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LIMITING BEHAVIOUR FOR ARRAYS OF UPPER EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract

For triangular arrays $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ of upper extended negatively dependent random variables weakly mean dominated by a random variable X and sequences $\{b_n\}$ of positive constants, conditions are given to guarantee an almost sure finite upper bound to $\sum_{k=1}^{n}(X_{n,k} - \mathbb{E}X_{n,k})/\sqrt{b_n \operatorname{Log} n}$, where $\operatorname{Log} n := \max\{1, \log n\}$, thus getting control over the limiting rate in terms of the prescribed sequence $\{b_n\}$ and permitting us to weaken or strengthen the assumptions on the random variables.

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1. Introduction

Strong limit theorems play a central role in probability theory and the classical strong laws have been extended to more general assumptions on the random variables. One of those interesting extensions involves arrays of random variables. In 1989, Hu *et al.* established the Marcinkiewicz–Zygmund strong law of large numbers for arrays of independent and identically distributed zero-mean random variables (see [6]). Three years later, Hu and Weber (see [7]) showed that the classical Hartman–Wintner law of the iterated logarithm is no longer valid for arrays of independent and identically distributed zero-mean random variables. In fact, Hu and Weber found a new rate of convergence in what has become known later as the 'law of the logarithm' (see [8] and the recent paper [5]):

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{n} X_{n,k} = \sqrt{2} \quad \text{a.s.} \tag{1.1}$$

for every triangular array $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ of independent and identically distributed random variables satisfying $\mathbb{E}X_{1,1} = 0$, $\mathbb{E}X_{1,1}^2 = 1$ and $\mathbb{E}X_{1,1}^4 < \infty$. In 1994, Qi [10] improved Hu and Weber's result proving that for arrays of independent and identically distributed random variables, (1.1) holds if and only if $\mathbb{E}X_{1,1} = 0$, $\mathbb{E}X_{1,1}^2 = 1$ and $\mathbb{E}X_{1,1}^4(\operatorname{Log}|X_{1,1}|)^{-2} < \infty$, where $\operatorname{Log} x$ denotes $\max\{1, \log x\}$. In 1996, Sung [11] took a further step proving (1.1) for (truncated) triangular arrays of independent zero-mean random variables. In addition, Sung gave an example in which (1.1) could fail if the truncation of $X_{n,k}$ is not tight enough (in particular, if $|X_{n,k}| \leqslant Cn/\sqrt{\log n}$ almost surely for any $1 \leqslant k \leqslant n$, $n \geqslant 1$ and some $C > 0$).

Our purpose in this note is to obtain

$$\limsup_{n\to\infty} \frac{1}{\sqrt{b_n \operatorname{Log} n}} \sum_{k=1}^{n} (X_{n,k} - \mathbb{E}X_{n,k}) \leqslant C \quad \text{a.s.} \tag{1.2}$$

for some constant $C > 0$ and for some sequences of positive constants $\{b_n\}$, while relaxing at the same time the assumptions on the triangular array of random variables $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$. Specifically, we shall prove (1.2) when the array of random variables $\{X_{n,k}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ is upper extended negatively dependent and weakly mean dominated by a random variable X.

We next state the definitions required in this paper. A random triangular array $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ is *weakly mean dominated* by a random variable X if, for some $C > 0$,

$$\frac{1}{n} \sum_{k=1}^{n} \mathbb{P}\{|X_{n,k}| > t\} \leqslant C\mathbb{P}\{|X| > t\}$$

for all $t > 0$ and every $n \geqslant 1$ (see [4]). If a triangular array of random variables is stochastically dominated by a random variable X (see, for instance, [12]), then it is weakly mean dominated; however, the converse is not true. Random variables $X_1, \dots, X_n$ are said to be *upper extended negatively dependent (UEND)* if there is a constant $M > 0$ such that

$$\mathbb{P}(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) \leqslant M \prod_{i=1}^{n} \mathbb{P}(X_i > x_i) \tag{1.3}$$

holds for all real numbers $x_1, \dots, x_n$ (see [2]). A sequence of random variables $\{X_n : n \geqslant 1\}$ is said to be upper extended negatively dependent if, for each $n \geqslant 1$, the random variables $X_1, \dots, X_n$ are upper extended negatively dependent. We say that a triangular array $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ of random variables is *row-wise upper extended negatively dependent* if for each fixed $n \geqslant 1$, the random variables $X_{n,1}, \dots, X_{n,n}$ are upper extended negatively dependent. A triangular array $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ of random variables is said to be upper extended negatively dependent if it is row-wise upper extended negatively dependent and the constant M in (1.3) is the same for each row.

Given a positive monotone sequence of constants $\{b_n\}$, a continuous monotone function $b(\cdot)$ on $[0, \infty[$ is called a *monotone extension* of $\{b_n\}$ if $b(n) = b_n$ (see [3, page 90]).

Associated to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we shall consider the space $\mathscr{L}_p$ $(p > 0)$ of all measurable functions X (necessarily random variables) for which $\mathbb{E}|X|^p < \infty$.

Throughout, C will denote a positive constant, which is not necessarily the same on each appearance. The symbol $\lfloor x \rfloor$ will be used to indicate the largest integer not greater than x.

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2. Main result

The main result of this paper is the following theorem.

Theorem 2.1. *Let $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of upper extended negatively dependent random variables weakly mean dominated by a random variable $X \in \mathscr{L}_1$, $\{a_n\}$ a positive increasing sequence of constants with increasing extension $a(\cdot)$ and $\{b_n\}$ a positive nondecreasing sequence of constants with nondecreasing extension $b(\cdot)$. If:*

(a) $\ell := \limsup_{n\to\infty} a_n \sqrt{\operatorname{Log} n / b_n} < \infty$;
(b) $\sum_{k=1}^{n} \mathbb{E}X_{n,k}^2 \leqslant b_n$;
(c) $\mathbb{E}a^{-1}(|X|) < \infty$;
(d) $\int_1^\infty 1/\operatorname{Log} u \int_{u-1}^\infty \mathbb{P}\{a^{-1}(|X|) > t\}\, dt\, du < \infty$;
(e) $\int_0^\infty \mathbb{P}\{|X| > t\} \int_0^{\lfloor a^{-1}(t) \rfloor} (u+1)/\sqrt{b(u) \operatorname{Log} u}\, du\, dt < \infty$;

then

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2b_n \operatorname{Log} n}} \sum_{k=1}^{n} (X_{n,k} - \mathbb{E}X_{n,k}) \leqslant \ell + \sqrt{2 + \ell^2} \quad a.s.$$

The next corollary allows us to identify a sufficient moment condition on X in order to derive an almost sure finite upper bound for the row-wise sum elements of the triangular array under the rate of the 'law of the logarithm'.

Corollary 2.2. *If $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ is a triangular array of upper extended negatively dependent random variables weakly mean dominated by a (nonnull) random variable X such that $\mathbb{E}X^4 \operatorname{Log} X < \infty$, then*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n \operatorname{Log} n}} \sum_{k=1}^{n} (X_{n,k} - \mathbb{E}X_{n,k}) \leqslant \sqrt{C\mathbb{E}X^2}(1 + \sqrt{3}) \quad a.s.$$

for some positive constant C.

3. Lemmas and proofs

The first auxiliary lemma extends the properties of upper extended negatively dependent sequences of random variables (described in [2, Lemma 2.2]) to triangular arrays.

Lemma 3.1. *If $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ is a triangular array of upper extended negatively dependent random variables and $f_{n,1}, \ldots, f_{n,n}$ are real functions, all monotone nondecreasing, then the triangular array $\{f_{n,k}(X_{n,k}) : 1 \leqslant k \leqslant n, n \geqslant 1\}$ is upper extended negatively dependent. Furthermore, if $f_{n,1}, \ldots, f_{n,n}$ are also positive,*

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then there exists a constant $M > 0$ such that

$$\mathbb{E}\Big[\prod_{k=1}^{n} f_{n,k}(X_{n,k})\Big] \leqslant M \prod_{k=1}^{n} \mathbb{E} f_{n,k}(X_{n,k})$$

for each $n \geqslant 1$.

PROOF. Assume that $f_{n,1}, \ldots, f_{n,n}$ are all monotone nondecreasing. Since $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ is a triangular array of upper extended negatively dependent random variables,

$$\mathbb{P}(X_{n,1} > x_{n,1}, X_{n,2} > x_{n,2}, \ldots, X_{n,n} > x_{n,n}) \leqslant M \prod_{k=1}^{n} \mathbb{P}(X_{n,k} > x_{n,k}) \tag{3.1}$$

for any $n \geqslant 1$ with $M > 0$ independent of n. For every $1 \leqslant k \leqslant n$ and each real number $y_{n,k}$, the event $\{f_{n,k}(X_{n,k}) > y_{n,k}\}$ is equivalent to either $\{X_{n,k} > x_{n,k}\}$ or $\{X_{n,k} \geqslant x_{n,k}\}$ with $x_{n,k} = \inf\{x : f_{n,k}(x) > y_{n,k}\}$ (in the latter case, $\{X_{n,k} \geqslant x_{n,k}\}$ can be approximated by $\{X_{n,k} > x^*_{n,k}\}$ as $x^*_{n,k} \to x^+_{n,k}$). From the continuity of the probability measure and (3.1),

$$\mathbb{P}[f_{n,1}(X_{n,1}) > y_{n,1}, \ldots, f_{n,n}(X_{n,n}) > y_{n,n}] \leqslant M \prod_{k=1}^{n} \mathbb{P}[f_{n,k}(X_{n,k}) > y_{n,k}]$$

with the same constant $M > 0$. The remaining statement follows from the well-known formula for positive random variables $Y_1, \ldots, Y_n$,

$$\mathbb{E}(Y_1 \cdots Y_n) = \int_0^{\infty} \cdots \int_0^{\infty} \mathbb{P}(Y_1 > y_1, \ldots, Y_n > y_n)\, dy_1 \cdots dy_n. \qquad \square$$

The result above is a Bernstein inequality (see [9, page 57]) for arrays of upper extended negatively dependent random variables.

LEMMA 3.2. *If $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ is an array of zero-mean upper extended negatively dependent random variables such that $|\mathbb{E}X^m_{n,k}| \leqslant m! a_n^{m-2} \mathbb{E}X^2_{n,k}/2 < \infty$, with $a_n > 0$, for all $1 \leqslant k \leqslant n$, $n \geqslant 1$ and every $m \geqslant 2$, then*

$$\forall \varepsilon > 0, \quad \mathbb{P}\Big\{\sum_{k=1}^{n} X_{n,k} > \varepsilon\Big\} \leqslant M \exp\Big[-\frac{\varepsilon^2}{2(\varepsilon a_n + \sum_{k=1}^{n} \mathbb{E}X^2_{n,k})}\Big]$$

for some $M > 0$ (independent of n).

PROOF. Fixing $a_n > 0$ and $0 < t_n < 1/a_n$,

$$\begin{aligned}
\mathbb{E}\exp(t_n X_{n,k}) &= 1 + \frac{t_n^2}{2!}\mathbb{E}X^2_{n,k} + \frac{t_n^3}{3!}\mathbb{E}X^3_{n,k} + \cdots + \frac{t_n^m}{m!}\mathbb{E}X^m_{n,k} + \cdots \\
&\leqslant 1 + \frac{t_n^2}{2}\mathbb{E}X^2_{n,k} + \frac{t_n^3}{2} a_n \mathbb{E}X^2_{n,k} + \cdots + \frac{t_n^m}{2} a_n^{m-2}\mathbb{E}X^2_{n,k} + \cdots \\
&\leqslant 1 + \frac{t_n^2}{2(1 - a_n t_n)}\mathbb{E}X^2_{n,k} \\
&\leqslant \exp\Big[\frac{t_n^2}{2(1 - a_n t_n)}\mathbb{E}X^2_{n,k}\Big].
\end{aligned}$$

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Since $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ is upper extended negatively dependent,

$$\mathbb{E}\exp\Big(t_n \sum_{k=1}^{n} X_{n,k}\Big) \leqslant M \prod_{k=1}^{n} \mathbb{E}\exp(t_n X_{n,k}) \leqslant M \exp\Big[\frac{t_n^2}{2(1-a_n t_n)} \sum_{k=1}^{n} \mathbb{E}X_{n,k}^2\Big]$$

for some $M > 0$ via Lemma 3.1 with $f_{n,k}(x) = e^{t_n x}$ $(0 < t_n < 1/a_n)$. From the Chebyshev inequality,

$$\begin{aligned}\mathbb{P}\Big\{\sum_{k=1}^{n} X_{n,k} > \varepsilon\Big\} &\leqslant M \exp(-\varepsilon t_n)\mathbb{E}\exp\Big(t_n \sum_{k=1}^{n} X_{n,k}\Big) \\ &\leqslant M \exp\Big[-\varepsilon t_n + \frac{t_n^2}{2(1-a_n t_n)} \sum_{k=1}^{n} \mathbb{E}X_{n,k}^2\Big]\end{aligned}$$

and, taking $t_n = \varepsilon/(\varepsilon a_n + \sum_{k=1}^{n} \mathbb{E}X_{n,k}^2)$,

$$\mathbb{P}\Big\{\sum_{k=1}^{n} X_{n,k} > \varepsilon\Big\} \leqslant M \exp\Big[-\frac{\varepsilon^2}{2(\varepsilon a_n + \sum_{k=1}^{n} \mathbb{E}X_{n,k}^2)}\Big]. \qquad \square$$

Lemma 3.3. *Let $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of zero-mean upper extended negatively dependent random variables and $\{a_n\}$, $\{b_n\}$ sequences of positive constants. If:*

(i) $|X_{n,k}| \leqslant a_n$ *almost surely for every* $1 \leqslant k \leqslant n$, $n \geqslant 1$;
(ii) $\sum_{k=1}^{n} \mathbb{E}X_{n,k}^2 \leqslant b_n$, $n \geqslant 1$;
(iii) $\ell := \limsup_{n\to\infty} a_n \sqrt{\operatorname{Log} n / b_n} < \infty$;

then

$$\limsup_{n\to\infty} \frac{1}{\sqrt{b_n \operatorname{Log} n}} \sum_{k=1}^{n} X_{n,k} \leqslant \ell + \sqrt{2 + \ell^2} \quad a.s.$$

Proof. Fix an arbitrary $\varepsilon > 0$. Since

$$\mathbb{E}|X_{n,k}^m| = \Big|\int_{-a_n}^{a_n} t^m \, d\mathbb{P}\{X_{n,k} \leqslant t\}\Big| \leqslant a_n^{m-2}\mathbb{E}X_{n,k}^2$$

for any $1 \leqslant k \leqslant n$, $n \geqslant 1$ and all integers $m \geqslant 2$,

$$\begin{aligned}&\mathbb{P}\Big\{\frac{1}{\sqrt{b_n \operatorname{Log} n}} \sum_{k=1}^{n} X_{n,k} > \ell + \sqrt{2+\ell^2} + \varepsilon\Big\} \\ &\quad\leqslant M \exp\left\{-\frac{\big(\ell + \sqrt{2+\ell^2} + \varepsilon\big)^2}{2\Big[\big(\ell + \sqrt{2+\ell^2} + \varepsilon\big)a_n \sqrt{\frac{\operatorname{Log} n}{b_n}} + 1\Big]} \operatorname{Log} n\right\}\end{aligned}$$

according to Lemma 3.2. Hence,

$$\liminf_{n\to\infty} \frac{(\ell + \sqrt{2+\ell^2} + \varepsilon)^2}{2\Big[(\ell + \sqrt{2+\ell^2} + \varepsilon)a_n \sqrt{\frac{\operatorname{Log} n}{b_n}} + 1\Big]}$$
$$= \frac{(\ell + \sqrt{2+\ell^2} + \varepsilon)^2}{2(\ell + \sqrt{2+\ell^2} + \varepsilon)\limsup_{n\to\infty} a_n \sqrt{\frac{\operatorname{Log} n}{b_n}} + 2}$$
$$= \frac{(\ell + \sqrt{2+\ell^2} + \varepsilon)^2}{2(\ell + \sqrt{2+\ell^2} + \varepsilon)\ell + 2} > 1,$$

which yields

$$\sum_{n=1}^{\infty} \mathbb{P}\Big\{\frac{1}{\sqrt{b_n \operatorname{Log} n}} \sum_{k=1}^{n} X_{n,k} > \ell + \sqrt{2+\ell^2} + \varepsilon\Big\} < \infty$$

and the result is a direct consequence of the Borel–Cantelli lemma. □

Remark 3.4. Lemma 3.3 improves [11, Lemma 1]. Indeed, an almost sure finite upper bound to $\sum_{k=1}^{n} X_{n,k}/\sqrt{n \operatorname{Log} n}$ is obtained when $\{X_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ is an array of zero-mean upper extended negatively dependent random variables.

Proof of Theorem 2.1. Setting

$$X'_{n,k} = X_{n,k} I_{\{|X_{n,k}| \leqslant a_n\}} + a_n I_{\{X_{n,k} > a_n\}} - a_n I_{\{X_{n,k} < -a_n\}},$$
$$X''_{n,k} = X_{n,k} I_{\{|X_{n,k}| > a_n\}} + a_n I_{\{X_{n,k} < -a_n\}} - a_n I_{\{X_{n,k} > a_n\}},$$

we have $X'_{n,k} + X''_{n,k} = X_{n,k}$. The triangular array $\{X'_{n,k} - \mathbb{E}X'_{n,k} : 1 \leqslant k \leqslant n, n \geqslant 1\}$ is upper extended negatively dependent since the function $g_L(t) = \max(\min(t, L), -L)$, which describes the truncation at level L, is nondecreasing. Since

$$|X'_{n,k} - \mathbb{E}X'_{n,k}| \leqslant 2a_n$$

and

$$\sum_{n=1}^{n} \mathbb{E}|X'_{n,k} - \mathbb{E}X'_{n,k}|^2 \leqslant 2b_n,$$

Lemma 3.3 guarantees that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2b_n \operatorname{Log} n}} \sum_{k=1}^{n} (X'_{n,k} - \mathbb{E}X'_{n,k}) \leqslant \ell + \sqrt{2+\ell^2} \quad \text{a.s.} \tag{3.2}$$

Now, we shall demonstrate that

$$\frac{1}{\sqrt{b_n \operatorname{Log} n}} \sum_{k=1}^{n} (X''_{n,k} - \mathbb{E}X''_{n,k}) \xrightarrow{\text{a.s.}} 0. \tag{3.3}$$

We have $|X''_{n,k}| \leqslant |X_{n,k}| I_{\{|X_{n,k}| > a_n\}}$ and

$$\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}|X_{n,k}| I_{\{|X_{n,k}| > a_n\}} \leqslant C\mathbb{E}|X| I_{\{|X| > a_n\}},$$

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since $\{X_{n,k}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ is weakly mean dominated by X. Thus,

$$\mathbb{P}\Big\{\frac{1}{\sqrt{b_n \operatorname{Log} n}} \sum_{k=1}^{n} |X_{n,k} - \mathbb{E}X_{n,k}| > \varepsilon\Big\} \leqslant \frac{2}{\varepsilon\sqrt{b_n \operatorname{Log} n}} \sum_{k=1}^{n} \mathbb{E}|X_{n,k}| \leqslant \frac{Cn}{\varepsilon\sqrt{b_n \operatorname{Log} n}} \mathbb{E}|X| I_{\{|X|>a_n\}}$$

and it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{b_n \operatorname{Log} n}} \mathbb{E}|X| I_{\{|X|>a_n\}} < \infty. \tag{3.4}$$

Integrating by parts,

$$\mathbb{E}|X| I_{\{|X|>a_n\}} = a_n \mathbb{P}\{|X| > a_n\} + \int_{a_n}^{\infty} \mathbb{P}\{|X| > t\}\, dt,$$

so that (3.4) becomes

$$\sum_{n=1}^{\infty} \Big(\frac{na_n}{\sqrt{b_n \operatorname{Log} n}} \mathbb{P}\{|X| > a_n\} + \frac{n}{\sqrt{b_n \operatorname{Log} n}} \int_{a_n}^{\infty} \mathbb{P}\{|X| > t\}\, dt \Big). \tag{3.5}$$

Recalling that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{na_n}{\sqrt{b_n \operatorname{Log} n}} \mathbb{P}\{|X| > a_n\} &\leqslant C \sum_{n=1}^{\infty} \frac{n}{\operatorname{Log} n} \mathbb{P}\{|X| > a_n\} \\
&\leqslant C \sum_{n=1}^{\infty} \mathbb{P}\{|X| > a_n\} \int_0^n \frac{1}{\operatorname{Log} u}\, du \\
&\leqslant C \int_1^{\infty} \mathbb{P}\{|X| > a(t-1)\} \int_0^t \frac{1}{\operatorname{Log} u}\, du\, dt \\
&\leqslant C \int_0^1 \int_1^{\infty} \mathbb{P}\{|X| > a(t-1)\}\, dt \frac{1}{\operatorname{Log} u}\, du \\
&\quad + C \int_1^{\infty} \frac{1}{\operatorname{Log} u} \int_u^{\infty} \mathbb{P}\{|X| > a(t-1)\}\, dt\, du \\
&= C\mathbb{E}a^{-1}(|X|) + C \int_1^{\infty} \frac{1}{\operatorname{Log} u} \int_{u-1}^{\infty} \mathbb{P}\{a^{-1}(|X|) > t\}\, dt\, du
\end{aligned}$$

for some $C > 0$ and

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{\sqrt{b_n \operatorname{Log} n}} \int_{a_n}^{\infty} \mathbb{P}\{|X| > t\}\, dt &= \int_0^{\infty} \mathbb{P}\{|X| > t\} \sum_{\{n: a_n \leqslant t\}} \frac{n}{\sqrt{b_n \operatorname{Log} n}}\, dt \\
&\leqslant \int_0^{\infty} \mathbb{P}\{|X| > t\} \int_0^{\lfloor a^{-1}(t) \rfloor} \frac{u+1}{\sqrt{b(u) \operatorname{Log} u}}\, du\, dt,
\end{aligned}$$

we establish the convergence of the series (3.5). From the Borel–Cantelli lemma, we obtain the convergence of (3.3), which, together with (3.2), yields the result. □

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PROOF OF COROLLARY 2.2. From [4, Lemma 2.1],

$$\sum_{k=1}^{n} \mathbb{E}X_{n,k}^2 \leqslant nC\mathbb{E}X^2$$

for some $C > 0$. Putting $b_n = Cn\mathbb{E}X^2$ and $a_n = \sqrt{b_n/\operatorname{Log} n}$ yields $\ell = 1$. Since $2t^2 \operatorname{Log} t/(C\mathbb{E}X^2)$ is an asymptotic inverse of $a(t) = Ct\mathbb{E}X^2$ (see [1, page 28]),

$$\int_0^\infty \mathbb{P}\{a^{-1}(|X|) > t\}\,dt \leqslant C\int_0^\infty \mathbb{P}\{|X| > y\} y \operatorname{Log} y\,dy,$$

$$\begin{aligned}\int_1^\infty \frac{1}{\operatorname{Log} u}\int_{u-1}^\infty \mathbb{P}\{a^{-1}(|X|) > t\}\,dt\,du &= \int_0^\infty \mathbb{P}\{|X| > a(t)\}\int_1^{t+1}\frac{1}{\operatorname{Log} u}\,du\,dt \\ &\leqslant C\int_0^\infty \mathbb{P}\{|X| > y\} y^3 \operatorname{Log} y\,dy\end{aligned}$$

and

$$\int_0^\infty \mathbb{P}\{|X| > t\}\int_0^{\lfloor a^{-1}(t)\rfloor}\frac{u+1}{\sqrt{b(u)\operatorname{Log} u}}\,du\,dt \leqslant C\int_0^\infty \mathbb{P}\{|X| > t\} t^3\sqrt{\operatorname{Log} t}\,dt.$$

Using [9, Lemma 2.4, page 61], it follows that assumptions (c)–(e) are fulfilled and this completes the proof. □

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