# A FORMULA FOR THE RESOLVENT OF A REYNOLDS OPERATOR

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## 1. Introduction

Let $\mathfrak{A}$ be a complex Banach algebra, possibly non-commutative, with identity $e$. By a Reynolds operator we mean here a bounded linear operator $T : \mathfrak{A} \to \mathfrak{A}$ satisfying the Reynolds identity

$$[*] \qquad Tx \cdot Ty = T(Tx \cdot y + x \cdot Ty) - T(Tx \cdot Ty)$$

for all $x, y \in \mathfrak{A}$. We prove that under certain conditions the resolvent of $T$, $R(\rho, T) = (\rho I - T)^{-1}$, has the form

$$(1.1) \qquad R(\rho, T)x = \frac{x}{\rho} + \frac{1}{\rho^2} T \left[ x \exp\left( -\frac{1}{\rho} s \right) \right] \cdot \exp\left( \frac{1}{\rho} s \right)$$

where $s = -\log(e - Te)$ and $\exp y = e + y + y^2/2! + \cdots$.

The conditions are, roughly, that a suitable logarithm of $e - Te$ exists in $\mathfrak{A}$; and the argument then shows that $T$ is quasinilpotent. The formula holds in particular if the spectral radius of $Te$ is less than 1: the logarithm $-s$ in question is then the sum of the usual series. The existence of $\log(e - Te)$ implies that 1 is in the resolvent set of $Te$; thus Reynolds operators for which $Te = e$ (as required by some authors) are excluded from the discussion. A precise statement of sufficient conditions under which the formula is valid is given in the theorem in § 3, below.

Similar formulae are also known for the resolvents of averaging operators, antiderivations and summation operators. We make some comparisons in § 4.

Since Reynolds operators exist which are not quasinilpotent and can have infinitely many points in their spectra, a formula for the resolvent of an arbitrary Reynolds operator must be expected to be somewhat more complicated than (1.1). Such a formula would be of some interest, but none is known at present.

*Notation.* We write $\mathfrak{B}(\mathfrak{A})$ for the Banach algebra of all bounded linear operators on $\mathfrak{A}$, with identity $I$, and $\text{Res}(a)$, $\text{Sp}(a)$ for the resolvent set



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and spectrum respectively of $a \in \mathfrak{A}$, and likewise for $\mathfrak{B}(\mathfrak{A})$. The spectral radius of $a$ is

$$\nu(a) = \lim_{n\to\infty} ||a^n||^{1/n}.$$

Throughout §§ 1, 2 and 3 of the paper, $T$ is assumed to be a Reynolds operator. We write

$$t = Te.$$

## 2

The proof we give of (1.1) is by verification, since this method leads to the largest range of validity for $\rho$. On the other hand, a more natural-seeming if longer argument is to deduce the formula by studying certain functions of $t$, first finding a formula for

$$f_\lambda = R(\lambda, T)e \tag{2.1}$$

in terms of $t$, and then obtaining (1.1) in the form

$$R(\rho, T)x = \frac{1}{\rho}x + \frac{1}{\rho^2}T[x(e+(1-\lambda)f_\lambda)^{-1}]\cdot(e+(1-\lambda)f_\lambda) \tag{2.2}$$

where

$$\rho = \frac{\lambda}{1-\lambda}, \qquad \lambda = \frac{\rho}{1+\rho}. \tag{2.3}$$

This is the type of proof used in [3] and [5] to obtain the formulae mentioned in § 4 below. It explains to some extent the form of formula obtained, and the reason for the condition on $\mathrm{Sp}(t)$.

We begin by sketching this argument. Since for large $\lambda$,

$$R(\lambda, T)e = \frac{e}{\lambda} + \sum_{n=1}^{\infty}\frac{T^{n-1}t}{\lambda^{n+1}},$$

we look for a formula for $T^n t$, which turns out to require first a formula for $T(t^n)$.

To this end, we observe that

$$T[(n+1)t^n - nt^{n+1}] = t^{n+1} \tag{2.4}$$

for $n = 0, 1, 2, \cdots$. Case $n = 0$ is the definition of $t$; case $n = 1$ is got by the substitution $x = y = e$ in [$*$], and the inductive proof of the general statement is made by putting $(n+1)t^n - nt^{n+1}$, $e$ in [$*$].

Let

$$v = Tt = T^2e.$$

Writing (2.4) as

$$T\left(\frac{t^{n+1}}{n+1}\right) = T\left(\frac{t^n}{n}\right) - \frac{t^{n+1}}{n(n+1)},$$

we prove easily that

$$T\left(\frac{t^n}{n}\right) = v - \left(\frac{t^2}{1\cdot 2} + \frac{t^3}{2\cdot 3} + \cdots + \frac{t^n}{(n-1)n}\right) \tag{2.5}$$

for $n = 1, 2, 3, \cdots$.

At this point we need some restriction upon the spectrum of $t$ in order to find $v$ as a function of $t$. Suppose $\nu(t) < 1$. Then $||t^n/n|| \to 0$ as $n \to \infty$, also the series $\sum t^n/(n-1)n$ converges absolutely; and so (2.5) gives

$$v = \sum_{n=1}^{\infty} \frac{t^{n+1}}{n(n+1)}. \tag{2.6}$$

Defining $\log(e-t)$ by the series $-\sum_{n=1}^{\infty} t^n/n$, we can write this

$$v = t + (e-t)\log(e-t). \tag{2.7}$$

We now prove the following formula, valid if $\nu(t) < 1$:

$$T^k(t) = \sum_{n=1}^{\infty} \frac{t^{n+1}}{n(n+1)} \gamma_{n-1}^{(k-1)} \tag{2.8}$$

for $k = 1, 2, \cdots$, where the $\gamma$'s are defined inductively by

$$\begin{aligned} &\gamma_j^{(0)} = 1 && (j \geqq 0),\\ &\gamma_0^{(k)} = 0 && (k \geqq 1),\\ &\gamma_j^{(1)} = \sum_{m=1}^{j} \frac{1}{m} && (j \geqq 1),\\ &\gamma_j^{(k)} = \sum_{m=1}^{j} \frac{\gamma_{m-1}^{(k-1)}}{m} && (j \geqq 1,\ k \geqq 1), \end{aligned} \tag{2.9}$$

and where the series converges absolutely in $\mathfrak{A}$.

The proof is by induction. Case $k = 1$ is (2.6). Assume case $k$; then

$$\begin{aligned} T^{k+1}(t) &= \sum_{n=1}^{\infty} \frac{T(t^{n+1})}{n(n+1)} \gamma_{n-1}^{(k-1)} \\ &= \sum_{n=1}^{\infty} \frac{\gamma_{n-1}^{(k-1)}}{n} \left(v - \sum_{m=1}^{n} \frac{t^{m+1}}{m(m+1)}\right) \\ &= \lim_{N\to\infty} \left[\sum_{n=1}^{N} \frac{\gamma_{n-1}^{(k-1)}}{n} \left(v - \sum_{m=1}^{N} \frac{t^{m+1}}{m(m+1)}\right) \right.\\ &\qquad \left. + \sum_{m=1}^{N} \frac{t^{m+1}}{m(m+1)} \sum_{n=1}^{m-1} \frac{\gamma_{n-1}^{(k-1)}}{n}\right] \\ &= \lim_{N\to\infty} \left[\frac{T(t^N)}{N} \sum_{n=1}^{N} \frac{\gamma_{n-1}^{(k-1)}}{n} + \sum_{m=1}^{N} \frac{t^{m+1}}{m(m+1)} \gamma_{m-1}^{(k)}\right]. \end{aligned}$$

To show that (2.8) holds for the case $k+1$, and so complete the proof, it remains to verify that

$$\lim_{N\to\infty} \frac{T(t^N)}{N} \gamma_N^{(k)} = 0. \tag{2.10}$$

This in turn is proved by induction on $k$. Consider the case $k = 1$. Since $\nu(t) < 1$, we have $||t^n||^{1/n} = \varepsilon_n$ where $\varepsilon_n \to l$, $l < 1$, and so $||t^n|| < l_0^n$ for large enough $n$, where $l_0 < 1$. Thus the norm of the lefthand side of (2.10) is less than or equal to

$$||T|| \frac{l_0^N}{N} (\log N+\gamma)$$

where $\gamma$ is Euler's constant: (2.10) follows, for $k = 1$. To prove (2.10) by induction one can use the inequality

$$\left|1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\frac{1}{n}-\log n-\gamma-\frac{1}{2n}\right| < \frac{1}{12n^2}$$

to show that $\gamma_n^{(k)} < A_k(\gamma+\log n)^k$ for some numbers $A_k$ independent of $n$, and then argue as above. This establishes (2.8).

REMARK. It can be verified that $\gamma_j^{(k)} = 0$ for $k > j$. Then the last equation of (2.9) becomes

$$\gamma_j^{(k)} = \sum_{m=k}^{j} \frac{\gamma_{m-1}^{(k-1)}}{m} \qquad (j \geqq k \geqq 1), \quad 0 \quad (k > j \geqq 1).$$

For large $\lambda$ we can now write

$$\begin{aligned} R(\lambda, T)e &= \frac{e}{\lambda} + \frac{t}{\lambda^2} + \sum_{k=1}^{\infty} \frac{T^k(t)}{\lambda^{k+2}} \\ &= \frac{e}{\lambda} + \frac{t}{\lambda^2} + \sum_{n=1}^{\infty} \frac{t^{n+1}}{n(n+1)} \sum_{k=1}^{n} \frac{\gamma_{n-1}^{(k-1)}}{\lambda^{k+2}} \end{aligned} \tag{2.11}$$

after substitution from (2.8) and reversal of order of summation. Write, for fixed $\lambda$,

$$\psi_n = \sum_{k=1}^{n} \frac{\gamma_{n-1}^{(k-1)}}{\lambda^{k+2}} \qquad (n = 1, 2, 3, \cdots).$$

Then

$$\begin{aligned} \psi_n &= \frac{1}{\lambda^3}\left(\gamma_{n-1}^{(0)} + \sum_{k=2}^{\infty} \frac{1}{\lambda^{k-1}} \sum_{j=1}^{n-1} \frac{\gamma_{j-1}^{(k-2)}}{j}\right) \\ &= \frac{1}{\lambda^3} + \frac{1}{\lambda} \sum_{j=1}^{n-1} \frac{\psi_j}{j}, \end{aligned}$$

whence

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$$\psi_n - \psi_{n-1} = \frac{1}{\lambda}\frac{\psi_{n-1}}{n-1},$$

and so

$$\psi_1 = \frac{1}{\lambda^3}, \quad \psi_n = \frac{1}{\lambda^3}\prod_{q=1}^{n-1}\left(1+\frac{1}{\lambda q}\right) \qquad (n = 2, 3, \cdots).$$

Substitution for $\psi_n$ in the last term of (2.11), leads to the formula, valid for $|\lambda| > \nu(T)$:

$$R(\lambda, T)e = \begin{cases} \dfrac{e}{\lambda-1} - \dfrac{1}{\lambda(\lambda-1)}\, {}_2F_1\left(\dfrac{1-\lambda}{\lambda}, \beta; \beta; t\right) & (\text{if } \lambda \neq 1), \\ e - \log(e-t) \quad (\text{if } 1 \in \operatorname{Res}(T) \text{ and } \lambda = 1). \end{cases}$$

The first series is easily identified as a binomial expansion, and (making that interpretation of the power) we have, for $\lambda \neq 1$,

$$R(\lambda, T)e = \frac{e}{\lambda-1} - \frac{1}{\lambda(\lambda-1)}(e-t)^{(\lambda-1)/\lambda}.$$

This is the required formula for $f_\lambda$. The proof of (2.2) now goes roughly as in the proof of the theorem in § 3 below, but with $c_\rho$ defined by

$$c_\rho = \lambda(e+(1-\lambda)f_\lambda) \quad (\rho \neq -1), \quad c_{-1} = e-t,$$

where $\lambda$ and $\rho$ are related by (2.3). We shall not anticipate here the proof of this second part.

The foregoing argument requires $\nu(t) < 1$, and does not give information about $\operatorname{Res}(T)$. In the next section we relax the first condition, though still requiring $1 \in \operatorname{Res}(t)$, and can show that $\operatorname{Sp}(T) = \{0\}$.

## 3. The main theorem

Given $x \in \mathfrak{A}$, we define $\log x$ to be any solution $y$ in $\mathfrak{A}$ of $\exp y = x$. The solution is of course in general not unique. As used in § 2, $\log(e-t)$ conforms to this definition, since it was there assumed that $\nu(t) < 1$; but $\log(e-t)$ may also exist for $\nu(t) \geqq 1$, provided $1 \in \operatorname{Res}(t)$, for the elements of $\mathfrak{A}$ having logarithms lie in the maximal group of regular elements. See [4].

The following discussion avoids specifying $\log(e-t)$ by its power series in $t$. First, we remark that in the context of § 2, the element $s' = -\log(e-t)$ satisfies the two equations $Ts' = s'-t$, $Tt = t-(e-t)s'$. The second of these is (2.7), while the first can be proved by applying $T$ to the power series for $s'$ and using (2.5), rather as in the proof of (2.8).

Definition. Let $s$ be any simultaneous solution in $\mathfrak{A}$ of the equations

(3.1a) $$Ts = s-t,$$

(3.1b) $$Tt = t-e(e-t)s.$$

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It will be shown subsequently that the solution is unique, and that $-s$ is a logarithm of $e-t$: thus the existence of a solution implies that $1 \in \operatorname{Res}(t)$. We no longer require the convergence of $\sum t^n/n$ or of the series in (2.6); $v$ $(= Tt)$ is now given by (3.1b).

Lemma 1. *The elements s and t commute.*

Proof. Put $s$, $e$ in Reynold's identity [*], and again put $e$, $s$ in [*], and use (3.1a). The result follows. //

We need a formula for $T(\exp \alpha s)$, and to this end first compute $T(s^n)$.

Lemma 2. *For* $n = 1, 2, \cdots$, *we have*

$$(a)_n \qquad T(s^n) = (-1)^n n! \left(t + \sum_{k=1}^{n} \frac{(-s)^k}{k!}\right),$$

$$(b)_n \qquad T(ts^{n-1}) = (-1)^{n-1}(n-1)! \left(t + \sum_{k=1}^{n} \frac{(-s)^k}{k!}\right) + \frac{ts^n}{n}.$$

Proof. The proof is by complete induction on the two equations simultaneously. Cases $(a)_1$ and $(b)_1$ are (3.1a) and (3.1b) respectively. For $(a)_2$ we can take $s$, $s$ in [*] and use (3.1); for $(b)_2$ take $e$, $s^2$ in [*] and use $(a)_1$, $(a)_2$, $(b)_1$. (Note that $T(t^2) = 2v - t^2$, by (2.5), and since $v$ is given by (3.1b), the formula for $T(t^2)$ in terms of $t$, $s$ uses $(b)_1$.)

Assume $(a)_1$, $(a)_{n-1}$ and $(b)_1$, $(b)_2, \cdots, (b)_{n-1}$, and put $s$, $s^{n-1}$ in [*]; a rather long and tedious computation leads to $(a)_n$. Then put $e$, $s^n$ in [*] and use $(a)_1$, $(a)_2, \cdots, (a)_n$, $(b)_1$, $(b)_2, \cdots, (b)_{n-1}$; a similar computation leads to $(b)_n$. We omit the details. //

Lemma 3. *The element* $-s$ *is a logarithm of* $e-t$, *that is*

$$\exp(-s) = e - t,$$

*so that* $1 \in \operatorname{Res}(t)$.

Proof. Divide $(a)_n$ by $(-1)^n n!$, and let $n \to \infty$, noting that $\|s^n/n!\| \to 0$ and $T$ is assumed continuous. //

Lemma 4. *We have*

$$T(\exp(\alpha s)) = \begin{cases} \dfrac{1}{1+\alpha}(t - e + \exp(\alpha s)) & (\alpha \neq -1), \quad (3.2) \\ s \exp(-s) & (\alpha = -1). \quad (3.3) \end{cases}$$

Proof.

$$T(\exp(\alpha s)) = t + \sum_{k=1}^{\infty} \frac{\alpha^k}{k} T(s^k)$$

$$= \lim_{N \to \infty} \left[\sum_{k=0}^{N} (-\alpha)^k t + \sum_{n=1}^{N} \frac{(-s)^n}{n!} \sum_{k=n}^{N} (-\alpha)^k\right]$$

by Lemma 2. Assume $\alpha \neq -1$. Using the formula for the sum of the geometric series and making some rearrangement, we get

$$T\left(\exp(\alpha s)\right) = \frac{1}{1+\alpha}\left(t-e+\exp(\alpha s)\right)$$
$$+\frac{1}{1+\alpha}\lim_{N\to\infty}(-1)^N\alpha^{N+1}\left[t+\sum_{n=1}^{N}\frac{(-s)^n}{n!}\right].$$

Lemma 3 shows that the expression in square brackets tends to 0 as $N \to \infty$; it is easy to verify from Lagrange's form for the remainder in the exponential series that the last term is 0. This proves (3.2); and the proof of (3.3) is similar (or one can use Lemma 3 and (3.1b) directly. //

For notational convenience let us write

$$c_\rho = \exp\left(\frac{1}{\rho}s\right), \qquad (\rho \neq 0)$$

so that $c_\rho^{-1} = c_{-\rho}$ and $c_{-1} = e-t$. Lemma 4 becomes

$$Tc_\rho = \frac{\rho}{1+\rho}(t-e+c_\rho) \quad (\rho \neq -1), \quad Tc_{-1} = sc_{-1}. \tag{3.4}$$

THEOREM. *If a solution $s$ of equations* (3.1a), (3.1b) *exists, then $T$ is quasinilpotent, that is,* $\mathrm{Sp}(T) = \{0\}$; *and for all $x \in \mathfrak{A}$ and all $\rho \neq 0$,*

$$R(\rho, T)x = \frac{1}{\rho}x + \frac{1}{\rho^2}T\left[x\exp\left(-\frac{1}{\rho}s\right)\right]\cdot\exp\left(\frac{1}{\rho}s\right) \tag{3.5}$$

$$= \frac{1}{\rho}x + \frac{1}{\rho^2}\exp\left(\frac{1}{\rho}s\right)\cdot T\left[\exp\left(-\frac{1}{\rho}s\right)x\right]. \tag{3.6}$$

PROOF. Define the operator $K_\rho$ for $\rho \neq 0$ by

$$K_\rho x = \frac{1}{\rho}x + \frac{1}{\rho^2}T(xc_{-\rho})\cdot c_\rho.$$

Clearly $K_\rho \in \mathfrak{B}(\mathfrak{A})$. Now

$$(\rho I - T)K_\rho x = x + \frac{1}{\rho}T(xc_{-\rho})\cdot c_\rho - \frac{1}{\rho}Tx - \frac{1}{\rho^2}T[T(xc_{-\rho})\cdot c_\rho]. \tag{3.7}$$

To simplify the last term, write $p = T(xc_{-\rho})$, supposing $x$ and $\rho$ given. If $\rho \neq -1$, put $xc_{-\rho}$ and $c_\rho - \rho(1+\rho)^{-1}e$ in [*]; using (3.4), one finds after simplification

$$T[T(xc_{-\rho})\cdot c_\rho] = T(pc_\rho) = \rho(pc_\rho - Tx).$$

Substituting this in (3.7), we get

$$(\rho I - T)K_\rho x = x, \tag{3.8}$$

when $\rho \neq -1$. If $\rho = -1$, put instead $xc_1$ and $c_{-1}+t = e$ in [*]; one finds

$$T(pc_{-1}) = pt - T(xc_1 t),$$

and then (3.7) becomes

$$\begin{aligned}(-I-T)K_{-1}x &= x - pc_{-1} + Tx - pt + T(xc_1 t)\\ &= x - p(e-t) - pt + T[x(c_1 t + e)] \text{ since } c_{-1} = e-t\\ &= x - p + T(xc_1) \text{ since } c_1 t + e = c_1\\ &= x.\end{aligned}$$

Thus (3.8) holds for all $\rho \neq 0$.

Again

$$(3.9)\quad K_\rho(\rho I - T)x = x + \frac{1}{\rho} T(xc_{-\rho}) \cdot c_\rho - \frac{1}{\rho} Tx - \frac{1}{\rho^2} T(Tx \cdot c_{-\rho}) \cdot c_\rho .$$

To simplify the last term, suppose first $\rho \neq 1$ and put $x$, $c_{-\rho} - \rho(\rho-1)^{-1}e$ in [*], to find

$$T(Tx \cdot c_{-\rho}) \cdot c_\rho = \rho T(xc_{-\rho}) - \rho Tx \cdot c_{-\rho}.$$

Substituting this in (3.9), we get

$$(3.10)\qquad K_\rho(\rho I - T)x = x,$$

when $\rho \neq 1$. If $\rho = 1$, we have $c_{-1} = e - t$, and to simplify (3.9) it suffices to take $x$, $e$ in [*]. Thus (3.10) holds for all $\rho \neq 0$. Then (3.9) and (3.10) show that $(\rho I - T)^{-1}$ exists, equals $K_\rho$, and so belongs to $\mathfrak{B}(\mathfrak{A})$; so $\rho \in \operatorname{Res}(T)$ and (3.5) follows. The form (3.6) results from the use of the same substitutions in [*], in the reverse order. //

COROLLARY 1. *The solution* $s$ *of* (3.1a) *and* (3.1b), *when it exists, is unique.*

PROOF. If a solution exists, then by the theorem $\operatorname{Sp}(T) = \{0\}$. Let $s$, $s_1$ be two solutions, and write $z = s - s_1$; then $Tz = z$. Since $1 \notin \operatorname{Sp}(T)$, $z = 0$; therefore $s = s_1$. //

COROLLARY 2. *If* $\nu(t) < 1$, *then* $T$ *is quasi-nilpotent.*

PROOF. As pointed out in the second paragraph in this section, $\nu(t) < 1$ implies the existence of the solution $s'$ for (3.1). //

## 4. Some related formulae

For comparison with these results, we list corresponding formulae for some other classes of operators, specified by identities akin to the Reynolds identity. The similarities, as well as the differences, are quite striking.

In each case the operator is denoted by $T$ and is assumed to lie in $\mathfrak{B}(\mathfrak{A})$; the defining identity is stated, and is assumed to hold for all $x$, $y \in \mathfrak{A}$. We write $t = Te$, $f_\lambda = R(\lambda, T)e$.

1° *Antiderivation* $Tx \cdot Ty = T(Tx \cdot y + x \cdot Ty)$. Here $\mathrm{Sp}\,(T) = \{0\}$, and for all $\lambda \neq 0$,

$$R(\lambda, T)x = \frac{1}{\lambda}\, x + \frac{1}{\lambda^2}\, T(xf_\lambda^{-1}) \cdot f_\lambda,$$

where $f_\lambda = 1/\lambda \exp\,(t/\lambda)$.

Also $T(t^n) = t^{n+1}/(n+1)$ $(n = 1, 2, \cdots)$. See [1] and [5].

2° *Summation operator* $Tx \cdot Ty = T(Tx \cdot y + x \cdot Ty - xy)$. Here $\mathrm{Sp}\,(T) \subseteqq \{0, 1\}$, and for all $\lambda \neq 0, 1$,

$$R(\lambda, T)x = \frac{1}{\lambda}\, x + \frac{1}{\lambda(\lambda-1)}\, T(xf_\lambda^{-1}) \cdot f_\lambda$$

where

$$f_\lambda = \frac{1}{\lambda} \exp \left[ t \log \left( \frac{\lambda}{\lambda - 1} \right) \right].$$

Also

$$T(t^n) = \frac{1}{n+1}\, \Phi_{n+1}(e+t) \qquad (n = 1, 2, \cdots),$$

where $\Phi_n$ denotes the $n$th Bernoulli polynomial. See [6]. (That the formula for $R(\lambda, T)x$ holds for all $\lambda \neq 0, 1$ can be verified, as in the proof of the theorem in § 3; one uses (4.2), (4.5) of [6]. For $x = e$ the result occurs in [1], p. 17. See also [7], where a fairly complete description of the spectral properties of summation operators is given.)

3° *Averaging operator* $T(Tx \cdot y) = Tx \cdot Ty = T(x \cdot Ty)$. Here

$$R(\lambda, T)x = \frac{1}{\lambda}\, x + \frac{1}{\lambda}\, Tx \cdot f_\lambda$$

where

$$f_\lambda = R(\lambda, t).$$

The spectral properties of $T$ can be related closely to those of $t$, and the formula leads to a fairly detailed analysis of $\mathrm{Sp}\,(T)$, particularly when $\mathfrak{A} = C(X)$, $X$ compact. See [3]. Also

$$T(t^n) = t^{n+1} \qquad (n = 1, 2, \cdots).$$

4° *Reynolds operator* $Tx \cdot Ty = T(Tx \cdot y + x \cdot Ty - Tx \cdot Ty)$. In the restricted case considered in this paper, $\mathrm{Sp}(T) = \{0\}$ and

$$R(\lambda, T)x = \frac{1}{\lambda}x + \frac{1}{\lambda^2}T\left[x\left(e+\frac{1}{1+\lambda}f_{\lambda/(1+\lambda)}\right)^{-1}\right]\cdot\left(e+\frac{1}{1+\lambda}f_{\lambda/(1+\lambda)}\right),$$

where

$$f_\mu = \frac{e}{\mu-1} - \frac{1}{\mu(\mu-1)}\exp\left[\frac{\mu-1}{\mu}\log(e-t)\right].$$

Also

$$\frac{1}{n}T(t^n) = t+(e-t)\log(e-t)-\left(\frac{t^2}{1\cdot 2}+\frac{t^3}{2\cdot 3}+\cdots+\frac{t^n}{(n-1)n}\right)$$

$$(n = 1, 2, 3, \cdots).$$

## References

[1] F. V. Atkinson, 'Some aspects of Baxter's functional equation', *J. of Math. Anal. and Applic.* 7 (1963), 1—30.

[2] G. Baxter, 'An analytic problem whose solution follows from a simple algebraic identity', *Pac. J. of Math.* 10 (1960), 731—742.

[3] J. L. B. Gamlen and J. B. Miller, 'Averaging and Reynolds operators on Banach algebras, II. Spectral properties of averaging operators', *J. of Math. Anal. and Applic.* (to appear).

[4] E. Hille, 'On roots and logarithms of elements of a complex Banach algebra', *Math. Annalen* 136 (1958), 46—57.

[5] J. B. Miller, 'Some properties of Baxter operators', *Acta Math. Acad. Sci. Hungar.* 17 (1966), 387—400.

[6] J. B. Miller, 'Averaging and Reynolds operators on Banach algebras, I. Representation by derivations and antiderivations', *J. of Math. Anal. and Applic.* 14 (1966), 527—548.

[7] J. B. Miller, 'Baxter operators and endomorphisms on Banach algebras' (to appear).

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Melbourne