one of the classical simple Lie algebras, and since this has dimension 36 and F_4 has dimension 52, it is surprising to learn that $\mathscr{D}(M_4^8(C))$ has dimension 20. In fact, we have:

THEOREM 1.1. *$\mathscr{D}(M_4^8(C))\simeq A_1\oplus A_1\oplus G_2$, where A_1 and G_2 are the classical and exceptional simple Lie algebras, respectively.*

2. **The general derivation.** A multiplication table for the basis $E_1,\ldots,E_{52}$ of $M_4^8(C)$ was computed. In order for a linear transformation T to be a derivation, it is necessary and sufficient that

$$T(E_i\circ E_j)=T(E_i)\circ E_j+E_i\circ T(E_j),\qquad i,j=1,\ldots,52 \tag{1}$$

Thus, letting $X=(x_{ij})$, $i,j=1,\ldots,52$, we were able with the aid of a computer to determine relations among the x_{ij} which must hold for X to represent a derivation.

Let

$$\begin{aligned} S_1&=\{(1,5),(1,13),(1,21),(2,29),(2,37),(3,45)\}\\ S_2&=\{(6,7),\ldots,(6,12),(7,8),\ldots,(7,12),(9,10),(9,11),(9,12)\}\\ S_3&=\{(2,5),(3,13),(4,21),(3,29),(4,37),(4,45)\}\end{aligned}\tag{2}$$

[1] This author was chiefly responsible for the computer programmes used in the preparation of this paper.

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MULTIPLICATION TABLE FOR $\mathscr{D}(M_4^8(C))$

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}
D_1	0	$-\frac{1}{2}D_4$	$-\frac{1}{2}D_5$	$\frac{1}{2}D_2$	$\frac{1}{2}D_3$	0	0	0	0	0	0
D_2	$\frac{1}{2}D_4$	0	$-\frac{1}{2}D_6$	$-\frac{1}{2}D_1$	0	$\frac{1}{2}D_3$	0	0	0	0	0
D_3	$\frac{1}{2}D_5$	$\frac{1}{2}D_6$	0	0	$-\frac{1}{2}D_1$	$-\frac{1}{2}D_2$	0	0	0	0	0
D_4	$-\frac{1}{2}D_2$	$\frac{1}{2}D_1$	0	0	$-\frac{1}{2}D_6$	$\frac{1}{2}D_5$	0	0	0	0	0
D_5	$-\frac{1}{2}D_3$	0	$\frac{1}{2}D_1$	$\frac{1}{2}D_6$	0	$-\frac{1}{2}D_4$	0	0	0	0	0
D_6	0	$-\frac{1}{2}D_3$	$\frac{1}{2}D_2$	$-\frac{1}{2}D_5$	$\frac{1}{2}D_4$	0	0	0	0	0	0
D_7	0	0	0	0	0	0	0	$-D_{13}$	$-D_{14}$	$-D_{11}-D_{15}$	$D_{10}-D_{16}$
D_8	0	0	0	0	0	0	D_{13}	0	$-D_{11}$	$-2D_{12}$	D_9
D_9	0	0	0	0	0	0	D_{14}	D_{11}	0	$-D_{18}$	$-2D_8-2D_{19}$
D_{10}	0	0	0	0	0	0	$D_{11}+D_{15}$	$2D_{12}$	D_{18}	0	$-D_{20}$
D_{11}	0	0	0	0	0	0	$-D_{10}+D_{16}$	$-D_9$	$2D_8+2D_{19}$	D_{20}	0
D_{12}	0	0	0	0	0	0	D_{17}	$-2D_{10}$	D_{20}	$2D_8$	D_{18}
D_{13}	0	0	0	0	0	0	$-D_8$	D_7	$-D_{16}$	$-D_{17}$	$-D_{12}+D_{14}$
D_{14}	0	0	0	0	0	0	$-D_9$	$-D_{10}$	D_7	D_8	$-D_{13}+D_{18}$
D_{15}	0	0	0	0	0	0	$-D_{10}+D_{16}$	D_9+D_{17}	$-D_8-D_{19}$	D_7-D_{20}	0
D_{16}	0	0	0	0	0	0	$-D_{11}-D_{15}$	D_{12}	D_{13}	0	D_7-D_{20}
D_{17}	0	0	0	0	0	0	$-D_{12}$	$-D_{11}-D_{15}$	0	D_{13}	D_8+D_{19}
D_{18}	0	0	0	0	0	0	D_{19}	D_{20}	$-D_{10}$	D_9	$-D_{12}$
D_{19}	0	0	0	0	0	0	$-D_{18}$	0	$-D_{11}$	D_{12}	D_9
D_{20}	0	0	0	0	0	0	0	$-D_{18}$	$-D_{12}$	$-D_{11}$	D_{10}

D_{12}	D_{13}	D_{14}	D_{15}	D_{16}	D_{17}	D_{18}	D_{19}	D_{20}	
0	0	0	0	0	0	0	0	0	D_1
0	0	0	0	0	0	0	0	0	D_2
0	0	0	0	0	0	0	0	0	D_3
0	0	0	0	0	0	0	0	0	D_4
0	0	0	0	0	0	0	0	0	D_5
0	0	0	0	0	0	0	0	0	D_6
$-D_{17}$	D_8	D_9	$D_{10}-D_{16}$	$D_{11}+D_{15}$	D_{12}	$-D_{19}$	D_{18}	0	D_7
$2D_{10}$	$-D_7$	D_{10}	$-D_9-D_{17}$	$-D_{12}$	$D_{11}+D_{15}$	$-D_{20}$	0	D_{18}	D_8
$-D_{20}$	D_{16}	$-D_7$	D_8+D_{19}	$-D_{13}$	0	D_{10}	D_{11}	D_{12}	D_9
$-2D_8$	D_{17}	$-D_8$	$-D_7+D_{20}$	0	$-D_{13}$	$-D_9$	$-D_{12}$	D_{11}	D_{10}
$-D_{18}$	$D_{12}-D_{14}$	$D_{13}-D_{18}$	0	$-D_7+D_{20}$	$-D_8-D_{19}$	D_{12}	$-D_9$	$-D_{10}$	D_{11}
0	$-D_{11}-D_{15}$	0	$D_{13}-D_{18}$	D_8	$-D_7$	$-D_{11}$	D_{10}	$-D_9$	D_{12}
$D_{11}+D_{15}$	0	D_{15}	$-D_{14}$	$-2D_{17}$	$2D_{16}$	0	$-D_{20}$	D_{19}	D_{13}
0	$-D_{15}$	0	$2D_{13}-2D_{18}$	$-D_{19}$	$-D_{20}$	D_{15}	D_{16}	D_{17}	D_{14}
$-D_{13}+D_{18}$	D_{14}	$-2D_{13}+2D_{18}$	0	$-D_{20}$	D_{19}	$-D_{14}$	$-D_{17}$	D_{16}	D_{15}
$-D_8$	$2D_{17}$	D_{19}	D_{20}	0	$-2D_{13}$	D_{17}	$-D_{14}$	$-D_{15}$	D_{16}
D_7	$-2D_{16}$	D_{20}	$-D_{19}$	$2D_{13}$	0	$-D_{16}$	D_{15}	$-D_{14}$	D_{17}
D_{11}	0	$-D_{15}$	D_{14}	$-D_{17}$	D_{16}	0	$-2D_{20}$	$2D_{19}$	D_{18}
$-D_{10}$	D_{20}	$-D_{16}$	D_{17}	D_{14}	$-D_{15}$	$2D_{20}$	0	$-2D_{18}$	D_{19}
D_9	$-D_{19}$	$-D_{17}$	$-D_{16}$	D_{15}	D_{14}	$-2D_{19}$	$2D_{18}$	0	D_{20}

Notice that an integer j appearing in the second component of an element of S_1 appears just once, and occurs again exactly once in the second component of an element of S_3. Thus there is a natural pairing of elements in S_1 with those in S_3. With this in mind, the matrix X representing the general derivation of $M_4^8(C)$ has the following form:

$$\begin{aligned} x_{ij} &= -x_{kj} \quad \text{for} \quad (i,j)\in S_3, (k,j)\in S_1 \\ x_{ji} &= -x_{ij} \quad \text{for} \quad (i,j)\in S_1 \cup S_3 \end{aligned} \tag{3}$$

Except for the cases of (3), the matrix is skew-symmetric. The 7×7 (skew-symmetric) submatrix with $(1, 1)$-entry in the $(6, 6)$-position of X, has entries satisfying:

$$\begin{aligned} x_{8,9} &= x_{6,11}-x_{7,10} \\ x_{8,10} &= x_{6,12}+x_{7,9} \\ x_{8,11} &= x_{7,12}-x_{6,9} \\ x_{8,12} &= -x_{6,10}-x_{7,11} \\ x_{10,11} &= x_{6,7}+x_{9,12} \\ x_{10,12} &= x_{6,8}-x_{9,11} \\ x_{11,12} &= x_{7,8}+x_{9,10} \end{aligned} \tag{4}$$

and is repeated six times along the main diagonal with a single 0 separating each repetition. The remaining entries in X are 0, except for some entries on four diagonals above and parallel to the main diagonal (and their four negative transposes). Let $D^{\ell}_{i,j}$ be the set $\{x_{i+k,j+k}:k=1,\ldots,\ell\}$. Then these non-zero entries are of the following form:

$$D^8_{4,12} = D^8_{36,44} = \{\tfrac{1}{2}x_{2,29}\}, \qquad D^8_{12,20} = D^8_{28,36} = \{\tfrac{1}{2}x_{3,45}\} \tag{5}$$

$$\begin{aligned} &D^8_{4,20} = \{\tfrac{1}{2}x_{2,37}\}, \qquad D^{16}_{12,28} = \{\tfrac{1}{2}x_{1,5}\} \\ &\qquad\qquad x_{29,45} = \tfrac{1}{2}x_{2,37}, \qquad D^7_{29,45} = \{-\tfrac{1}{2}x_{2,37}\} \end{aligned} \tag{6}$$

$$x_{5,29} = \tfrac{1}{2}x_{1,13}, \qquad D^7_{5,29} = \{-\tfrac{1}{2}x_{1,13}\}, \qquad D^8_{20,44} = \{\tfrac{1}{2}x_{1,13}\} \tag{7}$$

$$x_{5,37} = x_{13,45} = \tfrac{1}{2}x_{1,21}, \qquad D^7_{5,37} = D^7_{13,45} = \{-\tfrac{1}{2}x_{1,21}\} \tag{8}$$

3. **Proof of theorem.** We define a basis $D_1,\ldots,D_{20}$ for $\mathscr{D}(M_4^8(C))$ in the following way: $D_1,\ldots,D_6$ are those derivations obtained from the general derivation X by setting in turn, for each $(i,j)\in S_1$, $x_{i,j}=1$ and $x_{k,\ell}=0$ if $(k,\ell)\neq(i,j)$. $D_7,\ldots D_{20}$ are obtained from S_2 in the same way. A computer was again called upon to evaluate $D_iD_j-D_jD_i$ for $i,j=1,\ldots,20$, thus producing the multiplication table for the derivation algebra. We have attached this table to our article.

It is immediately apparent that the algebra is a direct sum of the algebra L_1 with basis $D_1,\ldots,D_6$ and the algebra L_2 with basis $D_7,\ldots,D_{20}$. We investigate first L_1.

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Letting $\langle\ ,\ \rangle$ denote the Killing form, one checks that $\langle D_i, D_i\rangle=-1, i=1,\ldots,6$ and $\langle D_i, D_j\rangle=0$ for $i\neq j$, $i,j\in\{1,\ldots,6\}$. It follows that L_1 is semi-simple. Now let i denote $(-1)^{1/2}$ and

$$\begin{aligned} &H_1 = -2i(D_1+D_6), && H_2 = 2i(D_6-D_1) \\ &X_1 = iD_2+D_3+D_1-iD_5, && X_2 = iD_2-D_3+D_4+iD_5 \\ &Y_1 = -iD_2+D_3+D_4+iD_5, && Y_2 = -iD_2-D_3+D_4-iD_5 \end{aligned} \tag{9}$$

Defining J_i to be $FX_i\oplus FY_i\oplus FH_i$, $i=1, 2$, one verifies first that $L_1\simeq J_1\oplus J_2$, and then notes that $J_1\simeq J_2\simeq A_1$.

For L_2, we have $\langle D_i, D_i\rangle=-16$, $i=7,\ldots,20$, and

$$\begin{aligned} &\langle D_{12}, D_{14}\rangle = \langle D_{10}, D_{16}\rangle = \langle D_{13}, D_{18}\rangle = \langle D_7, D_{20}\rangle = -8; \\ &\langle D_{11}, D_{15}\rangle = \langle D_9, D_{17}\rangle = \langle D_8, D_{19}\rangle = 8 \end{aligned} \tag{10}$$

From this, it is again easy to check that L_2 is semi-simple, and that $H=FD_7+FD_{20}$ is a Cartan subalgebra. Letting $\alpha_1(D_7)=i$, $\alpha_1(D_{20})=-i$, $\alpha_2(D_7)=0$, $\alpha_2(D_{20})=i$, and extending α_1 and α_2 to H by linearity, it is straightforward to show that the positive roots of L_2 are α_1, α_2, $\alpha_1+\alpha_2$, $\alpha_1+2\alpha_2$, $\alpha_1+3\alpha_2$, $2\alpha_1+3\alpha_2$, and hence $L_2\simeq G_2$.

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References

1. A. A. Albert, *A Theory of Commutative Power-Associative Algebras*, Trans. Amer. Math. Soc. **69** (1950), 503–527.

2. N. Jacobson, "Lie Algebras", Interscience, New York, 1962.

3. E. Kleinfeld, A Characterization of the Cayley Numbers, "Studies in Modern Algebra" (A. A. Albert, Editor), Studies in Mathematics, Vol. 2, Math. Assoc. of America, Prentice-Hall, New Jersey, 1963, 126–143.

4. R. D. Schafer, *The Exceptional Simple Jordan Algebras*, Amer. J. Math. **70** (1948), 82–94.

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