THE PROPAGATION OF A CONSTANT STRENGTH SHOCK THROUGH A SIMPLE WAVE

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1. Introduction

When a shock wave moving through an isentropic ideal gas catches up with, and passes into a simple expansion wave, the shock decays. Because of this the gas will not be isentropic in the region behind the shock. The problem of determining the motion of the gas in this region is as yet unsolved. In this paper we introduce a simple compression wave behind the shock which catches up with it at the instant of its entry into the leading expansion wave. This second wave is chosen so as to counteract the decaying effect of the first, and keep the shock strength constant throughout the motion. We assume the first wave to be point-centred, and caused by the withdrawal of a piston at a finite velocity from a gas at rest in a shock tube. After a finite time the piston is halted causing the shock. The problem is then to determine the subsequent motion of the piston to produce a compression wave with the desired property.

2. The gas equations

For simplicity we take the gas to have adiabatic exponent 5/3. The three possible types of one-dimensional isentropic gas motion are then as follows (**1**).

(i) Steady motion. This is characterised by u = constant and c = constant, where u and c are the gas velocity and the sound speed respectively.

(ii) Simple waves. These are regions in which either r = constant or s = constant where r and s are the " Riemann invariants " defined by

$$r = \tfrac{1}{2}(3c+u), \quad s = \tfrac{1}{2}(3c-u).$$

(iii) General Motion. Here r and s are constant along the respective characteristics

$$\frac{dx}{dt} = u+c \quad (r-\text{characteristic})$$

$$\frac{dx}{dt} = u-c \quad (s-\text{characteristic})$$

and the flow is given by the equations

$$x-\tfrac{2}{3}(2r-s)t = \frac{\partial w}{\partial r}$$

$$x+\tfrac{2}{3}(2s-r)t = -\frac{\partial w}{\partial s},$$

where

$$w = \frac{f(r)+g(s)}{r+s}$$

and f, g are functions to be determined by the boundary conditions on the region.

In such a region the particle paths satisfy the condition

$$\Phi(r, s) \equiv (r+s)^5\left[\left(\frac{\partial}{\partial r}\right)^2 \frac{F(r)}{(r+s)^3} - \left(\frac{\partial}{\partial s}\right)^2 \frac{G(s)}{(r+s)^3}\right] = \text{constant}, \qquad (1)$$

where

$$F(r) = \int f(r)dr, \quad G(s) = \int g(s)ds.$$

3. The shock conditions

We use subscripts b and f to denote the values of the variables on the back and front of the shock respectively. Then the shock strength $z>1$ is defined by

$$p_b = zp_f$$

where p is the pressure. The relationships between other variables are then

$$u_b = u_f+Ac_f, \qquad (2)$$

$$c_b = Bc_f$$

and the speed of the shock itself is given by

$$U = u_f+Dc_f, \qquad (3)$$

where

$$A = \frac{3(z-1)}{[5(4z+1)]^{\frac{1}{2}}} \quad B = \left[\frac{z(4+z)}{1+4z}\right]^{\frac{1}{2}} \quad D = \left(\frac{4z+1}{5}\right)^{\frac{1}{2}}$$

are functions of z. When a shock moves into an isentropic gas, the region behind the shock will also be isentropic only if z remains constant throughout the motion. When this happens, A, B, D are constant, and (3) becomes a differential equation for the path of the shock. Equations (2) now give the boundary condition along this path which enable us to determine the region behind the shock.

4. The solution of the problem

The various regions of the flow are shown in fig. 1. Henceforward, numerical suffices will refer to the corresponding regions in the figure, capital letters to the

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respective points. At $t = 0$, the gas is at rest with $c = c_0$ in the region $x \geqq 0$ of the xt-plane. The piston is then withdrawn at a speed less than the critical speed, $3c_0$, at which a vacuum would form between piston and gas-front, to the point $(-x_A, t_A)$ where it is halted. We introduce a dimensionless constant

$$C = \frac{c_0 t_A}{x_A},$$

which, by the above condition on the speed, must be greater than $\frac{1}{3}$, and the dimensionless variables

$$\tilde{x} = \frac{x}{x_A}, \quad \tilde{t} = \frac{tc_0}{x_A} = C\frac{t}{t_A}, \quad \tilde{w} = \frac{w}{c_0 x_A},$$

$$\tilde{u} = \frac{u}{c_0}, \quad \tilde{c} = \frac{c}{c_0}, \quad \tilde{r} = \frac{r}{c_0}, \quad \tilde{s} = \frac{s}{c_0}.$$

Dropping the tilda, we find that all the equations in Sections **2** and **3** remain true for these new variables. We now set out to solve the regions of the flow in turn.

Region (0) is at rest with

$$u_0 = 0, \quad c_0 = 1, \quad r_0 = s_0 = \tfrac{3}{2},$$

and the bounding characteristic OD is

$$x = t. \qquad \text{...(4)}$$

Region (1) is a simple wave with

$$s_1 = s_0 = \tfrac{3}{2},$$

$$r_1 = \tfrac{3}{4}\left(\frac{x}{t}+1\right),$$

$$u_1 = \tfrac{3}{4}\left(\frac{x}{t}-1\right),$$

$$c_1 = \tfrac{1}{4}\left(\frac{x}{t}+3\right).$$

Region (2) is a region of steady motion in which

$$u_2 = -C^{-1},$$

$$c_2 = 1-(3C)^{-1},$$

and the r-characteristic OB has equation

$$x = (u_2+c_2)t$$

$$= \left(1-\frac{4}{3C}\right)t.$$

The piston is stopped at the point $A\,(-1, C)$. The path OA is therefore

$$Cx+t=0.$$

From A, the shock AB moves into the gas bringing it to rest in region (3). Thus from (2) we have

$$u_2+A\,c_2=0,$$

which is an equation for z having one root greater than unity. Thus we have found the shock strength and so the values of B and D. Hence we know

$$c_3=Bc_2$$

and the equation for the shock path AB,

$$x=(u_2+Dc_2)t-DCc_2.$$

This in turn gives us the co-ordinates of the point B as

$$x_B=D(D-1)^{-1}(C-\tfrac{4}{3}),$$

$$t_B=D(D-1)^{-1}C.$$

The point C at which the piston is to be restarted is determined by the fact that CB is to be an r-characteristic since the compression wave caused by restarting the piston must catch up with the shock at B. Thus the line CB is

$$x-x_B=(u_3+c_3)(t-t_B)$$

from which C is found to be the point

$$x_c=-1,$$

$$t_c=(BD+A-D)(BD-B)^{-1}C.$$

The shock now moves into the simple wave. Since we assume the shock strength to remain constant, the differential equation for its path becomes

$$4\frac{dx}{dt}=(3+D)\frac{x}{t}+3(D-1),$$

which has the solution

$$x+3t=kt^{\beta}$$

as noted in Mackie and Weir (2). Here $\beta=\tfrac{1}{4}(3+D)$ and the constant of integration k is known since the point B must lie on the shock. The intersection of this curve with the characteristic OD (4) is

$$x_D=t_D,$$

$$t_D=\left(\frac{4}{k}\right)^{\alpha},$$

where $\alpha = (\beta - 1)^{-1} = 4(D-1)^{-1}$. Here we see that the shock will pass through the simple wave in a finite time.

Beyond D, the shock is moving into the gas at rest. It is therefore a straight line given by

$$(x - x_D) = D(t - t_D).$$

We now consider the flow behind the shock. We have already seen that region (3) is a rest region while region (4) is a simple compressive wave with s_4 constant. Similarly region (7) is in a state of uniform motion with

$$u_7 = A, \quad c_7 = B,$$

and region (6) must therefore be a simple wave in which we have r_6 constant. The remaining region (5) is of type (iii), with boundary conditions along the curve BD.

These conditions are determined in terms of the Riemann invariants from the shock relations and the conditions in region (1). If we introduce new variables

$$R = r + \tfrac{3}{2}, \quad S = s - \tfrac{3}{2},$$

we find that the problem is to find a function w of the form

$$w = \frac{f(R) + g(S)}{R + S}$$

satisfying $\dfrac{\partial w}{\partial R} = LR^{\alpha+1}$, $\dfrac{\partial w}{\partial S} = MR^{\alpha+1}$ along the line $R = GS$,

where

$$L = \frac{1 - A - B}{A + 3B + 3}\, 2F^{\alpha}, \quad M = \frac{A - B - 1}{A + 3B + 3}\, 2F^{\alpha}, \quad G = \frac{1 + E}{1 - E}$$

and

$$F = \frac{8}{3kB(1+E)}, \quad E = \frac{A+3}{3B}.$$

From the homogeneity of these conditions we see that the required solution will be

$$w_5 = \frac{aR_5^{\alpha+3} + bS_5^{\alpha+3}}{R_5 + S_5}$$

for values of a and b which are found immediately on substitution. Then in this region we have

$$x + 3t = (R_5 + S_5)^{-3}\{aR_5^{\alpha+2}[2(\alpha+3)S_5^2 + \alpha R_5 S_5 - \alpha R_5^2] - bS_5^{\alpha+2}[2(\alpha+3)R_5^2 + \alpha R_5 S_5 - \alpha S_5^2]\},$$

$$t = -\tfrac{3}{2}(R_5 + S_5)^{-3}\{aR_5^{\alpha+2}[(\alpha+1)R_5 + (\alpha+3)S_5] + bS_5^{\alpha+2}[(\alpha+3)R_5 + (\alpha+1)S_5]\}.$$

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The characteristics BF and DF will be given parametrically by these equations with $S_5 = \frac{3}{2}(B-1)+\frac{1}{2}Bu_2$ and $R_5 = \frac{3}{2}(B+1)+\frac{1}{2}A$ respectively.

From (1), the trajectories in this region will satisfy

$$\Phi \equiv aR_5^{\alpha+2}\left[\frac{\alpha(\alpha+1)}{\alpha+4}R_5^2+2\alpha R_5S_5+(\alpha+3)S_5^2\right]$$
$$-bS_5^{\alpha+2}\left[\frac{\alpha(\alpha+1)}{\alpha+4}S_5^2+2\alpha R_5S_5+(\alpha+3)R_5^2\right]$$
$$= \text{constant}, \quad \ldots(5)$$

and one of these trajectories will be the required piston path.

Region (4) is a simple wave with $S_4 = \frac{3}{2}(B-1)+\frac{1}{2}Bu_2$. The r-characteristics are the straight lines

$$x = (u_4+c_4)t+K(r_4)$$
$$= \tfrac{1}{3}(4R_4-2S_4-9)t+K(R_4), \quad \ldots\ldots(6)$$

where $K(R_4)$ is found to be

$$K(R_4) = (R_4+S_4)^{-2}\{aR_4^{\alpha+2}[(\alpha+2)R_4+(\alpha+3)S_4]-bS_4^{\alpha+3}\}$$

since we know x and t in terms of R_4 along BF.

We may now substitute (6) in the differential equation for the particle paths

$$\frac{dx}{dt} = u_4 = R_4-S_4-3$$

and integrate to find

$$(R_4+S_4)^4t = -3aR_4^{\alpha+2}\left[\frac{(\alpha+1)(\alpha+2)}{\alpha+4}R_4^2+2(\alpha+1)R_4S_4+(\alpha+3)S_4^2\right]$$
$$-6bR_4S_4^{\alpha+3}+\text{constant} \quad \ldots(7)$$

and this, together with (6), gives the parametric equation for the particle paths. The particular value of the constant in this equation needed to give the piston path is found since this path passes through the point C at which $R_4 = R_3$ and $t = t_c$. Once this is known the coordinates of the point F can be found, and hence also the value of the constant in (5) corresponding to the piston path through region (5). This in turn enables us to find the coordinates of G.

Region (6) is a simple wave which may be solved in a similar fashion to region (4). Corresponding to (6) and (7) we have

$$x = \tfrac{1}{3}(2R_6-4S_6-9)t+(R_6+S_6)^{-2}\{aR_6^{\alpha+3}-bS_6^{\alpha+2}[(\alpha+2)S_6+(\alpha+3)R_6]\},$$
$$(R_6+S_6)^4t = -6aR_6^{\alpha+3}S_6$$
$$-3bS_6^{\alpha+2}\left[\frac{(\alpha+1)(\alpha+2)}{\alpha+4}S_6^2+2(\alpha+1)R_6S_6+(\alpha+3)R_6^2\right]+\text{constant},$$

where $R_6 = \frac{3}{2}(1+B)+\frac{1}{2}A$, and S_6 is the parameter. The value of the constant corresponding to the piston is found from conditions at G.

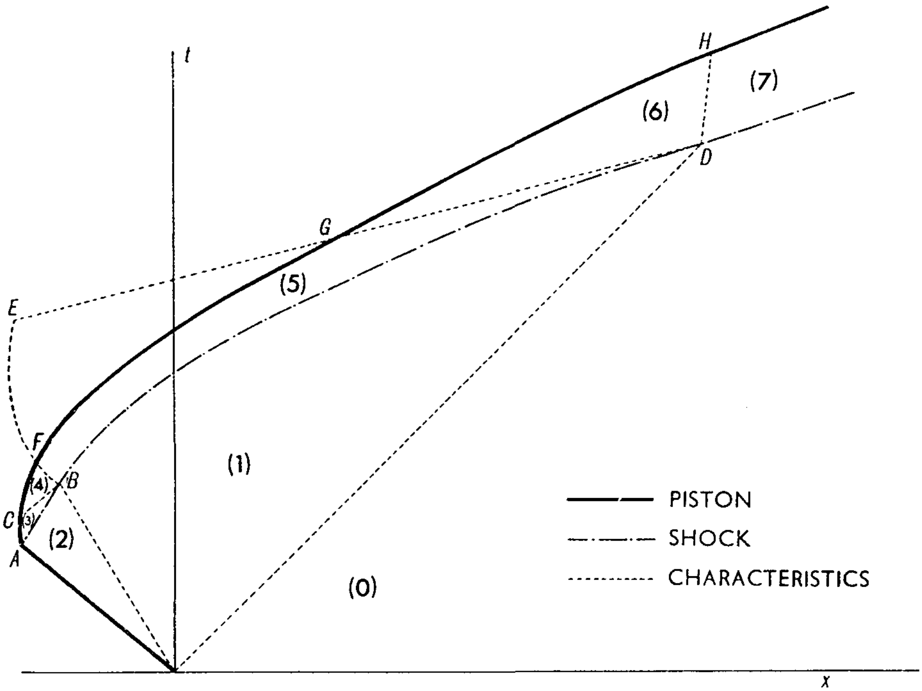

FIG. 1

z	2		5		11		20		50	
	x	t	x	t	x	t	x	t	x	t
A	−1·00	2·57	−1·00	1·19	−1·00	0·83	−1·00	0·69	−1·00	0·55
B	4·85	10·09	−0·29	2·32	−0·75	1·25	−0·86	0·91	−0·93	0·65
C	−1·00	4·26	−1·00	1·64	−1·00	1·03	−1·00	0·80	−1·00	0·60
D	51·34	51·34	8·14	8·14	3·47	3·47	2·20	2·20	1·31	1·31
E	−1·73	18·22	−1·50	4·49	−1·07	2·32	−0·89	1·62	−0·68	1·08
F	1·01	14·53	−0·71	2·80	−0·90	1·40	−0·94	0·99	−0·97	0·69
G	3·83	21·69	1·81	5·75	1·05	2·86	0·74	1·92	0·47	1·21
H	30·36	80·99	7·45	10·53	3·52	4·08	2·31	2·47	1·41	1·42

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Region (7) is in uniform motion with $u_7 = A$, $c_7 = B$.

The S-characteristic DH has equation

$$x - x_D = (A - B)(t - t_D)$$

and the point H is the intersection of this line with the piston path in region (6).

Finally the piston path beyond H is also a straight line given by

$$x - x_H = A(t - t_H).$$

In illustration, we list the coordinates of the various points in fig. 1 for different values of the shock strength z. Fig. 1 is approximately to scale for $z = 11$.

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