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RESTRICTION OF CHARACTERS TO SYLOW NORMALIZERS*

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Abstract. Suppose that G is a finite p-solvable group and let $\chi \in \mathrm{Irr}(G)$ be of p'-degree. In this note, we investigate when χ remains irreducible when restricted to $\mathbf{N}_G(P)$.

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1. Introduction. Suppose that G is a finite group and let P be a Sylow p-subgroup of G. In this note we are concerned about characters of G that remain irreducible when restricted to $\mathbf{N}_G(P)$. As an example of this situation, recall that if q is a power of p and $G = PSL(2, q)$, then every irreducible character of G of degree $q - 1$ restricts irreducibly to $\mathbf{N}_G(P)$. The situation is quite different, however, in the p-solvable case.

THEOREM A. *Suppose that G is p-solvable. Let $P \in \mathrm{Syl}_p(G)$ and let $\chi \in \mathrm{Irr}(G)$ be of p'-degree. If $\chi(1)$ is odd, then $\chi_{\mathbf{N}_G(P)}$ is irreducible if and only if $G = \ker(\chi)\mathbf{N}_G(P)$.*

Of course, if $G = \ker(\chi)\mathbf{N}_G(P)$, then it is trivial to see that χ restricts irreducibly to $\mathbf{N}_G(P)$, and our main work here is to show the converse.

The condition that $\chi(1)$ is odd is really necessary since $GL(2, 3)$ has a faithful character of degree 2 that restricts irreducibly to its 3-Sylow normalizers. Finally, if we consider characters χ of degree divisible by p, then we shall construct some easy examples where the conclusion of Theorem A fails.

2. Proofs. We begin with a general lemma on characters and group actions.

LEMMA 2.1. *Suppose that A acts on G and let $\chi \in \mathrm{Irr}(G)$ be A-invariant. If $\chi_{\mathbf{C}_G(A)}$ is irreducible, then $[G, A] \subseteq \ker(\chi)$.*

Proof. Let $\Gamma = GA$ be the semidirect product and let $C = \mathbf{C}_G(A)$. Note that $CA = C \times A$. Let $\eta = \chi_C \times 1_A \in \mathrm{Irr}(CA)$. By applying Mackey's theorem and Frobenius reciprocity, we obtain

$$[\chi, (\eta^\Gamma)_G] = [\chi, (\eta_C)^G] = [\chi, (\chi_C)^G] = [\chi_C, \chi_C] = 1\,.$$

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Hence, there exists (a unique) $\psi \in \mathrm{Irr}(\Gamma)$ over η and over χ that necessarily satisfies $\psi_G = \chi$ (by multiplicities). Now, we have that $\psi_C = \chi_C$ is irreducible, and therefore $\psi_{CA} = \eta$. Then $A \subseteq \ker(\psi)$. Hence $A^g \subseteq \ker(\psi)$, for all $g \in G$, and we deduce that $[G, A] \subseteq \ker(\psi)$. Then $[G, A] \subseteq \ker(\psi_G) = \ker(\chi)$, as desired. □

The following lemma is nearly trivial.

Lemma 2.2. *Suppose that $H \subseteq G$ and $\chi \in \mathrm{Irr}(G)$. If $\ker(\chi)H = G$, then χ_H is irreducible.*

Proof. If $\mathcal{X} : G \to \mathrm{GL}(n, \mathbf{C})$ affords χ, then we have that $\mathcal{X}(G) = \mathcal{X}(H)$, and the lemma easily follows. □

In the proof of Theorem A, we shall need a nontrivial fact on odd fully ramified sections.

Theorem 2.3. *Suppose that K and L are normal subgroups of G with K/L abelian of odd order. Let $\theta \in \mathrm{Irr}(K)$ be G-invariant such that $\theta_L = e\varphi$, where $\varphi \in \mathrm{Irr}(L)$ and $e^2 = |K : L|$. Then there exists a complement U of K/L in G such that, whenever $\chi \in \mathrm{Irr}(G)$ lies over θ, then χ_U is not irreducible.*

Proof. See Theorem 1.6 of [**2**]. □

We put all the work for proving Theorem A in the following result. In its proof, we shall use the Glauberman correspondence. Recall that if a p-group P acts on a p'-group L, then there is a canonical bijection

$$^* : \mathrm{Irr}_P(L) \to \mathrm{Irr}(\mathbf{C}_L(P)),$$

where $\mathrm{Irr}_P(L)$ is the set of irreducible P-invariant characters of L. In fact, if $\chi \in \mathrm{Irr}_P(L)$, then χ^* is the unique irreducible constituent of $\chi_{\mathbf{C}_L(P)}$ with multiplicity not divisible by p. (See Theorem 13.1 of [**1**].) In particular, notice that is S is another group acting on LP and normalizing P, then $\chi \in \mathrm{Irr}_P(L)$ is S-invariant if and only if χ^* is S-invariant.

Theorem 2.4. *Let G be a finite group, let $P \in \mathrm{Syl}_p(G)$ and suppose that K is a normal p'-subgroup of G such that $KP \lhd G$. Let $\chi \in \mathrm{Irr}(G)$ of p'-degree such that $\chi_{\mathbf{N}_G(P)}$ is irreducible. Let $\theta \in \mathrm{Irr}(K)$ be any irreducible constituent of χ_K. If $\chi(1)$ is odd, then $\theta_{\mathbf{C}_K(P)}$ is irreducible.*

Proof. We argue by double induction, first on $|G|$ and second on $|K|$. Write $N = \mathbf{N}_G(P)$ and $C = \mathbf{C}_K(P) = N \cap K$. By hypotheses, we have that $G = KN$. Write $\beta = \chi_N \in \mathrm{Irr}(N)$.

Let $\mu \in \mathrm{Irr}(KP)$ be over θ and under χ. Since χ has p'-degree, it follows that μ has p'-degree. Now, since

$$\frac{\mu(1)}{\theta(1)} \quad \text{divides} \quad |KP : K|$$

by Corollary 11.29 of [**1**], we conclude that $\mu_K = \theta$.

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We claim that μ is G-invariant. Otherwise, let T be the stabilizer of μ in G and let $\delta \in \mathrm{Irr}(T)$ be the Clifford correspondent of χ over μ (Theorem 6.11 of [**1**]). Then $\delta^G = \chi$ and, by Problem 5.2 of [**1**], we have that

$$\chi_N = (\delta^G)_N = (\delta_{\mathbf{N}_T(P)})^N$$

is irreducible. Hence, $\delta_{\mathbf{N}_T(P)}$ is irreducible. Since δ has odd p'-degree (because $\delta(1)$ divides $\chi(1)$) and δ lies over θ, by induction we shall have that θ_C is irreducible.

By the previous paragraph, we also conclude that θ is G-invariant. Now, let $\hat{\theta} \in \mathrm{Irr}(KP)$ be the canonical extension of θ to KP; (see page 220 of [**1**]). By uniqueness, notice that $\hat{\theta}$ is also G-invariant. By Gallagher's theorem (Corollary 6.17 of [**1**]), we may write $\mu = \hat{\theta}\delta$, where $\delta \in \mathrm{Irr}(KP/K)$ is G-invariant (by the uniqueness in Gallagher's theorem) and linear. Hence, we may write

$$\chi_{KP} = u\hat{\theta}\delta,$$

for some integer u. Now

$$\chi_{CP} = \beta_{CP} = u\hat{\theta}_{CP}\delta_{CP}\,.$$

By Clifford's theorem, we conclude that the character $\hat{\theta}_{CP}\delta_{CP}$ is a multiple of the sum of the different N-conjugates of some irreducible constituent of the character $\hat{\theta}_{CP}\delta_{CP}$.

Let $\theta^* \in \mathrm{Irr}(C)$ be the P-Glauberman correspondent of θ. Now $\chi_C = \beta_C$ contains θ_C. Hence, β_C contains θ^*. Since θ^* is N-invariant (because θ is N-invariant and the uniqueness in the Glauberman correspondence), we have that β_C is a multiple of θ^*, by Clifford's theorem. Hence, we deduce that

$$\theta_C = v\theta^*\,.$$

Therefore $\hat{\theta}_C$ is a multiple of θ^*

Assume now that p is odd. We have that the restriction of $\hat{\theta}$ to P is rational valued (by Corollary 13.4 of [**1**]). Now, since $\hat{\theta}(1)$ is odd, it follows that $\hat{\theta}_P$ contains some real valued character α. Since p is odd, $\alpha = 1_P$ and we conclude that $\hat{\theta}_P$ contains 1_P. Therefore, $\hat{\theta}_{CP}\delta_{CP}$ contains the N-invariant irreducible character $\xi\delta_{CP}$, where $\xi = \theta^* \times 1_P$. Therefore, $\hat{\theta}_{CP}\delta_{CP}$ is a multiple of $\xi\delta_{CP}$. Since δ is linear, we conclude that $\hat{\theta}_{CP}$ is a multiple of ξ. Then $P \subseteq \ker(\xi) \subseteq \ker(\hat{\theta})$ and therefore $P^K \subseteq \ker(\hat{\theta})$. Hence $[K, P] \subseteq \ker(\hat{\theta}) \cap K = \ker(\theta)$. Since $[K, P]C = K$, we conclude that θ_C is irreducible by Lemma 2.2.

Hence, we may assume that $p = 2$. Thus K is a group of odd order, and therefore solvable.

Now, we claim that $[K, P] = K$. Write $M = [K, P] \triangleleft G$. By coprime action, we have that $K = MC$. Hence, $MN = G$ and $MP \triangleleft G$. Suppose that $M < K$. Let $\nu \in \mathrm{Irr}(M)$ be under θ. By induction (since $|M| < |K|$), we have that $\nu_{\mathbf{C}_M(P)}$ is irreducible. Let $\epsilon \in \mathrm{Irr}(R)$ be the Clifford correspondent of θ over ν, where R is the stabilizer of ν in K. Since in this case $\nu_{\mathbf{C}_M(P)}$ is the Glauberman correspondent of ν, it follows by the

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comments preceding the statement of this theorem that $\mathbf{C}_R(P)$ is the stabilizer of $\nu_{\mathbf{C}_M(P)}$ in C. Now, by Corollary 4.2 of [**3**], we have that $\epsilon_{\mathbf{C}_R(P)} \in \mathrm{Irr}(\mathbf{C}_R(P))$. This character lies over $\nu_{\mathbf{C}_M(P)}$ and, by the Clifford correspondence, we have that

$$(\epsilon_{\mathbf{C}_R(P)})^C \in \mathrm{Irr}(C)\,.$$

Now

$$\theta_C = (\epsilon^K)_C = (\epsilon_{\mathbf{C}_R(P)})^C$$

is irreducible, proving the claim.

Let K/L be a chief factor of G. Since K is solvable, we have that K/L is abelian. Since $K = [K, P]$, by Fitting's lemma, we have that $C \subseteq L$. Now, let $\theta' \in \mathrm{Irr}(L)$ be the unique P-invariant character of L such that $(\theta')^* = \theta^*$. By the uniqueness of the Glauberman correspondence, we have that θ' is N-invariant. Now, by the going down theorem (Theorem 6.18 of [**1**]), we have two possibilities: either θ_L is irreducible or θ is fully ramified with respect to K/L.

Suppose first that $\theta_L = \theta'$ is irreducible. Let $W = LN < G$ and notice that $\chi_W = \chi' \in \mathrm{Irr}(W)$. Since $(\chi')_N$ is irreducible and has odd p'-degree, by induction we have that $\theta'_C \in \mathrm{Irr}(C)$. Then $\theta_C = (\theta')_C$ and the theorem is proved in this case. Hence we may assume that $\theta_L = e\theta'$ with $e^2 = |K : L|$. We claim that if U is a complement of K/L in G, then U is G-conjugate to NL. Since we may assume that $P \subseteq U$, then $U/L = \mathbf{N}_{G/L}(LP/L) = N/L$, and the claim easily follows. Now, by Theorem 2.3, we conclude that χ_{LN} is not irreducible. However, χ_N is irreducible and so it is χ_{LN}. This contradiction proves the theorem. □

Now we are ready to prove Theorem A, which we restate here.

THEOREM 2.5. *Suppose that G is p-solvable. Let $P \in \mathrm{Syl}_p(G)$ and let $\chi \in \mathrm{Irr}(G)$ of p'-degree. If $\chi(1)$ is odd, then $\chi_{\mathbf{N}_G(P)}$ is irreducible if and only if $G = \ker(\chi)\mathbf{N}_G(P)$.*

Proof. If $G = \ker(\chi)\mathbf{N}_G(P)$, then $\chi_{\mathbf{N}_G(P)}$ is irreducible by Lemma 2.2. To prove the converse, we argue by induction on $|G|$. Let $M = \ker(\chi)$ and $N = \mathbf{N}_G(P)$. Write $\bar{G} = G/M$ and use the bar convention. Then $\bar{N} = \mathbf{N}_{\bar{G}}(\bar{P})$. If $M > 1$, by induction we shall have $\bar{G} = \bar{N}$. Hence $\mathbf{N}_G(P)M = G$ and we shall be done. Hence we may assume that χ is faithful.

Since $\chi_N \in \mathrm{Irr}(N)$ has p'-degree and N has a normal Sylow p-subgroup P, it follows that $P' \subseteq \ker(\chi_N) \subseteq \ker(\chi) = 1$. Therefore, we have that P is abelian.

Let $K = \mathbf{O}_{p'}(G)$. Since $P/K \subseteq \mathbf{C}_{G/K}(\mathbf{O}_p(G/K)) \subseteq \mathbf{O}_p(G/K)$, we have that $PK \triangleleft G$. Now, let $\theta \in \mathrm{Irr}(K)$ be under χ. Since $\chi(1)$ is not divisible by p and $KP \triangleleft G$, notice that θ is P-invariant. By Theorem 2.4, we shall have that $\theta_{\mathbf{C}_K(P)}$ is irreducible. Hence, $[K, P] \subseteq \ker(\theta)$, by Lemma 2.1. Since $[K, P] \triangleleft G$, we shall have that $[K, P] \subseteq \ker(\theta)^g$, for all $g \in G$. Hence, $[K, P] \subseteq \ker(\chi) = 1$. Then $N = G$, as required. □

The conclusion of Theorem A fails if we allow the character χ to have degree divisible by p. For instance, choose any prime r such that p divides $r - 1$, and let G be the semidirect product of an extraspecial group E of order p^3 with a cyclic group C of order r, where the action of E on C has kernel D of index p. Now, let $\lambda \in \mathrm{Irr}(D)$

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such that $\mathbf{Z}(E)$ is not contained in the kernel of λ. Hence $\lambda^E \in \mathrm{Irr}(E)$. Now, let $\mu \in \mathrm{Irr}(C)$ be nontrivial. Let $\theta = \mu \times \lambda \in \mathrm{Irr}(CD)$ and notice that $\chi = \theta^G \in \mathrm{Irr}(G)$. Also, χ is faithful. In this case, $E \in \mathrm{Syl}_p(G)$ is self-normalizing and $\chi_E = \lambda^E$ is irreducible.

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