Proceedings of the Royal Society of Edinburgh, **153**, 1609–1629, *2023*
DOI:10.1017/prm.2022.63

Multiscale linearization of nonautonomous systems

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(Received 11 March 2022; accepted 29 August 2022)

We present sufficient conditions under which a given linear nonautonomous system and its nonlinear perturbation are topologically conjugated. Our conditions are of a very general form and provided that the nonlinear perturbations are well-behaved, we do not assume any asymptotic behaviour of the linear system. Moreover, the control on the nonlinear perturbations may differ along finitely many mutually complementary directions. We consider both the cases of one-sided discrete and continuous dynamics.

Keywords: Multiscale; linearization; nonautonomous dynamics

2020 *Mathematics subject classification:* Primary: 34D30
Secondary: 34D09

1. Introduction

Let us consider a (one-sided) linear nonautonomous difference equation given by

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{N}, \tag{1.1}$$

as well as its nonlinear perturbation

$$x_{n+1} = A_n x_n + f_n(x_n), \quad n \in \mathbb{N}, \tag{1.2}$$

where $A_n : X \to X$, $n \in \mathbb{N}$ is a sequence of invertible bounded linear operators acting on a Banach space X and $f_n : X \to X$, $n \in \mathbb{N}$ is a sequence of (nonlinear) maps. In this note we are interested in describing sufficient conditions under which the systems (1.1) and (1.2) are *topologically conjugated*, meaning that there exists a sequence of homeomorphisms $H_n : X \to X$, $n \in \mathbb{N}$ mapping the trajectories of (1.1) into trajectories of (1.2). Whenever such conjugacies exist, many important dynamical properties of the *nonlinear system* (1.2) can be obtained by studying the *linear system* (1.1), which in general is much easier to deal with.

Linearization problems, as the one described above, have a long history. As cornerstones of this theory (dealing with the case of autonomous dynamics), we refer to the works of Grobman [16, 17] and Hartman [18–20]. The first linearization results dealing with the case of infinite-dimensional dynamics are due to Palis [23] and Pugh [26]. The problem of formulating sufficient conditions under which the conjugacy exhibits higher regularity properties was first considered in the pioneering works of Sternberg [28, 29].

The first nonautonomous version of the Grobman–Hartman theorem was established by Palmer [24] for the case of continuous time. The discrete time version of his result (formulated in [1]) asserts that (1.1) and (1.2) are topologically conjugated provided that the following conditions hold:

- (1.1) admits an exponential dichotomy (see [12]);
- the nonlinear terms f_n are bounded and uniformly Lipschitz with a sufficiently small Lipschitz constant.

In addition, several authors obtained important extensions of the Palmer's theorem by relaxing assumptions related to the linear systems (1.1) (or its continuous counterpart). We refer to [3, 5, 6, 10, 21, 22, 27] and references therein. For recent results dealing with the higher regularity of conjugacies in nonautonomous linearization, we refer to [4, 7–9, 11, 13–15].

An ubiquitous assumption in most of those results is the existence of a decomposition of the phase space X into the stable and unstable directions along which (1.1) exhibits contraction and expansion, respectively. In other words, it is assumed that (1.1) exhibits some sort of dichotomic behaviour (although not necessarily of exponential nature). The key idea is that the lack of hyperbolicity can be compensated by properly controlling the 'size' of nonlinear terms f_n in (1.2). A notable exception is the work of Reinfelds and Šteinberga [27] in which the authors obtain a linearization result without any assumptions related to the asymptotic behaviour of (1.1). However, the conditions concerned with nonlinearities f_n are expressed in terms of a Green function corresponding to (1.1) which is still essentially given by decomposing X into two directions.

In this work, instead of considering a decomposition of X into just two directions, we allow for a decomposition of X into several directions with possible different behaviours of (1.1) along each of those directions. Our conditions are of general form as in [5, 27] and no asymptotic behaviour is required for the linear dynamics. On the other hand, the more 'non-hyperbolic' the linear system is (along certain direction), the more restrictive are the assumptions on the perturbations f_n (along that direction). In fact, we allow the presence of certain directions along which we do not impose any conditions on (1.1) and on the nonlinear perturbations f_n (besides requiring those to be continuous and bounded). In this general case, we obtain only a *quasi-conjugacy* between systems (1.1) and (1.2), meaning that they are conjugated except for a given deviation in the directions along which we have no control. To the best of our knowledge, this is the first time such a general result appears in the literature. We stress that our results are motivated by the recent paper by Pilyugin [25] which deals with the multiscale nonautonomous shadowing.

The paper is organized as follows. In § 2, we consider the case of discrete time, i.e. we establish sufficient conditions under which (1.1) and (1.2) are topologically conjugated. We discuss in detail the relationship between our result and related results in the literature and we provide an explicit example illustrating the strength of our result. Finally, in § 3 we establish an analogous result in the case of continuous time.

2. The case of discrete time

2.1. Preliminaries

Let $X = (X, |\cdot|)$ be an arbitrary Banach space and denote by $\mathcal{B}(X)$ the space of all bounded linear operators on X. By $\|\cdot\|$, we will denote the operator norm on $\mathcal{B}(X)$. Given a sequence $(A_n)_{n\in\mathbb{N}}$ of invertible operators in $\mathcal{B}(X)$ and $m, n \in \mathbb{N}$, let us consider the associated linear cocycle given by

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{if } m > n; \\ \mathrm{Id} & \text{if } m = n; \\ A_m^{-1}\cdots A_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

2.2. Multiscale

Let K be a finite set of the form $K = K^s \cup K^u \cup K^c$, where $K^i \cap K^j = \emptyset$ for $i, j \in \{s, u, c\}$, $i \neq j$. Suppose that for each $n \in \mathbb{N}$ there exists a family of projections P_n^k, $k \in K$ such that

$$\sum_{k\in K} P_n^k = \mathrm{Id}, \tag{2.1}$$

and

$$P_n^k P_n^l = 0 \ \text{ for every } k, l \in K,\ k \neq l.$$

In particular, by considering $X_k(n) = P_n^k(X)$, we have that

$$X = \bigoplus_{k\in K} X_k(n) \quad \text{for every } n \in \mathbb{N}.$$

REMARK 2.1. We observe that the notion of multiscale considered in this work is more general than the one considered by Pilyugin in [**25**]. Indeed, in the aforementioned work there is an extra assumption requiring that $A_n P_n^k = P_{n+1}^k A_n$ for every $n \in \mathbb{N}$ and $k \in K$. Moreover, in [**25**] K has the form $K = K^s \cup K^u$, i.e. $K^c = \emptyset$.

2.3. Standing assumptions

Given $k \in K^s \cup K^u$, take $\lambda_k > 0$ and let $(\mu_n^k)_{n\in\mathbb{N}}$ and $(\nu_n^k)_{n\in\mathbb{N}}$ be sequences of positive numbers such that:

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- for $k \in K^s$,

$$\sup_n \sum_{l=1}^{n} \|\mathcal{A}(n,l)P_l^k\|\nu_l^k < +\infty, \tag{2.2}$$

and

$$\sup_n \sum_{l=1}^{n} \|\mathcal{A}(n,l)P_l^k\|\mu_l^k \leqslant \lambda_k; \tag{2.3}$$

- for $k \in K^u$,

$$\sup_n \sum_{l=n+1}^{\infty} \|\mathcal{A}(n,l)P_l^k\|\nu_l^k < +\infty, \tag{2.4}$$

and

$$\sup_n \sum_{l=n+1}^{\infty} \|\mathcal{A}(n,l)P_l^k\|\mu_l^k \leqslant \lambda_k. \tag{2.5}$$

2.4. A linearization result

We are now ready to state our first main result.

THEOREM 2.2. *Let $f_n \colon X \to X$, $n \in \mathbb{N}$ be a sequence of maps such that $A_n + f_n$ is a homeomorphism for each $n \in \mathbb{N}$ and*

$$\|P_n^k f_{n-1}\|_\infty \leqslant \nu_n^k, \tag{2.6}$$

for every $k \in K^s \cup K^u$ and $n \geqslant 1$, where

$$\|P_n^k f_{n-1}\|_\infty := \sup\{|P_n^k f_{n-1}(x)| : x \in X\}.$$

Moreover, assume that for each $k \in K^s \cup K^u$, x, $y \in X$ and $n \in \mathbb{N}$,

$$|P_n^k f_{n-1}(x) - P_n^k f_{n-1}(y)| \leqslant \mu_n^k |x-y|. \tag{2.7}$$

Then, if

$$\sum_{k \in K^s \cup K^u} \lambda_k < 1, \tag{2.8}$$

i) *there exist sequences of continuous maps $H_n : X \to X$, $n \in \mathbb{N}$ and $\tau_n : X \to \bigoplus_{k \in K^c} X_k(n+1)$, $n \in \mathbb{N}$ such that*

$$H_{n+1} \circ A_n = (A_n + f_n) \circ H_n + \tau_n \circ H_n, \text{ for every } n \in \mathbb{N}. \tag{2.9}$$

In addition,

$$\sup_{n \in \mathbb{N}} \|H_n - \mathrm{Id}\|_\infty < +\infty \text{ and } \tau_n(x) = -\sum_{k \in K^c} P_{n+1}^k(f_n(x));$$

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ii) *there exist sequences of continuous maps* $\bar{H}_n : X \to X$, $n \in \mathbb{N}$ *and* $\bar{\tau}_n : X \to \bigoplus_{k\in K^c} X_k(n+1)$, $n \in \mathbb{N}$ *such that*

$$\bar{H}_{n+1} \circ (A_n + f_n) = A_n \circ \bar{H}_n + \bar{\tau}_n, \text{ for every } n \in \mathbb{N}. \tag{2.10}$$

In addition,

$$\sup_{n\in\mathbb{N}} \|\bar{H}_n - \mathrm{Id}\|_\infty < +\infty \text{ and } \bar{\tau}_n(x) = -\tau_n.$$

Moreover, in the case when either $K^c = \emptyset$ *or* $P_n^k f_{n-1} \equiv 0$ *for every* $k \in K^c$ *and* $n \geqslant 1$, *we have that* H_n *and* $\bar{H}_n$ *are homeomorphisms for each* $n \in \mathbb{N}$. *In addition,*

$$H_n \circ \bar{H}_n = \bar{H}_n \circ H_n = \mathrm{Id} \tag{2.11}$$

and

$$H_{n+1} \circ A_n = (A_n + f_n) \circ H_n, \tag{2.12}$$

for every $n \in \mathbb{N}$.

Remark 2.3. We observe that in the case when we have a good control of the perturbations along all the directions (i.e., $K^c = \emptyset$ or $P_n^k f_{n-1} \equiv 0$ for every $k \in K^c$ and $n \geqslant 1$), the previous result gives us a nonautonomous version of Grobman–Hartman's theorem. In the general case, however, we obtain a 'quasi-conjugacy' between systems (1.1) and (1.2), i.e. those are conjugated except for a given deviation (the factors τ_n and $\bar{\tau}_n$ in (2.9) and (2.10), respectively) in the directions along which we do not have any control.

Remark 2.4. Another important observation is the generality of theorem 2.2: we do not impose any condition on the linear maps $(A_n)_{n\in\mathbb{N}}$ but rather only on the allowed perturbations. Moreover, we allow for different levels of control on the perturbations along different directions. In particular, it generalizes previous results such as [**6**, theorem 2.1].

Remark 2.5. We note that the classical Palmer's theorem [**1**, **24**] corresponds to the particular case when:

- $|K^s| = |K^u| = 1$ and $K^c = \emptyset$;
- there exist D, $\lambda > 0$ such that

  $$\|\mathcal{A}(m,n)P_n\| \leqslant De^{-\lambda(m-n)} \quad \text{for } m \geqslant n,$$

  and

  $$\|\mathcal{A}(m,n)\tilde{P}_n\| \leqslant De^{-\lambda(n-m)} \quad \text{for } m \leqslant n,$$

  where $P_n = P_n^a$ and $\tilde{P}_n = P_n^b$ for $n \in \mathbb{N}$, $K^s = \{a\}$, $K^u = \{b\}$.

It is easy to verify that in this setting theorem 2.2 is applicable whenever (ν_l^k) and (μ_l^k) are constant sequences, and μ_l^k is sufficiently small.

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REMARK 2.6. Let K^s and K^u satisfy the same properties as in remark 2.5. We emphasize that in the case when $|K^c| = 1$, a result similar to theorem 2.2 was established (by using different techniques and under some additional assumptions) in [2, Theorem 3].

As an illustration of the broad applicability of Theorem 2.2 we provide the following simple example.

EXAMPLE 2.7. Take $X = \mathbb{R}^5$ and $K = \{1, 2, 3, 4, 5\}$. For each $k \in K$ and $n \in \mathbb{N}$, let P_n^k be the projection onto the k^{th} coordinate. Moreover, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of constant diagonal matrices given by

$$A_n = \text{diag}\left(\frac{1}{2}, 1, 1, 1, 2\right) \text{ for every } n \in \mathbb{N},$$

and consider $K^s = \{1, 2\}$, $K^c = \{3\}$ and $K^u = \{4, 5\}$. Take $\lambda_k = \frac{1}{5}$ for every $k \in K$ and

- $\nu_n^k = 1$ and $\mu_n^k = \frac{1}{10}$, for $k \in \{1, 5\}$ and $n \in \mathbb{N}$;
- $\nu_n^k = \frac{1}{2^n}$ and $\mu_n^k = \frac{1}{5\cdot 2^n}$, for $k \in \{2, 4\}$ and $n \in \mathbb{N}$.

Let $f_n \colon X \to X$, $f_n = (f_n^1, \ldots, f_n^5)$, $n \in \mathbb{N}$ be a sequence of continuous maps such that

- $\|f_{n-1}^k\|_\infty \leqslant 1$ and $Lip(f_{n-1}^k) \leqslant \frac{1}{10}$, for $k \in \{1, 5\}$ and $n \geqslant 1$;
- $\|f_{n-1}^k\|_\infty \leqslant \frac{1}{2^n}$ and $Lip(f_{n-1}^k) \leqslant \frac{1}{5\cdot 2^n}$, for $k \in \{2, 4\}$ and $n \geqslant 1$.

It is easy to verify that under the above assumptions, theorem 2.2 is applicable. Observe the different levels of control we allow along each direction: the more 'hyperbolic' a direction is, the less restrictive are the conditions on the perturbations along such a direction.

2.5. Proof of theorem 2.2

In this subsection we present the proof of theorem 2.2. For the sake of clarity, we will divide it into several steps.

Let $\mathcal{Y}$ denote the space of all sequences $\mathbf{h} = (h_n)_{n\in\mathbb{N}}$ of continuous maps $h_n : X \to X$ such that

$$\|\mathbf{h}\|_{\mathcal{Y}} := \sup_{n\in\mathbb{N}} \|h_n\|_\infty < +\infty.$$

It is easy to verify that $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space.

Construction of maps H_n:

Let us consider the operator $\mathcal{T} \colon \mathcal{Y} \to \mathcal{Y}$ given by

$$(\mathcal{T}\mathbf{h})_n(x) = \sum_{k\in K^s\cup K^u} (\mathcal{T}_k\mathbf{h})_n(x),$$

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where

- for $k \in K^s$, we set $(\mathcal{T}_k \mathbf{h})_0(x) = 0$ and

$$(\mathcal{T}_k \mathbf{h})_n(x) = \sum_{l=1}^{n} \mathcal{A}(n,l) P_l^k (f_{l-1}(\mathcal{A}(l-1,n)x + h_{l-1}(\mathcal{A}(l-1,n)x))),$$

  for $n \geqslant 1$;

- for $k \in K^u$, we define

$$(\mathcal{T}_k \mathbf{h})_n(x) = -\sum_{l=n+1}^{\infty} \mathcal{A}(n,l) P_l^k (f_{l-1}(\mathcal{A}(l-1,n)x + h_{l-1}(\mathcal{A}(l-1,n)x))),$$

  for every $n \in \mathbb{N}$ and $x \in X$.

Since $P_l^k = P_l^k P_l^k$, we have that

$$\begin{aligned} |(\mathcal{T}\mathbf{h})_n(x)| &\leqslant \sum_{k \in K^s \cup K^u} |(\mathcal{T}_k \mathbf{h})_n(x)| \\ &\leqslant \sum_{k \in K^s} |(\mathcal{T}_k \mathbf{h})_n(x)| + \sum_{k \in K^u} |(\mathcal{T}_k \mathbf{h})_n(x)| \\ &\leqslant \sum_{k \in K^s} \sum_{l=1}^{n} \|\mathcal{A}(n,l) P_l^k\| \cdot \|P_l^k f_{l-1}\|_\infty \\ &\quad + \sum_{k \in K^u} \sum_{l=n+1}^{\infty} \|\mathcal{A}(n,l) P_l^k\| \cdot \|P_l^k f_{l-1}\|_\infty, \end{aligned}$$

which combined with (2.2), (2.4) and (2.6) implies that

$$\sup_{n \in \mathbb{N}} \|(\mathcal{T}\mathbf{h})_n\|_\infty < +\infty.$$

Moreover, one can easily see that for every $\mathbf{h} \in \mathcal{Y}$ and $n \in \mathbb{N}$, $(\mathcal{T}\mathbf{h})_n$ is continuous. Hence, $\mathcal{T} \colon \mathcal{Y} \to \mathcal{Y}$ is well-defined. We now claim that $\mathcal{T} \colon \mathcal{Y} \to \mathcal{Y}$ is a contraction. Indeed, take $\mathbf{h}^i = (h_n^i)_{n \in \mathbb{Z}} \in \mathcal{Y}$, $i = 1, 2$. Using (2.3) and (2.7) we get that for each $k \in K^s$,

$$\begin{aligned} |(\mathcal{T}_k \mathbf{h}^1)_n(x) - (\mathcal{T}_k \mathbf{h}^2)_n(x)| &\leqslant \sum_{l=1}^{n} \|\mathcal{A}(n,l) P_l^k\| \mu_l^k \|h_{l-1}^1 - h_{l-1}^2\|_\infty \\ &\leqslant \lambda_k \|\mathbf{h}^1 - \mathbf{h}^2\|_{\mathcal{Y}}, \end{aligned}$$

for $x \in X$ and $n \in \mathbb{N}$. Similarly, for each $k \in K^u$,

$$\begin{aligned} |(\mathcal{T}_k \mathbf{h}^1)_n(x) - (\mathcal{T}_k \mathbf{h}^2)_n(x)| &\leqslant \sum_{l=n+1}^{\infty} \|\mathcal{A}(n,l) P_l^k\| \mu_l^k \|h_{l-1}^1 - h_{l-1}^2\|_\infty \\ &\leqslant \lambda_k \|\mathbf{h}^1 - \mathbf{h}^2\|_{\mathcal{Y}}, \end{aligned}$$

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for $x \in X$ and $n \in \mathbb{N}$. Consequently,

$$|(\mathcal{T}\mathbf{h}^1)_n(x) - (\mathcal{T}\mathbf{h}^2)_n(x)| \leqslant \sum_{k\in K^s\cup K^u} \lambda_k \|\mathbf{h}^1 - \mathbf{h}^2\|_{\mathcal{Y}}$$

for every $x \in X$ and $n \in \mathbb{N}$ and thus,

$$\|\mathcal{T}\mathbf{h}^1 - \mathcal{T}\mathbf{h}^2\|_{\mathcal{Y}} \leqslant \sum_{k\in K^s\cup K^u} \lambda_k \|\mathbf{h}^1 - \mathbf{h}^2\|_{\mathcal{Y}}.$$

Hence, by (2.8) we conclude that $\mathcal{T}$ is a contraction. Therefore, $\mathcal{T}$ has a unique fixed point $\mathbf{h} = (h_n)_{n\in\mathbb{N}} \in \mathcal{Y}$. Thus, we have that

$$h_{n+1}(A_n x) = (\mathcal{T}\mathbf{h})_{n+1}(A_n x) = \sum_{k\in K^s\cup K^u} (\mathcal{T}_k\mathbf{h})_{n+1}(A_n x), \tag{2.13}$$

for $x \in X$ and $n \in \mathbb{N}$. Now, for $k \in K^s$, $x \in X$ and $n \geqslant 1$, we have that

$$\begin{aligned}
&(\mathcal{T}_k\mathbf{h})_{n+1}(A_n x)\\
&\quad= \sum_{l=1}^{n+1} \mathcal{A}(n+1,l)P_l^k(f_{l-1}(\mathcal{A}(l-1,n+1)A_n x + h_{l-1}(\mathcal{A}(l-1,n+1)A_n x)))\\
&\quad= \sum_{l=1}^{n+1} \mathcal{A}(n+1,l)P_l^k(f_{l-1}(\mathcal{A}(l-1,n)x + h_{l-1}(\mathcal{A}(l-1,n)x)))\\
&\quad= A_n \sum_{l=1}^{n} \mathcal{A}(n,l)P_l^k(f_{l-1}(\mathcal{A}(l-1,n)x + h_{l-1}(\mathcal{A}(l-1,n)x)))\\
&\qquad+ P_{n+1}^k(f_n(x + h_n(x)))\\
&\quad= A_n(\mathcal{T}_k\mathbf{h})_n(x) + P_{n+1}^k(f_n(x + h_n(x))).
\end{aligned}$$

Similarly, for $k \in K^s$, $x \in X$ and $n = 0$, we observe that

$$(\mathcal{T}_k\mathbf{h})_1(A_0 x) = P_1^k(f_0(x + h_0(x))) = A_n(\mathcal{T}_k\mathbf{h})_0(x) + P_1^k(f_0(x + h_0(x))).$$

Finally, for $k \in K^u$, $x \in X$ and $n \in \mathbb{N}$, we have that

$$\begin{aligned}
&(\mathcal{T}_k\mathbf{h})_{n+1}(A_n x)\\
&\quad= -\sum_{l=n+2}^{\infty} \mathcal{A}(n+1,l)P_l^k(f_{l-1}(\mathcal{A}(l-1,n+1)A_n x + h_{l-1}(\mathcal{A}(l-1,n+1)A_n x)))\\
&\quad= -\sum_{l=n+2}^{\infty} \mathcal{A}(n+1,l)P_l^k(f_{l-1}(\mathcal{A}(l-1,n)x + h_{l-1}(\mathcal{A}(l-1,n)x)))\\
&\quad= -A_n \sum_{l=n+1}^{\infty} \mathcal{A}(n,l)P_l^k(f_{l-1}(\mathcal{A}(l-1,n)x + h_{l-1}(\mathcal{A}(l-1,n)x)))\\
&\qquad+ P_{n+1}^k(f_n(x + h_n(x)))\\
&\quad= A_n(\mathcal{T}_k\mathbf{h})_n(x) + P_{n+1}^k(f_n(x + h_n(x))).
\end{aligned}$$

Combining these observations with (2.1), (2.13) and the fact that $\mathbf{h}$ is a fixed point of $\mathcal{T}$, we obtain that

$$h_{n+1}(A_n x) = A_n h_n(x) + f_n(x + h_n(x)) - \sum_{k \in K^c} P_{n+1}^k(f_n(x + h_n(x))),$$

for $n \in \mathbb{N}$ and $x \in X$. Consequently, defining $H_n = \mathrm{Id} + h_n$, $n \in \mathbb{N}$, and $\tau_n : X \to \bigoplus_{k \in K^c} X_k(n+1)$ by $\tau_n(x) = -\sum_{k \in K^c} P_{n+1}^k(f_n(x))$ for $n \in \mathbb{N}$, we get that (2.9) holds.

Construction of maps $\bar{H}_n$:

We now consider $\bar{\mathbf{h}} = (\bar{h}_n)_{n \in \mathbb{N}} \in \mathcal{Y}$ given by

$$\bar{h}_n(x) = \sum_{k \in K^s \cup K^u} \bar{h}_n^k(x),$$

where

- for $k \in K^s$, we set $\bar{h}_0^k(x) = 0$ and

  $$\bar{h}_n^k(x) = -\sum_{l=1}^{n} \mathcal{A}(n,l) P_l^k f_{l-1}(\mathcal{F}(l-1,n)x) \quad \text{for } n \geqslant 1,$$

  where

  $$\mathcal{F}(m,n) = \begin{cases} F_{m-1} \circ \dots F_n & \text{for } m > n; \\ \mathrm{Id} & \text{for } m = n; \\ F_{m+1}^{-1} \circ \dots \circ F_n^{-1} & \text{for } m < n, \end{cases}$$

  and $F_n = A_n + f_n$, $n \in \mathbb{N}$;

- for $k \in K^u$ and $n \in \mathbb{N}$,

  $$\bar{h}_n^k(x) = \sum_{l=n+1}^{\infty} \mathcal{A}(n,l) P_l^k f_{l-1}(\mathcal{F}(l-1,n)x).$$

It follows easily from (2.2), (2.4) and (2.6) that indeed $\bar{\mathbf{h}} \in \mathcal{Y}$. Moreover, we observe that given $x \in X$ and $k \in K^s$, we have that

$$\begin{aligned} &\bar{h}_{n+1}^k(F_n(x)) \\ &= -\sum_{l=1}^{n+1} \mathcal{A}(n+1,l) P_l^k f_{l-1}(\mathcal{F}(l-1,n+1) F_n(x)) \\ &= -\sum_{l=1}^{n+1} \mathcal{A}(n+1,l) P_l^k f_{l-1}(\mathcal{F}(l-1,n)x) \\ &= -A_n \sum_{l=1}^{n} \mathcal{A}(n,l) P_l^k f_{l-1}(\mathcal{F}(l-1,n)x) - P_{n+1}^k f_n(x) \\ &= A_n \bar{h}_n^k(x) - P_{n+1}^k f_n(x), \end{aligned}$$

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for $n \geqslant 1$. Similarly, for $x \in X$ and $n = 0$, we observe that

$$\bar{h}_1^k(F_0(x)) = -P_1^k f_0(x) = A_0 \bar{h}_0^k(x) - P_1^k f_0(x).$$

Moreover, for $k \in K^u$, $x \in X$ and $n \in \mathbb{N}$, we have that

$$\begin{aligned}
&\bar{h}_{n+1}^k(F_n(x)) \\
&\quad = \sum_{l=n+2}^{\infty} \mathcal{A}(n+1, l) P_l^k f_{l-1}(\mathcal{F}(l-1, n+1) F_n(x)) \\
&\quad = \sum_{l=n+2}^{\infty} \mathcal{A}(n+1, l) P_l^k f_{l-1}(\mathcal{F}(l-1, n) x) \\
&\quad = A_n \sum_{l=n+1}^{\infty} \mathcal{A}(n, l) P_l^k f_{l-1}(\mathcal{F}(l-1, n) x) - P_{n+1}^k f_n(x) \\
&\quad = A_n \bar{h}_n^k(x) - P_{n+1}^k f_n(x).
\end{aligned}$$

Consequently, using (2.1) and recalling the definition of $\bar{\mathbf{h}}$, it follows that for every $n \in \mathbb{N}$ and $x \in X$,

$$\bar{h}_{n+1}(F_n(x)) = A_n \bar{h}_n(x) - f_n(x) + \sum_{k \in K^c} P_{n+1}^k f_n(x).$$

Thus, defining $\bar{H}_n = \mathrm{Id} + \bar{h}_n$, $n \in \mathbb{N}$, and $\bar{\tau}_n : X \to \bigoplus_{k \in K^c} X_k(n+1)$ by $\bar{\tau}_n(x) = \sum_{k \in K^c} P_{n+1}^k(f_n(x))$ for $n \in \mathbb{N}$, we conclude that (2.10) holds.

The cases when $K^c = \emptyset$ and $P_n^k f_{n-1} \equiv 0$ for every $k \in K^c$:

Suppose that either $K^c = \emptyset$ or $P_n^k f_{n-1} \equiv 0$ for every $k \in K^c$ and $n \in \mathbb{N}$. Hence, we have that $\tau_n = \bar{\tau}_n = 0$ for every $n \in \mathbb{N}$. In particular, (2.9) and (2.10) imply that

$$H_{n+1} \circ A_n = (A_n + f_n) \circ H_n \text{ and } \bar{H}_{n+1} \circ (A_n + f_n) = A_n \circ \bar{H}_n, \tag{2.14}$$

for every $n \in \mathbb{N}$. Hence, (2.12) holds. Moreover, it follows easily from (2.14) that

$$H_n(\mathcal{A}(n, m)x) = \mathcal{F}(n, m) H_m(x) \tag{2.15}$$

and

$$\bar{H}_n(\mathcal{F}(n, m)x) = \mathcal{A}(n, m) \bar{H}_m(x), \tag{2.16}$$

for every m, $n \in \mathbb{N}$.

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Recalling the definitions of $\bar{H}_n$ and H_n we get that for every $n \geqslant 1$ and $x \in X$,

$$
\begin{aligned}
\bar{H}_n(H_n(x)) &= H_n(x) + \bar{h}_n(H_n(x)) \\
&= x + h_n(x) + \bar{h}_n(H_n(x)) \\
&= x + \sum_{k \in K^s} \sum_{l=1}^{n} \mathcal{A}(n,l) P_l^k (f_{l-1}(\mathcal{A}(l-1,n)x + h_{l-1}(\mathcal{A}(l-1,n)x))) \\
&\quad - \sum_{k \in K^u} \sum_{l=n+1}^{\infty} \mathcal{A}(n,l) P_l^k (f_{l-1}(\mathcal{A}(l-1,n)x + h_{l-1}(\mathcal{A}(l-1,n)x))) \\
&\quad - \sum_{k \in K^s} \sum_{l=1}^{n} \mathcal{A}(n,l) P_l^k f_{l-1}(\mathcal{F}(l-1,n) H_n(x)) \\
&\quad + \sum_{k \in K^u} \sum_{l=n+1}^{\infty} \mathcal{A}(n,l) P_l^k f_{l-1}(\mathcal{F}(l-1,n) H_n(x)).
\end{aligned}
\tag{2.17}
$$

Now, by (2.15) it follows that

$$
\begin{aligned}
\mathcal{F}(l-1,n) H_n(x) &= H_{l-1}(\mathcal{A}(l-1,n)x) \\
&= \mathcal{A}(l-1,n)x + h_{l-1}(\mathcal{A}(l-1,n)x),
\end{aligned}
$$

which combined with (2.17) implies that $\bar{H}_n(H_n(x)) = x$ for every $x \in X$. The case when $n = 0$ can be treated similarly.

Our objective now is to show that $H_n(\bar{H}_n(x)) = x$ for every $x \in X$ and $n \in \mathbb{N}$. We start by observing that

$$
\begin{aligned}
H_n(\bar{H}_n(x)) &= \bar{H}_n(x) + h_n(\bar{H}_n(x)) \\
&= x + \bar{h}_n(x) + h_n(\bar{H}_n(x)).
\end{aligned}
$$

Consequently,

$$
H_n(\bar{H}_n(x)) - x = \bar{h}_n(x) + h_n(\bar{H}_n(x)). \tag{2.18}
$$

By analysing the right-hand side of (2.18), we have that

$$
\begin{aligned}
&\bar{h}_n(x) + h_n(\bar{H}_n(x)) \\
&\quad = - \sum_{k \in K^s} \sum_{l=1}^{n} \mathcal{A}(n,l) P_l^k f_{l-1}(\mathcal{F}(l-1,n)x) \\
&\qquad + \sum_{k \in K^u} \sum_{l=n+1}^{\infty} \mathcal{A}(n,l) P_l^k f_{l-1}(\mathcal{F}(l-1,n)x) \\
&\qquad + \sum_{k \in K^s} \sum_{l=1}^{n} \mathcal{A}(n,l) P_l^k (f_{l-1}(\mathcal{A}(l-1,n)\bar{H}_n(x) + h_{l-1}(\mathcal{A}(l-1,n)\bar{H}_n(x)))) \\
&\qquad - \sum_{k \in K^u} \sum_{l=n+1}^{\infty} \mathcal{A}(n,l) P_l^k (f_{l-1}(\mathcal{A}(l-1,n)\bar{H}_n(x) + h_{l-1}(\mathcal{A}(l-1,n)\bar{H}_n(x)))),
\end{aligned}
$$

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for $x \in X$ and $n \geqslant 1$. On the other hand, by using (2.16) we have that

$$\begin{aligned}\mathcal{A}(l-1,n)\bar{H}_n(x) + h_{l-1}(\mathcal{A}(l-1,n)\bar{H}_n(x)) &= H_{l-1}(\mathcal{A}(l-1,n)\bar{H}_n(x)) \\ &= H_{l-1}(\bar{H}_{l-1}(\mathcal{F}(l-1,n)x)).\end{aligned}$$

Thus, combining the previous observations we get that

$$\begin{aligned}&|\bar{h}_n(x) + h_n(\bar{H}_n(x))| \\ &\leqslant \sum_{k\in K^s}\sum_{l=1}^{n} \|\mathcal{A}(n,l)P_l^k\| \cdot |P_l^k f_{l-1}(H_{l-1}(\bar{H}_{l-1}(\mathcal{F}(l-1,n)x))) \\ &\quad - P_l^k f_{l-1}(\mathcal{F}(l-1,n)x)| \\ &\quad + \sum_{k\in K^u}\sum_{l=n+1}^{\infty} \|\mathcal{A}(n,l)P_l^k\| \cdot |P_l^k f_{l-1}(\mathcal{F}(l-1,n)x) \\ &\quad - P_l^k f_{l-1}(H_{l-1}(\bar{H}_{l-1}(\mathcal{F}(l-1,n)x)))| \\ &\leqslant \sum_{k\in K^s}\sum_{l=1}^{n} \|\mathcal{A}(n,l)P_l^k\|\mu_{l-1}^k |H_{l-1}(\bar{H}_{l-1}(\mathcal{F}(l-1,n)x))) - \mathcal{F}(l-1,n)x| \\ &\quad + \sum_{k\in K^u}\sum_{l=n+1}^{\infty} \|\mathcal{A}(n,l)P_l^k\|\mu_{l-1}^k |H_{l-1}(\bar{H}_{l-1}(\mathcal{F}(l-1,n)x))) - \mathcal{F}(l-1,n)x|.\end{aligned}$$

Therefore, using (2.18) it follows that

$$\begin{aligned}&|H_n(\bar{H}_n(x)) - x| \\ &\leqslant \sum_{k\in K^s}\sum_{l=1}^{n} \|\mathcal{A}(n,l)P_l^k\|\mu_{l-1}^k |H_{l-1}(\bar{H}_{l-1}(\mathcal{F}(l-1,n)x))) - \mathcal{F}(l-1,n)x| \\ &\quad + \sum_{k\in K^u}\sum_{l=n+1}^{\infty} \|\mathcal{A}(n,l)P_l^k|\mu_{l-1}^k |H_{l-1}(\bar{H}_{l-1}(\mathcal{F}(l-1,n)x))) - \mathcal{F}(l-1,n)x|.\end{aligned} \tag{2.19}$$

Now, since $\mathbf{h} = (h_n)_{n\in\mathbb{N}} \in \mathcal{Y}$ and $\bar{\mathbf{h}} = (\bar{h}_n)_{n\in\mathbb{N}} \in \mathcal{Y}$, it follows by (2.18) that $\mathbf{H} \circ \bar{\mathbf{H}} - \mathrm{Id} := (H_n \circ \bar{H}_n - \mathrm{Id})_{n\in\mathbb{N}} \in \mathcal{Y}$, which combined with (2.3), (2.5) and (2.19) implies that

$$\|\mathbf{H} \circ \bar{\mathbf{H}} - \mathrm{Id}\|_{\mathcal{Y}} \leqslant \sum_{k\in K^s\cup K^u} \lambda_k \|\mathbf{H} \circ \bar{\mathbf{H}} - \mathrm{Id}\|_{\mathcal{Y}}.$$

Thus, from (2.8) it follows that $\|\mathbf{H} \circ \bar{\mathbf{H}} - \mathrm{Id}\|_{\mathcal{Y}} = 0$, and consequently $H_n(\bar{H}_n(x)) = x$ for every $x \in X$ and $n \in \mathbb{N}$.

Summarizing, we have proved that (2.11) holds. In particular, we conclude that H_n and $\bar{H}_n$ are homeomorphisms for each $n \in \mathbb{N}$. The proof of theorem 2.2 is completed.

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3. The case of continuous time

The purpose of this section is to establish the version of theorem 2.2 for the case of continuous time. Let $A\colon [0, \infty) \to \mathcal{B}(X)$ and $f\colon [0, \infty) \times X \to X$ be continuous maps. We consider the associated semilinear differential equation

$$x' = A(t)x + f(t, x) \quad t \geqslant 0, \tag{3.1}$$

as well as the associated linear equation

$$x' = A(t)x \quad t \geqslant 0. \tag{3.2}$$

By $T(t, s)$ we will denote the evolution family associated to (3.2). Furthermore, $U(t, s)$ will denote the nonlinear evolution family corresponding to (3.1), i.e. $U(t, s)v = x(t)$, where $x\colon [0, \infty) \to X$ is the solution of (3.1) such that $x(s) = v$.

Let K be as in subsection 2.2. Suppose that for each $t \geqslant 0$ and $k \in K$ there is a projection $P^k(t)$ on X such that:

- $\sum_{k\in K} P^k(t) = \mathrm{Id}$;
- $P^k(t)P^l(t) = 0$ for $k, l \in K$, $k \neq l$;
- for $k \in K$, $t \mapsto P^k(t)$ is measurable.

Furthermore, we assume that there are Borel measurable functions $\mu^k, \nu^k\colon [0, \infty) \to [0, \infty)$ and positive numbers $\lambda_k > 0$, $k \in K^s \cup K^u$ such that:

- for $k \in K^s$,

$$\sup_t \int_0^t \|T(t, s)P^k(s)\|\nu^k(s)\, ds < +\infty, \tag{3.3}$$

  and

$$\sup_t \int_0^t \|T(t, s)P^k(s)\|\mu^k(s)\, ds \leqslant \lambda_k; \tag{3.4}$$

- for $k \in K^u$,

$$\sup_t \int_t^\infty \|T(t, s)P^k(s)\|\nu^k(s)\, ds < +\infty, \tag{3.5}$$

  and

$$\sup_t \int_t^\infty \|T(t, s)P^k(s)\|\mu^k(s)\, ds \leqslant \lambda_k. \tag{3.6}$$

The following is the version of theorem 2.2 in the present setting.

THEOREM 3.1. *Assume that the following conditions hold*:

- *for* $k \in K^s \cup K^u$ *and* $t \geqslant 0$,

$$\|P^k(t)f(t, \cdot)\|_\infty \leqslant \nu^k(t); \tag{3.7}$$

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- *for* $k \in K^s \cup K^u$, $t \geqslant 0$ *and* $x, y \in X$,

$$|P^k(t)f(t,x) - P^k(t)f(t,y)| \leqslant \mu^k(t)|x-y|; \tag{3.8}$$

- (2.8) *holds.*

Then,

- *there exists a continuous map* $H\colon [0,\infty)\times X \to X$ *such that if* $t \mapsto x(t)$ *is a solution of* (3.2)*, then* $t \mapsto H(t, x(t))$ *is a solution of*

$$x' = A(t)x + \sum_{k\in K^s\cup K^u} P^k(t)f(t,x); \tag{3.9}$$

- *there exists a continuous map* $\bar{H}\colon [0,\infty)\times X \to X$ *such that if* $t \mapsto y(t)$ *is a solution of* (3.1)*, then* $t \mapsto \bar{H}(t, y(t))$ *is a solution of*

$$x' = A(t)x + \sum_{k\in K^c} P^k(t)f(t,y(t))); \tag{3.10}$$

- *we have that*

$$\sup_t \|H(t,\cdot)-\mathrm{Id}\|_\infty < +\infty \quad \text{and} \quad \sup_t \|\bar{H}(t,\cdot)-\mathrm{Id}\|_\infty < +\infty. \tag{3.11}$$

Moreover, in the case when $K^c = \emptyset$ *or* $P^k(t)f(t,\cdot)\equiv 0$ *for* $t \geqslant 0$ *and* $k \in K^c$*, then* $H(t,\cdot)$ *and* $\bar{H}(t,\cdot)$ *are homeomorphisms for each* $t \geqslant 0$ *satisfying*

$$H(t,\bar{H}(t,x)) = \bar{H}(t,H(t,x)) = x, \tag{3.12}$$

$$H(t,T(t,s)x) = U(t,s)H(s,x) \quad \text{and} \quad \bar{H}(t,U(t,s)x) = T(t,s)\bar{H}(s,x), \tag{3.13}$$

for $t, s \geqslant 0$ *and* $x \in X$*.*

Proof. We follow closely the proof of theorem 2.2. Let $\mathcal{Y}$ denote the space of all continuous functions $h\colon [0,\infty)\times X \to X$ such that

$$\|h\|_{\mathcal{Y}} := \sup_{t\geqslant 0}\|h(t,\cdot)\|_\infty = \sup_{t,x}|h(t,x)| < +\infty.$$

Then, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space. We define an operator $\mathcal{T}\colon \mathcal{Y}\to\mathcal{Y}$ by

$$(\mathcal{T}h)(t,x) = \sum_{k\in K^s\cup K^u} (\mathcal{T}_k h)(t,x),$$

where

- for $k \in K^s$, we set

$$(\mathcal{T}_k h)(t,x) = \int_0^t T(t,s)P^k(s)(f(s,T(s,t)x + h(s,T(s,t)x)))\,ds,$$

for every $t \geqslant 0$ and $x \in X$;

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- for $k \in K^u$, we set

$$(\mathcal{T}_k h)(t,x) = -\int_t^\infty T(t,s)P^k(s)(f(s,T(s,t)x + h(s,T(s,t)x)))\,ds,$$

for every $t \geqslant 0$ and $x \in X$.

Observe that

$$\begin{aligned}|(\mathcal{T}h)(t,x)| &\leqslant \sum_{k\in K^s\cup K^u} |(\mathcal{T}_k h)(t,x)| \\ &\leqslant \sum_{k\in K^s} |(\mathcal{T}_k h)(t,x)| + \sum_{k\in K^u} |(\mathcal{T}_k h)(t,x)| \\ &\leqslant \sum_{k\in K^s} \int_0^t \|T(t,s)P^k(s)\| \cdot \|P^k(s)f(s,\cdot)\|_\infty\,ds \\ &\quad + \sum_{k\in K^u} \int_t^\infty \|T(t,s)P^k(s)\| \cdot \|P^k(s)f(s,\cdot)\|_\infty\,ds,\end{aligned}$$

which combined with (3.3), (3.5) and (3.7) implies that

$$\sup_{t\geqslant 0} \|(\mathcal{T}h)(t,\cdot)\|_\infty < +\infty.$$

This easily implies that $\mathcal{T}h \in \mathcal{Y}$. Take now h^1, $h^2 \in \mathcal{Y}$. By using (3.4) and (3.8) we get that for each $k \in K^s$,

$$\begin{aligned}|(\mathcal{T}_k h^1)(t,x) - (\mathcal{T}_k h^2)(t,x)| &\leqslant \int_0^t \|T(t,s)P^k(s)\|\mu^k(s)\|h^1(s,\cdot) - h^2(s,\cdot)\|_\infty\,ds \\ &\leqslant \lambda_k \|h^1 - h^2\|_{\mathcal{Y}},\end{aligned}$$

for $x \in X$ and $t \geqslant 0$. Similarly, by using (3.6) and (3.8), we have that for each $k \in K^u$,

$$\begin{aligned}|(\mathcal{T}_k h^1)(t,x) - (\mathcal{T}_k h^2)(t,x)| &\leqslant \int_t^\infty \|T(t,s)P^k(s)\|\mu^k(s)\|h^1(s,\cdot) - h^2(s,\cdot)\|_\infty\,ds \\ &\leqslant \lambda_k \|h^1 - h^2\|_{\mathcal{Y}},\end{aligned}$$

for $x \in X$ and $t \geqslant 0$. Consequently,

$$|(\mathcal{T}h^1)(t,x) - (\mathcal{T}h^2)(t,x)| \leqslant \sum_{k\in K^s\cup K^u} \lambda_k \|h^1 - h^2\|_{\mathcal{Y}}$$

for every $x \in X$ and $t \geqslant 0$ and thus,

$$\|\mathcal{T}h^1 - \mathcal{T}h^2\|_{\mathcal{Y}} \leqslant \sum_{k\in K^s\cup K^u} \lambda_k \|h^1 - h^2\|_{\mathcal{Y}}.$$

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Hence, by (2.8) we conclude that $\mathcal{T}$ is a contraction. Therefore, $\mathcal{T}$ has a unique fixed point $h \in \mathcal{Y}$. Therefore,

$$h(t, T(t,s)x) = (\mathcal{T}h)(t, T(t,s)x) = \sum_{k \in K^s \cup K^u} (\mathcal{T}_k h)(t, T(t,s)x), \tag{3.14}$$

for $x \in X$ and $t,\, s \geqslant 0$. Now, for $k \in K^s$, $x \in X$ and $t,\, s \geqslant 0$, we have that

$$\begin{aligned}
&(\mathcal{T}_k h)(t, T(t,s)x) \\
&\quad = \int_0^t T(t,r)P^k(r)(f(r, T(r,t)T(t,s)x + h(r, T(r,t)T(t,s)x)))\, dr \\
&\quad = \int_0^t T(t,r)P^k(r)(f(r, T(r,s)x + h(r, T(r,s)x)))\, dr \\
&\quad = T(t,s)\int_0^s T(s,r)P^k(r)(f(r, T(r,s)x + h(r, T(r,s)x)))\, dr \\
&\qquad + \int_s^t T(t,r)P^k(r)(f(r, T(r,s)x + h(r, T(r,s)x)))\, dr \\
&\quad = T(t,s)(\mathcal{T}_k h)(s,x) + \int_s^t T(t,r)P^k(r)(f(r, T(r,s)x + h(r, T(r,s)x)))\, dr.
\end{aligned}$$

Similarly, for $k \in K^u$, $x \in X$ and $t,\, s \geqslant 0$, we have that

$$\begin{aligned}
&(\mathcal{T}_k h)(t, T(t,s)x) \\
&\quad = -\int_t^\infty T(t,r)P^k(r)(f(r, T(r,t)T(t,s)x + h(r, T(r,t)T(t,s)x)))\, dr \\
&\quad = -\int_t^\infty T(t,r)P^k(r)(f(r, T(r,s)x + h(r, T(r,s)x)))\, dr \\
&\quad = -T(t,s)\int_s^\infty T(s,r)P^k(r)(f(r, T(r,s)x + h(r, T(r,s)x)))\, dr \\
&\qquad + \int_s^t T(t,r)P^k(r)(f(r, T(r,s)x + h(r, T(r,s)x)))\, dr \\
&\quad = T(t,s)(\mathcal{T}_k h)(s,x) + \int_s^t T(t,r)P^k(r)(f(r, T(r,s)x + h(r, T(r,s)x)))\, dr.
\end{aligned}$$

Combining these observations with (3.14) and the fact that h is a fixed point of $\mathcal{T}$, we obtain that

$$\begin{aligned}
h(t, T(t,s)x) &= T(t,s)h(s,x) + \int_s^t T(t,r)(f(r, T(r,s)x + h(r, T(r,s)x)))\, dr \\
&\qquad - \sum_{k \in K^c} \int_s^t T(t,r)P^k(r)(f(r, T(r,s)x + h(r, T(r,s)x)))\, dr,
\end{aligned}$$

for $t,\, s \geqslant 0$ and $x \in X$. By differentiating the above equality, we easily conclude that if $t \mapsto x(t)$ is a solution of (3.2), then $t \mapsto H(t,\, x(t))$ is a solution of (3.9), where $H(t,\, x) := x + h(t,\, x)$.

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We now consider $\bar{h} \in \mathcal{Y}$ given by

$$\bar{h}(t,x) = \sum_{k \in K^s \cup K^u} \bar{h}^k(t,x),$$

where

- for $k \in K^s$ and $t \geqslant 0$,

$$\bar{h}^k(t,x) := -\int_0^t T(t,s)P^k(s)f(s,U(s,t)x)\,ds;$$

- for $k \in K^u$ and $t \geqslant 0$,

$$\bar{h}^k(t,x) := \int_t^\infty T(t,s)P^k(s)f(s,U(s,t)x)\,ds.$$

It follows easily from (3.3), (3.5) and (3.7) that indeed $\bar{h} \in \mathcal{Y}$. Moreover, we observe that given $x \in X$ and $k \in K^s$, we have that

$$\begin{aligned}
&\bar{h}^k(t,U(t,s)x)\\
&\quad = -\int_0^t T(t,r)P^k(r)f(r,U(r,t)U(t,s)x)\,dr\\
&\quad = -\int_0^t T(t,r)P^k(r)f(r,U(r,s)x)\,dr\\
&\quad = -T(t,s)\int_0^s T(s,r)P^k(r)f(r,U(r,s)x)\,dr - \int_s^t T(t,r)P^k(r)f(r,U(r,s)x)\,dr\\
&\quad = T(t,s)\bar{h}^k(s,x) - \int_s^t T(t,r)P^k(r)f(r,U(r,s)x)\,dr,
\end{aligned}$$

for $t,\ s \geqslant 0$ and $x \in X$. Moreover, for $k \in K^u$, $x \in X$ and $t,\ s \geqslant 0$, we have that

$$\begin{aligned}
&\bar{h}^k(t,U(t,s)x)\\
&\quad = \int_t^\infty T(t,r)P^k(r)f(r,U(r,t)U(t,s)x)\,dr\\
&\quad = \int_t^\infty T(t,r)P^k(r)f(r,U(r,s)x)\,dr\\
&\quad = T(t,s)\int_s^\infty T(s,r)P^k(r)f(r,U(r,s)x)\,dr - \int_s^t T(t,r)P^k(r)f(r,U(r,s)x)\,dr\\
&\quad = T(t,s)\bar{h}^k(s,x) - \int_s^t T(t,r)P^k(r)f(r,U(r,s)x)\,dr.
\end{aligned}$$

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Consequently, it follows that for $t,\, s \geqslant 0$ we have that

$$\bar{h}(t, U(t,s)x) = T(t,s)\bar{h}(s,x) - \int_s^t T(t,r) f(r, U(r,s)x)\, dr + \sum_{k\in K^c} \int_s^t T(t,r)P^k(r) f(r, U(r,s)x)\, dr.$$

From this we easily conclude that if $t \mapsto y(t)$ is a solution of (3.1), then $t \mapsto \bar{H}(t,\, y(t))$ is a solution of (3.10), where $\bar{H}(t,\, x) = x + \bar{h}(t,\, x)$. Finally, we observe that since $h,\, \bar{h} \in \mathcal{Y}$, we have that (3.11) holds.

Suppose now that either $K^c = \emptyset$ or $P^k(t) f(t,\, \cdot) \equiv 0$ for every $k \in K^c$ and $t \in \mathbb{R}$. From the previous observations, we conclude that (3.13) holds for $t,\, s \geqslant 0$ and $x \in X$. Moreover, for every $t \geqslant 0$ and $x \in X$, we have that

$$\begin{aligned}
\bar{H}(t, H(t,x)) &= H(t,x) + \bar{h}(t, H(t,x)) \\
&= x + h(t,x) + \bar{h}(t, H(t,x)) \\
&= x + \sum_{k\in K^s} \int_0^t T(t,s)P^k(s)(f(s, T(s,t)x + h(s, T(s,t)x)))\, ds \\
&\quad - \sum_{k\in K^u} \int_t^\infty T(t,s)P^k(s)(f(s, T(s,t)x + h(s, T(s,t)x)))\, ds \\
&\quad - \sum_{k\in K^s} \int_0^t T(t,s)P^k(s) f(s, U(s,t)H(t,x))\, ds \\
&\quad + \sum_{k\in K^u} \int_t^\infty T(t,s)P^k(s) f(s, U(s,t)H(t,x))\, ds.
\end{aligned}$$

By applying (3.13), we conclude that $\bar{H}(t,\, H(t,\, x)) = x$ for $x \in X$ and $t \geqslant 0$.

We now claim that $H(t,\, \bar{H}(t,\, x)) = x$ for $x \in X$ and $t \geqslant 0$. Observe that

$$H(t, \bar{H}(t,x)) - x = \bar{h}(t,x) + h(t, \bar{H}(t,x)). \tag{3.15}$$

For $t \geqslant 0$, we have that

$$\begin{aligned}
&\bar{h}(t,x) + h(t, \bar{H}(t,x)) \\
&= - \sum_{k\in K^s} \int_0^t T(t,s)P^k(s) f(s, U(s,t)x)\, ds \\
&\quad + \sum_{k\in K^u} \int_t^\infty T(t,s)P^k(s) f(s, U(s,t)x)\, ds \\
&\quad + \sum_{k\in K^s} \int_0^t T(t,s)P^k(s)(f(s, T(s,t)\bar{H}(t,x) + h(s, T(s,t)\bar{H}(t,x))))\, ds \\
&\quad - \sum_{k\in K^u} \int_t^\infty T(t,s)P^k(s)(f(s, T(s,t)\bar{H}(t,x) + h(s, T(s,t)\bar{H}(t,x))))\, ds.
\end{aligned}$$

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By (3.13), we have that

$$\begin{aligned}T(s,t)\bar{H}(t,x)+h(s,T(s,t)\bar{H}(t,x)) &= H(s,T(s,t)\bar{H}(t,x))\\ &= H(s,\bar{H}(s,U(s,t)x)),\end{aligned}$$

and thus

$$\begin{aligned}&|\bar{h}(t,x)+h(t,\bar{H}(t,x))|\\ &\leqslant \sum_{k\in K^s}\int_0^t \|T(t,s)P^k(s)\|\cdot|P^k(s)f(s,H(s,\bar{H}(s,U(s,t)x)))\\ &\quad - P^k(s)f(s,U(s,t)x)|\,ds\\ &\quad + \sum_{k\in K^u}\int_t^\infty \|T(t,s)P^k(s)\|\cdot|P^k(s)f(s,U(s,t)x)\\ &\quad - P^k(s)f(s,H(s,\bar{H}(s,U(s,t)x)))|\,ds\\ &\leqslant \sum_{k\in K^s}\int_0^t \|T(t,s)P^k(s)\|\mu^k(s)|H(s,\bar{H}(s,U(s,t)x)))-U(s,t)x|\,ds\\ &\quad + \sum_{k\in K^u}\int_t^\infty \|T(t,s)P^k(s)\|\mu^k(s)|H(s,\bar{H}(s,U(s,t)x)))-U(s,t)x|\,ds.\end{aligned}$$

Therefore, using (3.15) it follows that

$$\begin{aligned}&|H(t,\bar{H}(t,x))-x|\\ &\leqslant \sum_{k\in K^s}\int_0^t \|T(t,s)P^k(s)\|\mu^k(s)|H(s,\bar{H}(s,U(s,t)x)))-U(s,t)x|\,ds\\ &\quad + \sum_{k\in K^u}\int_t^\infty \|T(t,s)P^k(s)|\mu^k(s)|H(s,\bar{H}(s,U(s,t)x)))-U(s,t)x|\,ds.\end{aligned}$$

Set

$$G(t,x)=H(t,\bar{H}(t,x))-x, \quad t\geqslant 0,\ x\in X.$$

Now, since h, $\bar{h}\in\mathcal{Y}$, it follows from (3.15) that $G\in\mathcal{Y}$, which combined with (3.4) and (3.6) implies that

$$\|G\|_{\mathcal{Y}}\leqslant \sum_{k\in K^s\cup K^u}\lambda_k\|G\|_{\mathcal{Y}}.$$

Thus, from (2.8) it follows that $G=0$, and consequently $H(t,\ \bar{H}(t,\ x))=x$ for every $x\in X$ and $t\geqslant 0$. We conclude that (3.12) holds which completes the proof of the theorem. □

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EXAMPLE 3.2. Let X and K be as in example 2.7. For $t \geqslant 0$ and $k \in K$, let $P^k(t)$ be the projection onto the k^{th} coordinate. Moreover, take

$$A(t) = \operatorname{diag}(-1, 0, 0, 0, 1) \quad t \geqslant 0,$$

and consider $K^s = \{1, 2\}$, $K^c = \{3\}$ and $K^u = \{4, 5\}$. Take $\lambda_k = \frac{1}{5}$ for every $k \in K$ and

- $\nu^k(t) = 1$ and $\mu^k(t) = \frac{1}{5}$, for $k \in \{1, 5\}$ and $t \geqslant 0$;
- $\nu^k(t) = e^{-t}$ and $\mu_n^k = \frac{1}{5}e^{-t}$, for $k \in \{2, 4\}$ and $t \geqslant 0$.

Let $f\colon [0, \infty) \times X \to X$, $f = (f^1, \ldots, f^5)$ be a continuous map such that

- $\|f^k(t, \cdot)\|_\infty \leqslant 1$ and $Lip(f^k(t, \cdot)) \leqslant \frac{1}{5}$, for $k \in \{1, 5\}$ and $t \geqslant 0$;
- $\|f^k(t, \cdot)\|_\infty \leqslant e^{-t}$ and $Lip(f^k(t, \cdot)) \leqslant \frac{1}{5}e^{-t}$, for $k \in \{2, 4\}$ and $t \geqslant 0$.

It is easy to verify that under the above assumptions, theorem 3.1 is applicable.

Acknowledgments

L.B. was partially supported by a CNPq-Brazil PQ fellowship under Grants No. 306484/2018-8 and 307633/2021-7. D.D. was supported in part by Croatian Science Foundation under the project IP-2019-04-1239 and by the University of Rijeka under the projects uniri-prirod-18-9 and uniri-prprirod-19-16.

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