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Elliptic Springer theory

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Abstract

We introduce an elliptic version of the Grothendieck–Springer sheaf and establish elliptic analogues of the basic results of Springer theory. From a geometric perspective, our constructions specialize geometric Eisenstein series to the resolution of degree-zero, semistable G-bundles by degree-zero B-bundles over an elliptic curve E. From a representation theory perspective, they produce a full embedding of representations of the elliptic or double affine Weyl group into perverse sheaves with nilpotent characteristic variety on the moduli of G-bundles over E. The resulting objects are principal series examples of elliptic character sheaves, objects expected to play the role of character sheaves for loop groups.

1. Introduction

Let G be a connected reductive complex algebraic group, $B \subset G$ a Borel subgroup, $N \subset B$ its unipotent radical, and $H = B/N$ the universal Cartan. Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$, and $\mathfrak{h}$ denote the corresponding Lie algebras. Let W denote the Weyl group.

1.1 Springer theory

Let us begin by recalling the standard sheaf-theoretic formulations [Lus81, BM81, KL80] of Springer theory [Spr76, Spr78, Spr82]. We will emphasize an interpretation in terms of geometric Eisenstein series [Lau88, BG96] but in the nonstandard setting of singular curves of arithmetic genus one. (See [Nad11] for a parallel symplectic treatment of Springer theory, and [BN08] for a survey of moduli of bundles on cubic curves.)

1.1.1 *Rational setting.* Let $X_{\mathfrak{g}}$ be the flag variety of Borel subalgebras (or equivalently, subgroups), and $\mathcal{N} \subset \mathfrak{g}$ the nilpotent cone. The Springer resolution

$$\mu_{\mathcal{N}} : \widetilde{\mathcal{N}} \simeq T^* X_{\mathfrak{g}} \simeq \{v \in \mathfrak{b}\} \subset \mathcal{N} \times X_{\mathfrak{g}} \longrightarrow \mathcal{N}$$

is a semi-small map and so the Springer sheaf $S_{\mathcal{N}} = \mu_{\mathcal{N}!} \mathbb{C}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}]$ is perverse. The endomorphisms of $S_{\mathcal{N}}$ as a perverse sheaf are equivalent to the group algebra $\mathbb{C}[W]$ of the Weyl group. The Grothendieck–Springer resolution

$$\mu_{\mathfrak{g}} : \widetilde{\mathfrak{g}} \simeq \{v \in \mathfrak{b}\} \subset \mathfrak{g} \times X_{\mathfrak{g}} \longrightarrow \mathfrak{g}$$

is a small map and its restriction to the regular semisimple locus $\mathfrak{g}^{\mathrm{rs}} \subset \mathfrak{g}$ is the natural W-cover. Hence the Grothendieck–Springer sheaf $S_{\mathfrak{g}} = \mu_{\mathfrak{g}!} \mathbb{C}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}]$ is the middle extension of the natural $\mathbb{C}[W]$-regular local system over $\mathfrak{g}^{\mathrm{rs}} \subset \mathfrak{g}$. Thus the endomorphisms of $S_{\mathfrak{g}}$ as a perverse sheaf are

equivalent to the group algebra $\mathbb{C}[W]$. Furthermore, under any invariant isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$, one can identify $S_\mathcal{N}$ and $S_\mathfrak{g}$ as Fourier transforms of each other [Gin83, HK84]. Thus Springer theory compatibly realizes representations of W as perverse sheaves on $\mathcal{N}$ and $\mathfrak{g}$.

Taking quotients by the adjoint action of G, one can identify the equivariant Grothendieck–Springer resolution with the natural induction map of adjoint quotients

$$\mu_\mathfrak{g} : \widetilde{\mathfrak{g}}/G \simeq \mathfrak{b}/B \longrightarrow \mathfrak{g}/G.$$

For a cuspidal cubic curve $E_{\mathrm{cusp}} \simeq \mathbb{G}_a \cup \{\infty\}$ (or any simply-connected projective curve of arithmetic genus one), this admits the interpretation as the induction map

$$\mu_\mathfrak{g} : \mathrm{Bun}_B^0(E_{\mathrm{cusp}}) \longrightarrow \mathrm{Bun}_G^{\mathrm{ss},0}(E_{\mathrm{cusp}})$$

from degree-zero B-bundles to degree-zero, semistable G-bundles. Any degree-zero, semistable bundle pulls back to the trivial bundle along the normalization map $\mathbb{P}^1 \to E_{\mathrm{cusp}}$, and the Lie algebra appears as descent data at the cusp $\{\infty\}$. The correspondence of adjoint quotients

$$\mathfrak{h}/H \xleftarrow{\nu_\mathfrak{g}} \mathfrak{b}/B \xrightarrow{\mu_\mathfrak{g}} \mathfrak{g}/G$$

admits the interpretation as the correspondence of moduli of bundles

$$\mathrm{Bun}_H^0(E_{\mathrm{cusp}}) \xleftarrow{\nu_\mathfrak{g}} \mathrm{Bun}_B^0(E_{\mathrm{cusp}}) \xrightarrow{\mu_\mathfrak{g}} \mathrm{Bun}_G^{\mathrm{ss},0}(E_{\mathrm{cusp}}).$$

The W-action on the Grothendieck–Springer sheaf $S_\mathfrak{g} = \mu_{\mathfrak{g}!}\nu_\mathfrak{g}^* \mathbb{C}_{\mathfrak{h}/H}[\dim \mathfrak{g}]$ reflects the functional equation of the geometric Eisenstein series construction applied to the constant sheaf.

Remark 1.1. In addition to the endomorphisms of $S_\mathfrak{g}$ as a perverse sheaf, the differential graded algebra of endomorphisms of $S_\mathfrak{g}$ as an equivariant complex can be calculated:

$$\mathrm{End}_{D_c^\flat(\mathfrak{g}/G)}(S_\mathfrak{g}) \simeq H^*(BH) \rtimes \mathbb{C}[W] \simeq \mathrm{Sym}^*(\mathfrak{h}^\vee[-2]) \rtimes \mathbb{C}[W].$$

Thus the full subcategory of $D_c^\flat(\mathfrak{g}/G)$ generated by $S_\mathfrak{g}$ is equivalent to the category of finitely-generated $H^*(BH) \rtimes \mathbb{C}[W]$-modules. (See [Rid13] for a definitive account and mixed version.)

Going further, the category of finitely-generated $H^*(BH) \rtimes \mathbb{C}[W]$-modules is equivalent to the full subcategory of $D_c^\flat((\mathfrak{h}/H)/W)$ generated by the pushforward $q_!\mathbb{C}_{\mathfrak{h}/H}$ of the constant sheaf along the natural quotient map

$$q : \mathfrak{h}/H \longrightarrow (\mathfrak{h}/H)/W.$$

The geometric Eisenstein series construction $\mu_{\mathfrak{g}!}\nu_\mathfrak{g}^*[\dim \mathfrak{g}]$ descends to a fully faithful embedding

$$D_c^\flat((\mathfrak{h}/H)/W) \hookrightarrow D_c^\flat(\mathfrak{g}/G)$$

which recovers Springer theory on the full subcategory generated by $q_!\mathbb{C}_{\mathfrak{h}/H}$. (See [Gun13] where the theory is in fact established in the group-theoretic setting and for all $\mathcal{D}$-modules.)

1.2 Trigonometric setting

Now let us expand our scope to the group-theoretic Grothendieck–Springer resolution

$$\mu_G : \widetilde{G} \simeq \{g \in B\} \subset G \times X_{\mathfrak{g}} \longrightarrow G.$$

One recovers the linear Grothendieck–Springer resolution $\mu_{\mathfrak{g}} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ by deforming to the normal cone of the identity of G. Taking quotients by the adjoint action of G, we obtain the natural induction map of adjoint quotients

$$\mu_G : \widetilde{G}/G \simeq B/B \longrightarrow G/G.$$

For a nodal cubic curve $E_{\mathrm{node}} \simeq \mathbb{G}_m \cup \{\infty\}$ (or any projective curve of arithmetic genus one with fundamental group $\mathbb{Z}$), this admits the interpretation as the induction map

$$\mu_G : \mathrm{Bun}_B^0(E_{\mathrm{node}}) \longrightarrow \mathrm{Bun}_G^{\mathrm{ss},0}(E_{\mathrm{node}})$$

from degree-zero B-bundles to degree-zero, semistable G-bundles. Any degree-zero, semistable bundle pulls back to the trivial bundle along the normalization map $\mathbb{P}^1 \to E_{\mathrm{node}}$, and the group appears as descent data identifying the two preimages of the node $\{\infty\}$. The correspondence of adjoint quotients

$$H/H \xleftarrow{\nu_G} B/B \xrightarrow{\mu_G} G/G$$

admits the interpretation as the correspondence of moduli of bundles

$$\mathrm{Bun}_H^0(E_{\mathrm{node}}) \xleftarrow{\nu_G} \mathrm{Bun}_B^0(E_{\mathrm{node}}) \xrightarrow{\mu_G} \mathrm{Bun}_G^{\mathrm{ss},0}(E_{\mathrm{node}}).$$

We will focus on the geometric Eisenstein series construction

$$\mu_{G!}\nu_G^*[\dim G] : D_c^\flat(H/H) \longrightarrow D_c^\flat(G/G).$$

The fundamental group $\pi_1(H)$ is the coweight lattice $\Lambda_H = \mathrm{Hom}(\mathbb{G}_m, H)$ with spectrum the dual torus $H^\vee = \mathrm{Spec}\,\mathbb{C}[\Lambda_H]$. Thus finite-rank W-equivariant local systems on H correspond to finite-dimensional representations of the affine Weyl group $W_{\mathrm{aff}} = \Lambda_H \rtimes W$ which in turn correspond to W-equivariant coherent sheaves on $H^\vee$ with finite support.

Starting from a finite-rank local system $\mathcal{L}$, the corresponding Grothendieck–Springer sheaf

$$S_{G,\mathcal{L}} = \mu_{G!}\nu_G^*\mathcal{L}[\dim G]$$

is a perverse sheaf.

Example 1.2. If we begin with the trivial local system $\mathcal{L}_0$ with its tautological W-equivariant structure, the endomorphisms of the resulting Grothendieck–Springer sheaf

$$S_{G,0} \simeq \mu_{G!}\mathbb{C}_{\widetilde{G}}[\dim G]$$

are equivalent to the group algebra $\mathbb{C}[W]$.

If we begin with the universal local system $\mathcal{L}_{\mathrm{univ}}$ corresponding to the natural W_{aff}-representation $\mathbb{C}[\Lambda_H]$ and in turn to the structure sheaf $\mathcal{O}_{H^\vee}$, we obtain the universal Grothendieck–Springer sheaf

$$S_{G,\mathrm{univ}} = \mu_{G!}\nu_G^*\mathcal{L}_{\mathrm{univ}}[\dim G].$$

Although $S_{G,\mathrm{univ}}$ is not constructible, it is cohomologically bounded, and informally one can view it as a perverse sheaf with endomorphisms equivalent to the group algebra $\mathbb{C}[W_{\mathrm{aff}}]$.

Remark 1.3. Suppose a W-equivariant local system $\mathcal{L}$ comes from a finite-dimensional $\mathbb{C}[\Lambda_H]^W$-module, or in other words, a coherent sheaf on $H^\vee /\!/ W$ with finite support. Then we may lift $\mathcal{L}$ to a module over the Harish-Chandra center $\mathfrak{z}\mathfrak{g} \simeq \mathbb{C}[\mathfrak{h}]^W$, or in other words, to a coherent sheaf on $\mathfrak{h}^\vee /\!/ W$. If we view this lift as a generalized eigenvalue for $\mathfrak{z}\mathfrak{g}$, the $\mathcal{D}$-module corresponding to $S_{G,\mathcal{L}}$ is equivalent to the Harish-Chandra system of differential equations on G where the differential operators from $\mathfrak{z}\mathfrak{g}$ are prescribed to act by this generalized eigenvalue.

In summary, the geometric Eisenstein series construction $\mu_{G!}\nu_G^*[\dim G]$ descends to a fully faithful embedding

$$\mathrm{Loc}_{\mathrm{fin}}(H/W) \simeq \mathbb{C}[W_{\mathrm{aff}}]\text{-mod}_{\mathrm{fin}} \simeq \mathrm{Coh}_{\mathrm{fin}}(H^\vee/W) \hookrightarrow \mathrm{Perv}(G/G)$$

with domain finite-rank W-equivariant local systems on H, or equivalently, finite-dimensional W_{aff}-representations, or equivalently, W-equivariant coherent sheaves on $H^\vee$ with finite support, into G-equivariant perverse sheaves on G.

Remark 1.4. Going beyond abelian categories, one can keep track of the derived structure of H-equivariance, and also allow arbitrary W-equivariant constructible complexes on H/H. The geometric Eisenstein series construction $\mu_{G!}\nu_G^*[\dim G]$ descends to a fully faithful embedding

$$D_c^\flat((H/H)/W) \hookrightarrow D_c^\flat(G/G).$$

For example, the differential graded algebra of endomorphisms of $S_{G,0}$ is the twisted product $H^*(H/H) \rtimes \mathbb{C}[W]$, and that of $S_{G,\mathrm{univ}}$ is the twisted product $H^*(BH) \rtimes \mathbb{C}[W_{\mathrm{aff}}]$. (See [Gun13] where the theory is in fact established for all $\mathcal{D}$-modules.)

Remark 1.5. Continuing with the derived structure of H-equivariance in the picture, the derived category of local systems on $H/H \simeq H \times BH$ is equivalent to modules over $\mathbb{C}[\Lambda_H] \otimes \mathrm{Sym}^*(\mathfrak{h}[1])$ via the identification $H_{-*}(H) \simeq \mathrm{Sym}^*(\mathfrak{h}[1])$. Observe that the Langlands parameter moduli $\mathrm{Loc}_{H^\vee}(E_{\mathrm{node}})$ of $H^\vee$-local systems on E_{node} admits the presentation

$$\mathrm{Loc}_{H^\vee}(E_{\mathrm{node}}) \simeq H^\vee \times \mathrm{SpecSym}^*(\mathfrak{h}[1]) \times BH^\vee.$$

Thus one can view the natural domain of the geometric Eisenstein series embedding as W-equivariant coherent sheaves with finite support and trivial $H^\vee$-equivariant structure on $\mathrm{Loc}_{H^\vee}(E_{\mathrm{node}})$. (A similar interpretation applies in the rational setting discussed earlier, where the moduli $\mathrm{Loc}_{H^\vee}(E_{\mathrm{cusp}})$ is more simply the product $\mathrm{SpecSym}^*(\mathfrak{h}[1]) \times BH^\vee$.)

1.3 Elliptic Springer theory

From a geometric viewpoint, this paper extends the above story from the rational and trigonometric settings to the elliptic setting of smooth elliptic curves. Via the restriction of geometric Eisenstein series to degree-zero semistable bundles, we introduce an elliptic version of Grothendieck–Springer sheaves and calculate their endomorphisms. For degree-zero semistable bundles in arithmetic genus one, the induction map from B-bundles to G-bundles is already proper, and so there is no need for the intricacies of Laumon or Drinfeld compactifications. The Weyl group symmetry of the construction is a simple form of the functional equation and admits a straightforward verification. The construction also works universally over the moduli of smooth elliptic curves (compatibility with the rational and trigonometric constructions at the boundary).

As discussed below in more detail, our primary motivation stems from understanding character sheaves for loop groups in the guise of perverse sheaves with nilpotent singular support on the moduli of G-bundles on a smooth elliptic curve E. Within this framework, the Grothendieck–Springer sheaves produced by the following theorem form the *elliptic principal series*, or geometric avatars of principal series representations of loop groups. To simplify the story, let us assume that the derived group $[G, G] \subset G$ is simply connected.

THEOREM 1.6. *For a smooth elliptic curve E, there is a fully faithful embedding*

$$\mathbb{C}[W_E]\text{-mod}_{\mathrm{fin}} \hookrightarrow \mathrm{Perv}_{\mathcal{N}}(\mathrm{Bun}_G^{\mathrm{ss}}(E))$$

from finite-dimensional representations of the elliptic or double affine Weyl group

$$W_E = (\pi_1(E) \otimes \Lambda_H) \rtimes W$$

to perverse sheaves with nilpotent singular support on the moduli of semistable G-bundles on E.

Remark 1.7. The domain category $\mathbb{C}[W_E]\text{-mod}_{\mathrm{fin}}$ of the theorem has two Langlands dual descriptions: on the one hand, it is equivalent to finite-rank W-equivariant local systems on the automorphic moduli $\mathrm{Bun}_H^0(E)$ of degree-zero H-bundles on E; on the other hand, it is equivalent to W-equivariant coherent sheaves with finite support and trivial equivariance for automorphisms on the Langlands parameter moduli $\mathrm{Loc}_{H^\vee}(E)$. One can view the theorem as a small but interesting part of the geometric Langlands correspondence for the elliptic curve E.

Remark 1.8. Geometric Eisenstein series in genus one, as well as modified versions such as the elliptic Grothendieck–Springer sheaves of the theorem, are objects of the elliptic Hall category, introduced and studied in depth by Schiffmann and Vasserot [Sch11, Sch12, SV13, SV11, SV12]. Notably, the trace functions of geometric Eisenstein series are identified with Macdonald's symmetric functions. The elliptic Hall algebra, the Grothendieck group spanned by geometric Eisenstein series, is identified with a variant of Cherednik's double affine Hecke algebra and related to K-groups of Hilbert schemes of points. One can view elliptic Springer theory as a categorical aspect of elliptic Hall algebras.

Remark 1.9. With the results of this paper in hand, it is not difficult to adapt the arguments of the trigonometric case [Gun13] to understand the monadic structure of the elliptic Grothendieck–Springer construction and enhance the theorem to a statement on the level of derived categories. What results is a fully faithful embedding

$$D_c^\flat(\mathrm{Bun}_H^0(E)/W) \hookrightarrow D_c^\flat(\mathrm{Bun}_G^{\mathrm{ss}}(E))$$

which recovers the statement of the theorem at the level of abelian categories of local systems and perverse sheaves.

For a precise account of the functor of Theorem 1.6 and further details, the reader could proceed to §2. We conclude the present introduction with an informal discussion of motivation from the theory of character sheaves, in particular for loop groups. This is not needed to read §2 and the arguments in the subsequent sections.

1.3.1 *Brief recollection about character sheaves.* Our interest in Springer theory stems from its central role in Lusztig's theory of character sheaves [Lus85]. In the traditional group-theoretic setting, the Grothendieck–Springer sheaves $S_{G,\lambda}$ attached to W-invariant characters $\lambda : \Lambda_H \to \mathbb{C}^*$ provide the geometric avatars of principal series representations of finite groups of Lie type.

For each W-invariant character $\lambda : \Lambda_H \to \mathbb{C}^*$, the category $\mathcal{C}h_G^\lambda$ of character sheaves with central character λ forms the categorical Hochschild homology of the monoidal λ-monodromic Hecke category $\mathcal{H}_G^\lambda$ [BN09]. Any (sufficiently finite) $\mathcal{H}_G^\lambda$-module category defines a character object in the categorical Hochschild homology which is thus a character sheaf. The Grothendieck–Springer sheaf $S_{G,\lambda}$ is the trace of the unit of $\mathcal{H}_G^\lambda$, or equivalently, the character of the regular $\mathcal{H}_G^\lambda$-module. The Grothendieck–Springer sheaves collectively provide the principal series character sheaves underlying the principal series representations of finite groups of Lie type. More strikingly, every character sheaf appears as the trace of an endomorphism of a regular monodromic Hecke module category. Thus the entire spectrum of finite groups of Lie type is captured by the *categorified principal series.*

From the perspective of topological field theory, one can view each monoidal monodromic Hecke category as a quantization of G-gauge fields on an interval with B-reductions at the end points. From general principles, one expects its categorical Hochschild homology to be the analogous quantization of G-gauge fields on the circle, and thus the appearance of character sheaves is not surprising. But we will now turn to loop groups where this viewpoint leads to less evident conclusions.

1.3.2 *Towards character sheaves for loop groups.* While Lusztig's character sheaves account for characters of finite groups of Lie types, for p-adic groups, such a theory is still not available but highly desirable. One obvious approach is to pass from p-adic groups to loop groups (from mixed to equal characteristic) and then attempt to follow Lusztig's constructions. In particular, one might hope that the monoidal λ-monodromic affine Hecke categories $\mathcal{H}_{LG}^\lambda$ would be rich enough to produce all depth-zero characters. Unfortunately, with standard techniques, such a direct approach encounters serious obstructions in the infinite-dimensional and codimensional nature of adjoint orbits in loop groups. But as demonstrated most strikingly by Ngô in his proof of the fundamental lemma [Ngô10], Hitchin systems over global curves provide finite-dimensional models of this geometry.

Returning to the perspective of topological field theory, and in particular the geometric Langlands program as a quantization of four-dimensional super Yang–Mills theory, one might look specifically to the Hitchin system for an elliptic curve E to model the geometry of the adjoint quotient of the loop group. Indeed, this is not a new idea: it has precedents in Looijenga's (unpublished) identification of holomorphic G-bundles on the Tate curve E_q with twisted conjugacy classes in loop groups (see [EFK95] and Baranovsky and Ginzburg's refinement for semistable bundles and integral twisted conjugacy classes [BG96]). Thus it is reasonable to look for a theory of character sheaves for loop groups in the geometry of the moduli $\mathrm{Bun}_G(E)$ of G-bundles on an elliptic curve E. (This is discussed for example by Schiffmann [Sch12] who attributes the idea to Ginzburg.)

With the preceding motivation and geometric reformulation of character sheaves [MV88, Gin89] in mind, we propose the following definition.

DEFINITION 1.10. Given a smooth elliptic curve E, an *elliptic character sheaf* is a constructible complex on the moduli $\mathrm{Bun}_G(E)$ with singular support in the global nilpotent cone (the 0-fiber of the Hitchin system). We denote the category of elliptic character sheaves by $\mathcal{C}h_G(E)$.

While it appears difficult to construct a theory of character sheaves from the elliptic picture of twisted conjugacy classes, a more structured approach is available via the degeneration of the Tate curve $E_q \rightsquigarrow E_0$ to a nodal elliptic curve and then the passage to its normalization $\mathbb{P}^1 \simeq \tilde{E}_0$. This sequence offers an analogue of the horocycle transform for loop groups completely within the setting of finite-dimensional geometry. Via the geometry of degenerations and normalizations, we expect that the categorical Hochschild homology of the λ-monodromic affine Hecke category $\mathcal{H}^\lambda_{LG}$ is equivalent to the category of elliptic character sheaves with central character λ. Any (sufficiently dualizable) $\mathcal{H}^\lambda_{LG}$-module category has a character in the categorical Hochschild homology which will then be such an elliptic character sheaf.

Finally, the results of this paper independently produce the elliptic Grothendieck–Springer sheaf $S_{E,\lambda} \in D^\flat_c(\mathrm{Bun}^{\mathrm{ss}}_G(E))$ whose extension by zero is the elliptic character sheaf which should arise via the trace of the unit of $\mathcal{H}^\lambda_{LG}$, or equivalently, the character of the regular $\mathcal{H}^\lambda_{LG}$-module category.

1.3.3 *Langlands duality/mirror symmetry.* Let us mention further developments related to the preceding discussion.

Following Kazhdan and Lusztig [KL87, CG97], the affine Hecke algebra admits a Langlands dual presentation as equivariant coherent sheaves on the Steinberg variety of the dual group. Bezrukavnikov [Bez06, Bez12] categorifies this to an equivalence of the monoidal affine Hecke category (consisting of unipotent bimonodromic sheaves on the affine flag variety) with equivariant coherent sheaves on the Steinberg variety.

From this Langlands dual starting point, one can ask what are the geometric avatars of characters. In joint work with Preygel [BNP13], we provide the following answer for the 'global' version of the affine Hecke category, where we take into account all monodromies at once. Recall that fixing a basis of $\pi_1(E)$ produces an equivalence from the moduli $\mathrm{Loc}_{G^\vee}(E)$ of dual group local systems on the elliptic curve E to the derived stack of commuting pairs of dual group elements up to conjugacy: the two commuting elements are given by the monodromies of a local system around the basis of loops.

THEOREM 1.11 [BNP13]. *The Hochschild homology category of the global affine Hecke category is equivalent to the derived category $D^\flat_\mathcal{N}\mathrm{Coh}(\mathrm{Loc}_{G^\vee}(E))$ of coherent sheaves with nilpotent singular support on the moduli of dual group local systems on the elliptic curve E.*

Remark 1.12. Coherent sheaves with nilpotent singular support on moduli of local systems were introduced by [AG15] as the natural target for the spectral side of the geometric Langlands conjecture.

Remark 1.13. The commuting stack $\mathrm{Loc}_{G^\vee}(E)$ plays a central role in representation theory. We mention two recent exciting developments. Its K-groups play a central role (as the Langlands dual form of the elliptic Hall algebra) in the work of Schiffmann and Vasserot [SV13, SV11, SV12], with close ties to the theory of Macdonald polynomials and double affine Hecke algebras. The work of Ginzburg [Gin12] unveils and exploits a direct link between the (Lie algebra) commuting variety, Cherednik algebras and the Harish-Chandra system or Springer sheaf.

Remark 1.14. It is natural to ask which coherent sheaf on $\mathrm{Loc}_{G^\vee}(E)$ corresponds to the universal elliptic Grothendieck–Springer sheaf, or in other words, the character of the regular 'global' affine Hecke module. Following through the constructions, one finds the *coherent Grothendieck–Springer sheaf* resulting from the pushforward of the structure sheaf along the induction map $\mathrm{Loc}_{B^\vee}(E) \to \mathrm{Loc}_{G^\vee}(E)$. (In fact, a slight modification of this object matches where we take into account the natural linearity over $H^\vee$.)

In joint work with Helm (in preparation), we relate a natural q-deformation of this sheaf to the representation theory of affine Hecke algebras, and more broadly, relate all coherent character sheaves with representations of p-adic groups. We expect elliptic and coherent character sheaves to elucidate the depth-zero representation theory of p-adic groups.

2. Main statements

Throughout the rest of the paper, we will make the simplifying assumption that the derived group $[G, G] \subset G$ is simply connected.

Fix an elliptic curve E (or more precisely, a smooth projective curve of genus one; our assertions will not involve any choice of a base point). The geometry of G-bundles on E, in particular the stack of semistable bundles and its coarse moduli, is well understood, with an extensive literature going back to Atiyah [Ati57]. We have found the beautiful sources [Ram75, Ram96, Las98, FMW97, FM98] to be especially helpful.

Consider the Eisenstein diagram of stacks of principal bundles

$$\mathrm{Bun}_G(E) \xleftarrow{p_E} \mathrm{Bun}_B(E) \xrightarrow{q_E} \mathrm{Bun}_H(E).$$

Recall that $\mathrm{Bun}_H(E)$ is naturally a commutative group-stack, and there is a canonical equivalence

$$\mathrm{Bun}_H(E) \simeq \mathrm{Pic}(E) \otimes_{\mathbf{Z}} \Lambda_H$$

where $\Lambda_H = \mathrm{Hom}(\mathbb{G}_m, H)$ denotes the coweight lattice of H and $\mathrm{Pic}(E)$ denotes the Picard stack. In particular, the group of connected components $\pi_0 \mathrm{Bun}_H(E)$ is canonically isomorphic to Λ_H. We denote the neutral component of degree-zero H-bundles by

$$\mathcal{H}_E \subset \mathrm{Bun}_H(E).$$

Recall that the Weyl group W naturally acts on Λ_H and hence also on $\mathrm{Bun}_H(E)$ and hence in turn on $\mathcal{H}_E$. An H-bundle is said to be regular if it is not isomorphic to any of its W-translates. We denote the open substack of W-regular degree-zero H-bundles by

$$\mathcal{H}_E^{\mathrm{r}} \subset \mathcal{H}_E.$$

Observe that $\pi_0 \mathrm{Bun}_G(E)$ is canonically equivalent to Λ_H/R_G where $R_G \subset \Lambda_H$ denotes the coroot lattice of G. We say that a G-bundle is degree zero if it lies in the connected component of the trivial G-bundle. Consider the open substack of degree-zero semistable G-bundles

$$\mathcal{G}_E \subset \mathrm{Bun}_G(E).$$

According to [Las98, FM98], the coarse moduli of S-equivalent degree-zero semistable G-bundles admits the description

$$\mathcal{G}_E/\{S\text{-equivalence}\} \simeq [\mathcal{H}_E]/W$$

where $[\mathcal{H}_E]$ denotes the coarse moduli of degree-zero H-bundles, and we take the naive finite quotient by W on the right.

Consider further the open substack of regular semisimple degree-zero semistable G-bundles

$$\mathcal{G}_E^{\mathrm{rs}} \subset \mathcal{G}_E.$$

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Here 'regular' means that their automorphisms are as small as possible (of the dimension of H) and 'semisimple' means they are induced from a torus bundle. The above identification restricts to induce a canonical equivalence

$$\mathcal{G}_E^{\mathrm{rs}} \simeq \mathcal{H}_E^{\mathrm{r}}/W$$

where the W-action on $\mathcal{H}_E^{\mathrm{r}}$ is free (and so may be interpreted in any fashion).

Example 2.1. Take $G = SL_2$, so that $W \simeq \mathbb{Z}/2\mathbb{Z}$, $H \simeq \mathbb{G}_m$, and $\Lambda_H \simeq \mathbb{Z}$. Then $[\mathcal{H}_E] \simeq E$, and the naive quotient $E \simeq [\mathcal{H}_E] \to [\mathcal{H}_E]/W \simeq \mathbb{P}^1$ is the usual Weierstrass ramified two-fold cover. The moduli of semistable bundles lying above any of the four points of ramification is equivalent to the adjoint quotient of the nilpotent cone $\mathcal{N}/G$, and the semistable bundles lying above the complement of the four points form the regular semisimple locus.

Recall that $q_E : \mathrm{Bun}_B(E) \to \mathrm{Bun}_H(E)$ is a bijection on connected components. We say that a B-bundle is degree zero if it projects to a degree-zero H-bundle. We denote the connected component of degree-zero B-bundles by

$$\mathcal{B}_E \subset \mathrm{Bun}_B(E).$$

We have the natural restriction of the Eisenstein diagram

$$\mathcal{G}_E \xleftarrow{\ \mu_E\ } \mathcal{B}_E \xrightarrow{\ \nu_E\ } \mathcal{H}_E.$$

We refer to the map μ_E as the elliptic Grothendieck–Springer resolution.

Recall that a representable map $f : X \to Y$ of irreducible stacks is said to be small if f is proper, surjective, and, for all $k > 0$, we have $\mathrm{codim} f(X_k) > 2k$ where $X_k \subset X$ denotes the union of fibers of f of dimension k.

Theorem 2.2. (1) *The elliptic Grothendieck–Springer resolution*

$$\mu_E : \mathcal{B}_E \longrightarrow \mathcal{G}_E$$

is a small map.

(2) *The restriction of the elliptic Grothendieck–Springer resolution to the inverse image* $\mathcal{B}_E^{\mathrm{rs}} = \mu_E^{-1}(\mathcal{G}_E^{\mathrm{rs}})$ *of the regular semisimple locus fits into the Cartesian diagram of W-covers*

$$
\begin{array}{ccc}
\mathcal{B}_E^{\mathrm{rs}} & \xrightarrow{\ \mu_E^{\mathrm{rs}}\ } & \mathcal{G}_E^{\mathrm{rs}} \\
{\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\
\mathcal{H}_E^{\mathrm{r}} & \longrightarrow & \mathcal{H}_E^{\mathrm{r}}/W
\end{array}
$$

The proof of the theorem will be given in §4 after collecting some preliminaries in §3.

Remark 2.3. Some simple observations:

(1) The theorem and its proof extend universally over the moduli of elliptic curves.

(2) The theorem holds equally well over any projective curve of arithmetic genus one, and in fact the resulting objects are well known when no component is a smooth elliptic curve (so that the fundamental group of the curve is less than rank two). For example, over a cuspidal elliptic curve, one finds the Lie algebra version of the Grothendieck–Springer resolution; and over any nodal necklace of projective lines, in particular a nodal elliptic curve, one finds the group version.

The fundamental group $\pi_1(\mathcal{H}_E)$ is the tensor product lattice $\pi_1(E) \otimes \Lambda_H$ with spectrum two copies of the dual torus $H^\vee \times H^\vee = \mathrm{Spec}\,\mathbb{C}[\pi_1(E) \otimes \Lambda_H]$. Thus (finite-rank) W-equivariant local systems on $\mathcal{H}_E$ correspond to (finite-dimensional) representations of the elliptic Weyl group

$$W_E = (\pi_1(E) \otimes \Lambda_H) \rtimes W$$

which in turn correspond to W-equivariant quasicoherent sheaves on $H^\vee \times H^\vee$ (with finite global sections).

Starting from a finite-rank local system $\mathcal{L}$, we define the corresponding elliptic Grothendieck–Springer sheaf by the formula

$$S_{E,\mathcal{L}} = \mu_{E!} \nu_E^* \mathcal{L} \in D_c^\flat(\mathcal{G}_E).$$

For the elliptic curve E, the Hitchin system reduces to the natural map $T^* \mathrm{Bun}_G(E) \to \mathfrak{h}^* /\!/ W$ that assigns to a covector $\xi \in H^0(E, \mathfrak{g}_\mathcal{P}^*)$ at a bundle $\mathcal{P} \in \mathrm{Bun}_G(E)$ its characteristic polynomial (or equivalently, image under the geometric invariant quotient of the coadjoint action). We define the nilpotent cone $\mathcal{N} \subset T^* \mathrm{Bun}_G(E)$ to be the 0-fiber of the Hitchin system.

COROLLARY 2.4. *Let $\mathcal{L}$ be a finite-rank local system on $\mathcal{H}_E$, or equivalently, a finite-dimensional $\pi_1(\mathcal{H}_E)$-representation.*

(1) The elliptic Grothendieck–Springer sheaf $S_{E,\mathcal{L}}$ is the middle extension of its restriction to the regular semisimple locus.

(2) The elliptic Grothendieck–Springer construction factors into induction followed by a fully faithful embedding

$$\mathcal{S}_E : \mathbb{C}[\pi_1(\mathcal{H}_E)]\text{-mod}_{\mathrm{fin}} \longrightarrow \mathbb{C}[W_E]\text{-mod}_{\mathrm{fin}} \hookrightarrow \mathrm{Perv}_\mathcal{N}(\mathrm{Bun}_G^{\mathrm{ss}}(E))$$

into perverse sheaves with nilpotent singular support.

The corollary follows closely from the theorem. The proof of the corollary will be given in § 4 following the proof of the theorem.

Remark 2.5. If we begin with the universal local system $\mathcal{L}_{\mathrm{univ}}$ corresponding to the natural W_E-representation $\mathbb{C}[\pi_1(E) \otimes \Lambda_H]$ and in turn to the structure sheaf of $H^\vee \times H^\vee$, we obtain the universal Grothendieck–Springer sheaf

$$S_{E,\mathcal{L}_{\mathrm{univ}}} = \mu_{E!} \nu_E^* \mathcal{L}_{\mathrm{univ}}.$$

Although $S_{E,\mathcal{L}_{\mathrm{univ}}}$ is not constructible, it is cohomologically bounded, and informally one can view it as a perverse sheaf with endomorphisms equivalent to the group algebra $\mathbb{C}[W_E]$.

3. Tannakian reminders

Following the account in [BG02], we recall a useful way to translate Borel structures on principal bundles into linear algebra.

We continue with the setup of G a connected reductive group with simply connected derived group, $B \subset G$ a Borel subgroup, $N \subset B$ its unipotent radical, and $H = B/N$ the universal Cartan.

Let $\check{\Lambda}_H = \mathrm{Hom}(H, \mathbb{G}_m)$ denote the weight lattice, and $\check{\Lambda}_G^+ \subset \check{\Lambda}_H$ the cone of dominant weights. For $\lambda \in \check{\Lambda}_G$, let V^λ denote the irreducible G-representation of highest weight λ.

Recall that specifying the Borel subgroup $B \subset G$, or in other words, a point of the flag variety of G, is equivalent to specifying a collection of lines

$$L^\lambda \subset V^\lambda, \quad \text{for } \lambda \in \check{\Lambda}_G,$$

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satisfying the Plucker relations: the natural projections

$$V^\lambda \otimes V^\mu \longrightarrow V^{\lambda+\mu}, \quad \text{for } \lambda, \mu \in \check{\Lambda}_G,$$

restrict to isomorphisms

$$L^\lambda \otimes L^\mu \xrightarrow{\ \sim\ } L^{\lambda+\mu}.$$

Now let X be any base scheme. The moduli $\mathrm{Bun}_B(X)$ represents the following data: an S-point of $\mathrm{Bun}_B(X)$ is an S-point ξ of $\mathrm{Bun}_G(X)$, together with a collection of invertible subsheaves

$$\mathcal{L}^\lambda \subset \mathcal{V}_\xi^\lambda, \quad \text{for } \lambda \in \check{\Lambda}_G,$$

such that the quotients $\mathcal{V}_\xi^\lambda/\mathcal{L}^\lambda$ are flat over $S \times X$, and the collection satisfies the Plucker relations: the natural projections

$$\mathcal{V}_\xi^\lambda \otimes \mathcal{V}_\xi^\mu \longrightarrow \mathcal{V}_\xi^{\lambda+\mu}, \quad \text{for } \lambda, \mu \in \check{\Lambda}_G,$$

restrict to isomorphisms

$$\mathcal{L}^\lambda \otimes \mathcal{L}^\mu \xrightarrow{\ \sim\ } \mathcal{L}^{\lambda+\mu}.$$

Now suppose X is a smooth connected projective curve. Unlike the absolute case over a point where the flag variety is connected and proper, the fibers of the map $p : \mathrm{Bun}_B(X) \to \mathrm{Bun}_G(X)$ need not be connected or proper. Following Drinfeld, to compactify the connected components of the fibers of p, one can consider generalized B-structures. The moduli $\overline{\mathrm{Bun}}_B(X)$ of generalized B-bundles represents the following data: an S-point of $\overline{\mathrm{Bun}}_B(X)$ is an S-point ξ of $\mathrm{Bun}_G(X)$, together with a collection of invertible subsheaves

$$\mathcal{L}^\lambda \subset \mathcal{V}_\xi^\lambda, \quad \text{for } \lambda \in \check{\Lambda}_G,$$

such that the quotients $\mathcal{V}_\xi^\lambda/\mathcal{L}^\lambda$ are flat relative to S (but not necessarily over $S \times X$), and the collection satisfies the Plucker relations: the natural projections

$$\mathcal{V}_\xi^\lambda \otimes \mathcal{V}_\xi^\mu \longrightarrow \mathcal{V}_\xi^{\lambda+\mu}, \quad \text{for } \lambda, \mu \in \check{\Lambda}_G,$$

restrict to isomorphisms

$$\mathcal{L}^\lambda \otimes \mathcal{L}^\mu \xrightarrow{\ \sim\ } \mathcal{L}^{\lambda+\mu}.$$

The following statements can all be found in [BG02]. There is an evident open embedding $\mathrm{Bun}_B(X) \subset \overline{\mathrm{Bun}}_B(X)$ whose image is dense. The projection $q : \mathrm{Bun}_B(X) \to \mathrm{Bun}_H(X)$ extends to a projection $\overline{q} : \overline{\mathrm{Bun}}_B(X) \to \mathrm{Bun}_H(X)$. The map $p : \mathrm{Bun}_B(X) \to \mathrm{Bun}_G(X)$ extends to a representable map $\overline{p} : \overline{\mathrm{Bun}}_B(X) \to \mathrm{Bun}_G(X)$, and the restriction of $\overline{p}$ to any connected component is proper.

4. Proofs

We prove here our main statements made earlier: Theorem 2.2 (which also appeared in a less precise form as Theorem 1.6) as well as Corollary 2.4. The arguments are assembled from familiar statements about vector bundles on smooth projective curves. The Tannakian reminders in § 3 above allow one to reduce abstract statements about G-bundles to more concrete versions about vector bundles.

Let X be a smooth projective curve. We continue to use interchangeably the geometric language of vector bundles and line bundles and the algebraic language of locally free sheaves and invertible sheaves. It is worth pointing out one important distinction in terminology (which played a role above in the discussion of Drinfeld compactifications): an inclusion $\mathcal{W} \subset \mathcal{W}$ of locally free sheaves need not be a fiberwise inclusion of vector bundles; namely, the quotient $\mathcal{V}/\mathcal{W}$ may have torsion in the former case while it will be a vector bundle in the latter case.

Given a vector bundle $\mathcal{V} \to X$, we have its degree $\deg \mathcal{V}$ (first Chern class integrated over X) and rank $\operatorname{rk} \mathcal{V}$ (dimension). One says that $\mathcal{V}$ is semistable if it satisfies the slope inequality

$$\deg \mathcal{W}/\operatorname{rk} \mathcal{W} \leqslant \deg \mathcal{V}/\operatorname{rk} \mathcal{V}$$

for all proper nonzero vector subbundles $\mathcal{W} \subset \mathcal{V}$. In particular, if $\mathcal{V}$ has degree zero, then $\mathcal{W}$ must have nonpositive degree.

LEMMA 4.1. *Suppose $\mathcal{L} \subset \mathcal{V}$ is the inclusion of a degree-zero invertible sheaf into a degree-zero locally free sheaf on X. If $\mathcal{V}$ is semistable, then $\mathcal{V}/\mathcal{L}$ is locally free, and so $\mathcal{L} \subset \mathcal{V}$ is a vector subbundle.*

Proof. If $\mathcal{V}/\mathcal{L}$ contains torsion, then we may twist $\mathcal{L}$ to obtain a positive locally free subbundle of $\mathcal{V}$, in contradiction to the fact that $\mathcal{V}$ is degree zero and semistable. $\qquad\square$

Let $\mathcal{G}_X$ be the moduli of degree-zero, semistable G-bundles on X. Let $\mathcal{B}_X$ be the moduli of degree-zero B-bundles on X, where by 'degree' we mean the degree of the induced H-bundle.

LEMMA 4.2. *The induction map $\mu_X : \mathcal{B}_X \to \mathcal{G}_X$ is representable and proper.*

Proof. As established in [BG02], the Plucker presentation of §3 shows μ_X is representable. Then Lemma 4.1 shows it coincides with the Drinfeld compactification and hence is proper. $\qquad\square$

Let $x \in X$ be a geometric point.

LEMMA 4.3. *Suppose $\mathcal{V}$ is a degree-zero semistable vector bundle, and $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{V}$ are degree-zero line subbundles. If $\mathcal{L}_1|_x = \mathcal{L}_2|_x$ inside $\mathcal{V}|_x$, then $\mathcal{L}_1 = \mathcal{L}_2$ inside $\mathcal{V}$.*

Proof. Suppose not. Then the composition $\mathcal{L}_1 \subset \mathcal{V} \to \mathcal{V}/\mathcal{L}_2$ is an inclusion of a degree-zero invertible sheaf into a degree-zero semistable locally free sheaf. By Lemma 4.1, the composition must be an inclusion of vector bundles, in contradiction to the fact that it must also have a zero at x. $\qquad\square$

Let $\mathcal{G}_{X,x} = \mathcal{G}_E \times_{BG} BB$ be the moduli of a degree-zero semistable G-bundle ξ together with a flag in the fiber $\xi|_x$. We have a natural map $r_X : \mathcal{B}_X \to \mathcal{G}_{X,x}$ given by taking $\mu_X : \mathcal{B}_X \to \mathcal{G}_X$ in the first factor and the fiber at x in the second factor.

LEMMA 4.4. *The natural map $r_X : \mathcal{B}_X \to \mathcal{G}_{X,x}$ is a closed embedding.*

Proof. Apply Lemma 4.3 to the Plucker presentation of μ_X. $\qquad\square$

Let $\mathcal{Z}_X = \mathcal{B}_X \times_{\mathcal{G}_X} \mathcal{B}_X$ be the moduli of a pair of degree-zero B-bundles together with an isomorphism of their induced G-bundles.

LEMMA 4.5. *If X is an elliptic curve, then the irreducible components of $\mathcal{Z}_X$ are in natural bijection with the Weyl group W, and the dimension of each irreducible component is zero.*

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Proof. Fix $x \in X$ a geometric point, and consider the natural restriction map $\pi_X : \mathcal{Z}_X \to BB \times_{BG} BB \simeq B\backslash G/B$ given by taking fibers at x. By Lemma 4.4, for a given relative position $Y_w \subset B\backslash G/B$ represented by two Borel subgroups $B_1, B_2 \subset G$, the inverse image $\pi_X^{-1}(Y_w)$ is equivalent to the moduli of degree-zero $B_1 \cap B_2$-bundles. Using only that the canonical bundle of X is trivial, it is straightforward to check this moduli has the correct dimension by induction on the solvable filtration of $B_1 \cap B_2$. $\qquad\square$

Consider the composition $q_1 : \mathcal{Z}_X \to \mathcal{B}_X \to \mathcal{H}_X$ where the first map is projection along the first factor and the second is the usual projection.

LEMMA 4.6. *The restriction of q_1 to each irreducible component of $\mathcal{Z}_X$ has equidimensional fibers.*

Proof. The lemma follows from the proof of Lemma 4.5 where the components are shown to be moduli of degree-zero $B_1 \cap B_2$-bundles. $\qquad\square$

Now let us turn to the proofs of Theorem 2.2 as well as Corollary 2.4.

A representable map $f : X \to Y$ of irreducible stacks is said to be small (or semi-small) if f is proper, surjective, and, for all $k > 0$, we have $\mathrm{codim} f(X_k) > 2k$ (or $\mathrm{codim} f(X_k) \geqslant 2k$) where $X_k \subset X$ denotes the union of fibers of f of dimension k. Traditionally (see, for example, [DM09]) the prior definitions are made for a map of varieties or schemes, but they can be checked after smooth base change, and so extend naturally to stacks.

We will employ the following general strategy to establish that a map $f : X \to Y$ is small.

First, suppose f is proper, surjective, and $V \subset Y$ is an open substack such that $f : U = X \times_Y V \to V$ is finite. Let $X' = X\backslash U$, $Y' = Y\backslash V$, and $g : X' \to Y'$ be the restriction of f. It is immediate from the definitions that if g is semi-small, then f is small.

Second, to see that $g : X' \to Y'$ is semi-small, it suffices to show that the dimension of each irreducible component of $Z = X' \times_{Y'} X'$ is less than or equal to $\dim Y$.

Proof of Theorem 2.2. Fix a splitting $H \subset B$ of the projection $B \to H$ so that we obtain a maximal torus $H \subset B \subset G$. By [Las98, Theorem 4.16] or [FM98, Corollary 5.12], the induction from H-bundles to G-bundles provides an equivalence

$$\mathcal{H}_E^{\mathrm{r}}/W \xrightarrow{\sim} \mathcal{G}_E^{\mathrm{rs}}.$$

Strictly speaking, this is stated on the level of coarse moduli in [Las98, FM98], but the additional necessary ingredients for stacks are readily available. First, the induction map is defined on the level of stacks. Second, the automorphisms of a regular bundle are simply H (see, for instance, [FM98, Theorem 4.1]), and hence the induction map is an isomorphism on automorphisms as well.

For assertion (2), it now suffices to show that the natural map

$$\mathcal{B}_E^{\mathrm{rs}} \longrightarrow \mathcal{H}_E^{\mathrm{r}}$$

is an equivalence.

Given the splitting $H \subset B$, and corresponding map

$$\mathcal{H}_E^{\mathrm{r}} \longrightarrow \mathcal{B}_E^{\mathrm{rs}}$$

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it suffices to show, for any $\xi \in \mathcal{H}_E^r$, that all B-bundles lying over it are equivalent to the B-bundle it induces by the splitting.

To see the above assertion, we can appeal to the Plucker presentation of the previous section. Given a weight $\mu \in \check{\Lambda}_G$, let us denote by $\mathcal{L}_\xi^\mu \in \mathrm{Pic}^0(E)$ the associated degree-zero line bundle. Thanks to the Plucker presentation of the previous section, it suffices to show that for any weight $\mu \in \check{\Lambda}_G$, and any simple root $\alpha \in \check{\Lambda}_G$, any extension between the degree-zero line bundles $\mathcal{L}_\xi^\mu, \mathcal{L}_\xi^{\mu+\alpha}$ canonically splits. It follows immediately that any B-structure is unique.

Thus we are reduced to needing, for any simple root $\alpha \in \check{\Lambda}_G$, the vanishing $H^1(E, \mathcal{L}_\xi^\alpha) \simeq 0$. By Serre duality, this holds over an elliptic curve E if and only if the degree-zero line bundle $\mathcal{L}_\xi^\alpha$ is nontrivial. This is in fact an equivalent formulation of what it means for ξ to be regular. Let us check that it is indeed implied by the assumption that the W-orbit of ξ is a free W-orbit. Let $\sigma \in W$ be the simple reflection corresponding to the root α. Choose $\mu \in \check{\Lambda}_G$ regular and dominant, so that by assumption the degree-zero line bundles $\mathcal{L}_\xi^\mu, \mathcal{L}_\xi^{\sigma(\mu)}$ are not isomorphic. But then $\mathcal{L}_\xi^{\sigma(\mu)} \simeq \mathcal{L}_\xi^\mu \otimes (\mathcal{L}_\xi^\alpha)^k$, for some integer k, so we see that $\mathcal{L}_\xi^\alpha$ must be nontrivial.

Finally, for assertion (1), the above discussion, together with Lemmas 4.2, 4.5, and 4.6, shows that the restriction of μ_X to the complement of the regular semisimple locus is semi-small. $\square$

Proof of Corollary 2.4. Assertion (1) is a standard consequence of assertion (1) of Theorem 2.2. Namely, the pushforward of a (shifted) local system along a small resolution produces a perverse sheaf that is the middle extension of its generic pushforward (see, for example, [DM09]).

For assertion (2), let us first establish that the construction $\mathcal{S}_E$ produces complexes with nilpotent singular support. At a B-bundle $\mathcal{P}$, the pullback of covectors along μ_E is given by $H^0(E, \mathfrak{g}_{\mathcal{P}}^*) \to H^0(E, \mathfrak{b}_{\mathcal{P}}^*)$. The Hitchin system factors through the natural projection $H^0(E, \mathfrak{b}_{\mathcal{P}}^*) \to \mathfrak{h}^*$. Thus $\mathcal{S}_E$ will take local systems, i.e. sheaves with vanishing singular support on $\mathcal{H}_E$, to complexes with nilpotent singular support on $\mathcal{G}_E$. Once we have seen the asserted factorization of $\mathcal{S}_E$, it will follow immediately that all complexes in the image of the fully faithful embedding have nilpotent singular support, since every finite-dimensional $\mathbb{C}[W_E]$-module appears within an induction of a finite-dimensional $\mathbb{C}[\pi_1(\mathcal{H}_E)]$-module.

Recall that, in general, the middle extension off an open locus is fully faithful, as can be seen as follows. Restriction to the open locus provides a left inverse so the middle extension is faithful. If it were not full, then from the preceding observation we could find a nontrivial morphism restricting to zero over the open locus. But by the the minimality of the middle extension, it must equal the kernel of such a morphism, and hence the morphism must in fact be trivial.

Introduce notation for the embeddings $j_{\mathcal{H}/W} : \mathcal{H}_E^r/W \to \mathcal{H}_E/W$, $j_{\mathcal{G}} : \mathcal{G}_E^{rs} \to \mathcal{G}_E$. Note that restriction $j_{\mathcal{H}/W}^*$ to a dense open locus is fully faithful on local systems. Using the identification $\mathcal{G}_E^{rs} \simeq \mathcal{H}_E^r/W$, we can thus take the fully faithful embedding to be the composition $j_{\mathcal{G}!*}j_{\mathcal{H}/W}^*$ on local systems on $\mathcal{H}_E/W$.

Finally, for the factorization, introduce notation for the open embedding $j_{\mathcal{B}} : \mathcal{B}_E^{rs} \to \mathcal{B}_E$. We have seen that

$$\mathcal{S}_E \simeq j_{\mathcal{G}!*}\mu_{E!}^{rs}j_{\mathcal{B}}^*\nu_E^*.$$

Recall that we can drop the middle extension and work with the functor $\mu_{E!}^{\mathrm{rs}} j_{\mathcal{B}}^* \nu_E^*$. Introduce notation for the open embedding $j_{\mathcal{H}} : \mathcal{H}_E^{\mathrm{r}} \to \mathcal{H}_E$ and natural projection $\pi_{\mathcal{H}}^{\mathrm{r}} : \mathcal{H}_E^{\mathrm{r}} \to \mathcal{H}_E^{\mathrm{r}}/W$. We have the canonical isomorphism

$$\mu_{E!}^{\mathrm{rs}} j_{\mathcal{B}}^* \nu_E^* \simeq \pi_{\mathcal{H}!}^{\mathrm{r}} j_{\mathcal{H}}^*$$

thanks to the diagram of assertion (2) of Theorem 2.2.

Finally, note that restriction $j_{\mathcal{H}}^*$ to a dense open locus is fully faithful on local systems. And the pushforward $\pi_{\mathcal{H}!}^{\mathrm{r}}$ along a W-cover is precisely the induction map on local systems. $\qquad\square$

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