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Harmonicity of Holomorphic Maps Between Almost Hermitian Manifolds

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Abstract. In this paper we study holomorphic maps between almost Hermitian manifolds. We obtain a new criterion for the harmonicity of such holomorphic maps, and we deduce some applications to horizontally conformal holomorphic submersions.

1 Introduction

A map $f\colon M \to N$ between Riemannian manifolds is harmonic if it is a critical point of the energy density of f. In [3], J. Eells and J. H. Sampson proved that a holomorphic map between Kähler manifolds is a harmonic map. This result was generalized by A. Lichnerowicz for a holomorphic map between almost Kähler manifolds and also for a holomorphic map between a semi-Kähler manifold and a quasi-Kähler manifold (see [8]). Harmonic morphisms are harmonic maps which satisfy the additional condition of horizontal conformality. In [6], S. Gudmundsson and J. C. Wood obtained conditions for a holomorphic map between almost Hermitian manifolds to be a harmonic or a morphism harmonic, and they generalized a result of B. Watson on the harmonicity of certain almost Hermitian submersions (see [10, 11]).

In this paper we study holomorphic maps between almost Hermitian manifolds; we obtain a new criterion for the harmonicity of such holomorphic maps, and we deduce some applications to horizontally conformal holomorphic submersions.

In Section 2 we recall the definitions and some properties of harmonicity, almost Hermitian manifolds, and holomorphic and horizontally conformal maps. In Section 3 we give an expression for the tension field $\tau(f)$ of a holomorphic map $f\colon M \to N$ between almost Hermitian manifolds (see Proposition 3.1) which generalizes the one obtained by S. Gudmundsson and J. C. Wood in [6] when N is quasi-Kähler. Also we obtain the criterion for the harmonicity of holomorphic maps (Theorem 3.3). In section 4 we introduce, for a horizontally conformal holomorphic submersion $f\colon (M, J, g) \to (N, J', g')$, the tensor B defined by B. Watson and L. Vanhecke in [11] for almost Hermitian submersions, and we obtain relations between the divergences of the almost complex structures J and J' and the tensors B and $\tau(f)$ (Propositions 4.5 to 4.8). These results are applied in Section 5 in order to study the harmonicity, the minimality of the fibres, and the transference of structures on horizontally conformal holomorphic submersions, which generalize some results of S. Gudmundsson and J. C. Wood, and of B. Watson and L. Vanhecke for almost Hermitian submersions ([10] and [11]).

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2 Preliminaries

Let (M, g) and (N, g') be two Riemannian manifolds and $f\colon M \to N$ a smooth map. We denote by ∇ and ∇' the Riemannian connections on M and N, respectively and by ∇^f the pull-back of the Riemannian connection ∇' on N to the pull-back bundle $f^{-1}TN \to M$, which is given by $\nabla^f_X V = \nabla'_{f_*X} V$ for $X \in \mathfrak{X}(M)$ and V a C^∞-section on the induced bundle $f^{-1}TN$. The *tension field* $\tau(f)$ is the trace of the second fundamental form of f, *i.e,*

$$\tau(f) = \sum_{k=1}^{m} (\nabla^f_{e_k} f_*(e_k) - f_*(\nabla_{e_k} e_k)),$$

where $\{e_1, \dots, e_m\}$ is a local orthonormal basis for $\mathfrak{X}(M)$.

We recall that f is *harmonic* if it is a critical point of the energy density function of f. This condition is equivalent to $\tau(f) = 0$.

The map $f\colon M \to N$ is said to be a *harmonic morphism* if for each open subset U of N with $f^{-1}(U) \neq \varnothing$ and each harmonic function $h\colon U \to \mathbb{R}$ the composition $h \circ f\colon f^{-1}(U) \to \mathbb{R}$ is harmonic.

We recall that a map $f\colon M \to N$ between Riemannian manifolds of equal dimension is *conformal* if, for each point $x \in M$, the induced linear map (*i.e.,* the differential) $f_{*x}\colon T_xM \to T_{f(x)}N$ is conformal with respect to the Riemannian metrics. Horizontal conformality is a generalization of this concept to the case when the target manifold is of lower dimension than the domain. If $f\colon M \to N$ is a map between two Riemannian manifolds, and $x \in M$ is a non-degenerate point, we decompose the space T_xM into its *vertical space* $\mathcal{V}_x = \operatorname{Ker} f_{*x}$ and its *horizontal space* $\mathcal{H}_x = (\operatorname{Ker} f_{*x})^\perp$, *i.e.,* the orthogonal complement of $\mathcal{V}_x$, so that $T_xM = \mathcal{V}_x \oplus \mathcal{H}_x$. The map is said to be *horizontally conformal* if for each point $x \in M$ either the rank of f_{*x} is 0 (*i.e., x* is a critical point), or the restriction of f_{*x} to the horizontal space $\mathcal{H}_x$ is surjective and conformal (here x is regular point). This second property is equivalent to that $g'(f_*X, f_*Y) = \lambda^2 g(X, Y)$ for all horizontal vector fields X and Y. If we put $\lambda = 0$ at the critical points, $\lambda\colon M \to [0, \infty)$ is a continuous function which is smooth at regular points, but whose square λ^2 is smooth on the whole of M.

When $\dim M < \dim N$, the only horizontally conformal maps are the constant mappings. If $\lambda^2 \equiv 1$ the map is called a *Riemannian submersion* and if $\operatorname{grad} \lambda^2 \in \mathcal{V}$, f is called *horizontally homothetic*.

The following result is a characterization for harmonic morphisms (see [2] and [7]).

Theorem 2.1 *A smooth map $f\colon (M, g) \to (N, g')$ is a harmonic morphism if and only if it is a horizontally conformal harmonic map.*

The above theorem expresses in analytical and geometric terms an essentially analytical object and provides a handle on harmonic morphisms.

Also, in [1], P. Baird and J. Eells proved the following.

Theorem 2.2 *Let $f\colon (M, g) \to (N, g')$ be a non-constant horizontal conformal map. Then*

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(i) *If N is a surface then f is a harmonic morphism if and only if its fibres at regular points are minimal.*

(ii) *If* $\dim N \geq 3$, *then any two of the following conditions imply the third:*

(a) *f is harmonic,*

(b) *the fibres of f are minimal at regular points,*

(c) *f is horizontally homothetic,* i.e., $f_*(\operatorname{grad}(\lambda^2)) = 0$.

Now let (M, J, g) be an almost Hermitian manifold of dimension $2m$. Then we have $J^2 = -I$ and $g(JX, JY) = g(X, Y)$ for $X, Y \in \mathfrak{X}(M)$, I being the identity transformation. In such a case, every point x of M has a neighborhood U_x and local vector fields $X_1, \dots, X_m$ on U_x such that $\{X_1, \dots, X_m, JX_1, \dots, JX_m\}$ is a local orthonormal basis for $\mathfrak{X}(M)$, which is called a local Hermitian basis. The fundamental 2-form Φ of M is defined by $\Phi(X, Y) = g(X, JY)$, for $X, Y \in \mathfrak{X}(M)$.

In [4], A. Gray and L. Hervella obtained a complete classification of the almost Hermitian manifolds, where the different classes correspond to $U(n)$-invariant subspaces of the representation space $W_1 \oplus W_2 \oplus W_3 \oplus W_4$. Let us recall some well-known definitions of some classes of almost Hermitian manifolds involved here. An almost Hermitian manifold (M, J, g) is said to be:

- *Kähler* ($\equiv \{0\}$) if $\nabla J = 0$.
- *nearly Kähler* ($\equiv W_1$) if $(\nabla_X J)X = 0$
- *almost Kähler* ($\equiv W_2$) if $d\Phi = 0$.
- *quasi-Kähler* ($\equiv W_1 \oplus W_2$) if $(\nabla_X J)Y + \nabla_{JX} J)JY = 0$.
- *semi-Kähler* ($\equiv W_1 \oplus W_2 \oplus W_3$) if $\delta J = 0$, where δ denotes the codifferential in (M, g).

The relations among these classes are represented in the following diagram (where $\longrightarrow$ denotes strict inclusion).

$$\begin{tikzcd}
 & \text{nearly Kähler} \arrow[dr] & & \\
\text{Kähler} \arrow[ur] \arrow[dr] & & \text{quasi-Kähler} \arrow[r] & \text{semi-Kähler} \\
 & \text{almost Kähler} \arrow[ur] & &
\end{tikzcd}$$

Relations between almost Hermitian structures.

We recall that a map $f\colon M \to N$, between two almost Hermitian manifolds (M, J, g) and (N, J', g'), is holomorphic if $J' \circ f_* = f_* \circ J$.

The result obtained by Lichnerowicz on the harmonicity of holomorphic maps between a semi-Kähler manifold and a quasi-Kähler manifold was also induced by Gudmundsson and Wood in [6]. Indeed, they proved that if $f\colon M \to N$ is a holomorphic map and if N is quasi Kähler then,

$$\tau(f) = -f_*(J\delta J).$$

A smooth surjective map $f\colon M \to N$ is called a *Riemannian submersion* if f has maximal rank and $f_{*|\mathcal{H}}$ is a linear isometry, where $\mathcal{H}$ is of Horizontal distribution associated to f. We say that f is an *almost Hermitian submersion* if f is a Riemannian submersion which additionally is holomorphic. For a detailed study on almost Hermitian submersions we refer the reader to [10].

Riemannian submersions are special cases of horizontally conformal maps. If $f\colon M \to N$ is a horizontally conformal submersion and holomorphic then we say that f is a *horizontally conformal holomorphic submersion*, with dilation λ.

The fibres of a horizontally conformal holomorphic submersion are almost Hermitian manifolds, of dimension $2(m-n)$, with the induced structure by the total space (M, J, g), and which also is denoted by (J, g). In general, the horizontal distribution $\mathcal{H}(\mathcal{M})$ is not completely integrable.

If $f\colon (M, g) \to N$ is a submersion, we can introduce the *fundamental tensors* of f which are given by (see [9]):

$$T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF,$$

$$A_E F = h\nabla_{hE} vF + v\nabla_{hE} hF,$$

for all vector fields E and F on M, where h and v denote the horizontal and vertical projections, respectively.

If f is a horizontally conformal submersion we have

Proposition 2.3 **([5])** *Let $f\colon (M, g) \to (N, g')$ be a horizontally conformal submersion with dilation λ then,*

$$A_X Y = \frac{1}{2}\{v[X, Y] - \lambda^2 g(X, Y)\operatorname{grad}_v(\lambda^{-2})\}$$

for all X, Y horizontal vector fields.

3 Holomorphic Maps and Harmonicity

In this section we will obtain an expression for the tension field for any holomorphic map between almost Hermitian manifolds that we will use to deduce a characterization of its harmonicity.

If $f\colon (M, g) \to (N, g')$ is a smooth map, we denote by $\operatorname{trac}_g f^*(\nabla' J')$ the trace of the tensor field $f^*(\nabla' J')$ by g, which is given by

$$\operatorname{trac}_g f^*(\nabla' J') = \sum_{i=1}^{2m} (\nabla'_{f_* e_i} J') f_* e_i$$

where $\{e_1, \dots, e_{2m}\}$ is a local orthonormal basis on (M, g). Then we have,

Proposition 3.1 *Let $f\colon (M, J, g) \to (N, J', g')$ be a holomorphic map. Then the tension field $\tau(f)$ of f is given by*

$$\tau(f) = J'(\operatorname{trac}_g f^*(\nabla' J') - f_*(\delta J)),$$

Proof Let $\{X_1, \dots, X_m, JX_1, \dots, JX_m\}$ be a local Hermitian basis in M. Then

$$\tau(f) = \sum_{k=1}^{m} \big[\nabla^f_{X_k} f_*(X_k) - f_*(\nabla_{X_k} X_k) + \nabla^f_{JX_k} f_*(JX_k) - f_*(\nabla_{JX_k} JX_k) \big] .$$

Now,

$$\begin{aligned}
\delta J &= \sum_{k=1}^{m} \big[(\nabla_{X_k} J) X_k + (\nabla_{JX_k} J) JX_k \big] \\
&= \sum_{k=1}^{m} (\nabla_{X_k} JX_k - \nabla_{JX_k} X_k) - \sum_{k=1}^{m} J(\nabla_{X_k} X_k + \nabla_{JX_k} JX_k),
\end{aligned}$$

and then we have,

$$\sum_{k=1}^{m} (\nabla_{X_k} X_k + \nabla_{JX_k} JX_k) = J\delta J - J(\sum_{k=1}^{m} (\nabla_{X_k} JX_k - \nabla_{JX_k} X_k).$$

On the other hand, since that $f_*([X,Y]) = \nabla^f_X f_* Y - \nabla^f_Y f_* X$ for all $X, Y \in \mathfrak{X}(M)$, we have

$$f_*(\nabla_{X_k} JX_k - \nabla_{JX_k} X_k) = f_*([X_k, JX_k]) = \nabla^f_{X_k} f_*(JX_k) - \nabla^f_{JX_k} f_*(X_k).$$

Thus, we deduce

$$\begin{aligned}
\tau(f) &= \sum_{k=1}^{m} (\nabla^f_{X_k} f_*(X_k) + \nabla^f_{JX_k} f_*(JX_k)) - f_*(J\delta J) + f_*(J(\sum_{k=1}^{m} (\nabla_{X_k} JX_k - \nabla_{JX_k} X_k)) \\
&= \sum_{k=1}^{m} [\nabla^f_{X_k} f_*(X_k) + \nabla^f_{JX_k} f_*(JX_k) + J'(\nabla^f_{X_k} f_*(JX_k) - \nabla^f_{JX_k} f_*(X_k))] - f_*(J\delta J).
\end{aligned}$$

Furthermore, from definition of ∇^f and as f is holomorphic it follows that:

$$\begin{aligned}
&\sum_{k=1}^{m} [\nabla^f_{X_k} f_*(X_k) + \nabla^f_{JX_k} f_*(JX_k) + J'(\nabla^f_{X_k} f_*(JX_k) - \nabla^f_{JX_k} f_*(X_k))] \\
&= J'(\sum_{k=1}^{m} [\nabla'_{f_* X_k} J' f_* X_k - J' \nabla'_{f_* X_k} f_* X_k - \nabla'_{J' f_* X_k} f_* X_k - J' \nabla'_{J' f_* X_k} J' f_* X_k]) \\
&= J'(\sum_{k=1}^{m} [(\nabla'_{f_* X_k} J') f_* X_k + (\nabla'_{J' f_* X_k} J') J' f_* X_k]) \\
&= J'(\mathrm{trac}_g\, f^*(\nabla' J')).
\end{aligned}$$

This completes the proof. ∎

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If (N, J', g') is quasi-Kähler (or nearly Kähler or almost Kähler), then $\operatorname{trac}_g f^*(\nabla' J') = 0$, hence from the above proposition we deduce the following.

Corollary 3.2 [6] *Let $f\colon (M, J, g) \to (N, J', g')$ be a holomorphic map. If N is quasi-Kähler then the tension field $\tau(f)$ of f is given by*

$$\tau(f) = -J'(f_*(\delta J)).$$

From Proposition 3.1 we obtain a criterion for the harmonicity of holomorphic maps between almost Hermitian manifolds.

Theorem 3.3 *Let $f\colon (M, J, g) \to (N, J', g')$ be a holomorphic map. Then f is harmonic if and only if $\operatorname{trac}_g f^*(\nabla' J') = f_*(\delta J)$.*

From this theorem we deduce the following result of Lichnerowicz, [8].

Corollary 3.4 *Let $f\colon (M, J, g) \to (N, J', g')$ be a holomorphic map from a semi-Kähler manifold to a quasi-Kähler one. Then f is harmonic.*

4 Horizontally Conformal Holomorphic Submersions

Next, we will find the expressions between the divergences or codifferentials of the almost Hermitian structures of the total and base spaces and on the fibres of one horizontally conformal holomorphic submersion, and we will relate them to the tension field.

Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion. We define a tensor field B by

$$B(E, F) = v\nabla_{hE} JhF - v\nabla_{JhE} hF + h\nabla_{hE} JvF - h\nabla_{JhE} vF,$$

for all $E, F \in \mathfrak{X}(M)$. The tensor B was defined by B. Watson and L. Vanhecke for almost Hermitian submersions (see, [11]).

From this definition we have the following.

Proposition 4.1 *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion. Then for all horizontal vector fields X, Y on M,*

(i) $B(X, Y) = A_X JY - A_{JX} Y$,
(ii) $B(X, Y) - B(Y, X) = 2g(JX, Y)\lambda^2 \operatorname{grad}_V(\lambda^{-2})$,
(iii) $B(X, Y) = B(JX, JY)$.

Now, if M is quasi-Kähler we have the following.

Proposition 4.2 *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion. If M is quasi-Kähler then, for all horizontal vector fields X, Y on M,*

(i) $A_X JX = -\frac{\lambda^2}{2} g(X, X) J(\operatorname{grad}_V(\lambda^{-2}))$,
(ii) $B(X, Y) = \lambda^2(-g(X, Y) J(\operatorname{grad}_v(\lambda^{-2})) + g(JX, Y) \operatorname{grad}_v(\lambda^{-2}))$.

Proof If M is quasi-Kähler, then

$$\nabla_X JX - \nabla_{JX} X = J(\nabla_X X + \nabla_{JX} JX)$$

If X is horizontal, from vertical part of this equation we have

$$A_X JX - A_{JX} X = J(A_X X + A_{JX} JX).$$

Now, from definition of A we have that $A_X JX = -A_{JX} X$ and $A_X X = A_{JX} JX$. Thus

$$2A_X JX = 2JA_X X = -g(X,X)\lambda^2 J(\operatorname{grad}_V(\lambda^{-2})),$$

and

$$B(X,X) = -g(X,X)\lambda^2 J(\operatorname{grad}_V(\lambda^{-2})),$$

and hence we deduce that

$$B(X,Y) + B(Y,X) = -2g(X,Y)\lambda^2 J(\operatorname{grad}_V(\lambda^{-2})).$$

Now the result follows from the above proposition. ■

Also, we have the following.

Proposition 4.3 *Let $f\colon (M,J,g) \to (N,J',g')$ be a horizontally conformal holomorphic submersion. If M is quasi-Kähler then B vanishes on horizontal vector fields if and only if* $\operatorname{grad}_v(\lambda^{-2}) = 0$.

From this proposition one can easily deduce the following result of B. Watson and L. Vanhecke ([11]).

Proposition 4.4 *Let $f\colon (M,J,g) \to (N,J',g')$ be an almost Hermitian submersion. If M is quasi-Kähler then B vanishes on horizontal vector fields.*

Next we obtain relations between the divergences of the almost complex structures J and J'. We denote the divergence of J on the fibres of f by $\hat{\delta} J$.

Proposition 4.5 *Let $f\colon (M,J,g) \to (N,J',g')$ be a horizontally conformal holomorphic submersion. Then*

$$\nu(\delta J) = \hat{\delta} J + \frac{1}{2}\operatorname{trac} B + n\lambda^2 J(\operatorname{grad}_V(\lambda^{-2})).$$

Proof Let $\{E_1,\dots,E_{m-n}, JE_1,\dots,JE_{m-n}, F_1,\dots,F_n, JF_1,\dots,JF_n\}$ be a local Hermitian basis, being E_k vertical vector fields and F_k horizontal vector fields. Then

$$\begin{aligned}\nu(\delta J) &= \nu\Big(\sum_{i=1}^{m-n}[(\nabla_{E_i} J)E_i + (\nabla_{JE_i} J)JE_i]\Big) + \nu\Big(\sum_{j=1}^{n}[(\nabla_{F_j} J)F_j + (\nabla_{JF_j} J)JF_j)]\Big)\\ &= \nu\Big(\sum_{i=1}^{m-n}[(\nabla_{E_i} J)E_i + (\nabla_{JE_i} J)JE_i]\Big) + \nu\Big(\sum_{j=1}^{n}[(\nabla_{F_j} JF_j - \nabla_{JF_j} F_j)\\ &\qquad - J(\nabla_{F_j} F_j + \nabla_{JF_j} JF_j)]\Big).\end{aligned}$$

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Now

$$\nu\Big(\sum_{i=1}^{m-n}[(\nabla_{E_i}J)E_i + (\nabla_{JE_i}J)JE_i]\Big) = \hat{\delta}J,$$

$$\nu(\nabla_{F_j}JF_j - \nabla_{JF_j}F_j) = B(F_j, F_j),$$

and

$$\nu(\nabla_{F_j}F_j + \nabla_{JF_j}JF_j) = A_{F_j}F_j + A_{JF_j}JF_j = -\lambda^{-2}\operatorname{grad}_\nu(\lambda^{-2}).$$

Thus, using these relations in the expression of $\nu(\delta J)$ we obtain the result. ■

Proposition 4.6 *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion. Then*

$$\operatorname{trac}_g f^*(\nabla' J') = \lambda^2 \delta' J'.$$

Proof Let $\{E_1, \dots, E_{m-n}, JE_1, \dots, JE_{m-n}, F_1, \dots, F_n, JF_1, \dots, JF_n\}$ be a local Hermitian basis, being E_k vertical vector fields and F_k horizontal vector fields obtained as follows: if $\{F'_1, \dots, F'_n, J'F'_1 \dots, J'F'_n\}$ is a Hermitian basis on N we consider F^*_j the horizontal lifts of F'_j, $j = 1, \dots, n$ and normalize by setting $F_j = \lambda F^*_j$, where λ is the dilation of f. Then

$$\operatorname{trac}_g f^*(\nabla' J') = \sum_{j=1}^{n}[(\nabla'_{f_*F_j}J')f_*F_j + (\nabla'_{f_*JF_j}J')f_*JF_j] = \lambda^2\delta' J'. \qquad ■$$

From Propositions 3.1 and 4.6 we obtain the following.

Proposition 4.7 *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion. Then the tension field $\tau(f)$ of f is given by*

$$\tau(f) = \lambda^2 J'\delta' J' - J' f_*(\delta J).$$

Finally, combining Propositions 4.5 and 4.6 we have the following.

Proposition 4.8 *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion. Then*

$$\delta J = \hat{\delta}J + \tfrac{1}{2}\operatorname{trac} B + n\lambda^2 J(\operatorname{grad}_V(\lambda^{-2})) + \lambda^2(\delta' J')^* + (J'\tau(f))^*$$

where $(\delta' J')^*$ and $(J'\tau(f))^*$ are the horizontal lifts of $\delta' J'$ and $J'\tau(f)$ on M respectively.

5 Harmonicity, Minimality and Horizontally Conformal Holomorphic Semi-Kähler Submersions

In this section we will apply the results obtained in Section 4 previous to horizontally conformal holomorphic submersions with total or base space a semi-Kähler manifold. For these submersions we will study the harmonicity of f, the minimality of the fibres, and the transference of the almost Hermitian structures between the total and base spaces and the fibres.

First, from Proposition 4.7 we have the following.

Theorem 5.1 *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion with dilation λ. Then f is harmonic (and so a harmonic morphism) if and only if $f_*(\delta J) = \lambda^2 \delta' J'$.*

We can also deduce the following.

Proposition 5.2 [6] *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion with dilation λ. Then, any two of the following conditions imply the third:*

(i) *f is harmonic and so a harmonic morphism,*
(ii) *N is semi-Kähler.*
(iii) *$f_* \delta J = 0$.*

Corollary 5.3 *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion with dilation λ, with semi-Kähler total space M. Then N is semi-Kähler if and only if f is a harmonic morphism.*

Now, from the definition of the tension field $\tau(f)$ it is not hard to check that

$$\tau(f) = -2(m-n) f_*(H) + (n-1)\lambda^2 f_*(\mathrm{grad}_H(\lambda^{-2})),$$

where H is the mean curvature of the fibres. Then from Proposition 4.7 we have

$$J' f_*(\delta J) = \lambda^2 J' \delta' J' + 2(m-n) f_*(H) - (n-1)\lambda^2 f_*(\mathrm{grad}_H(\lambda^{-2})).$$

Thus, we obtain the following.

Proposition 5.4 *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion with dilation λ, with semi-Kähler total space M. Then, if $n \neq 1$ and $m \neq n$, any two of the following conditions imply the third:*

(i) *N is semi-Kähler.*
(ii) *The fibres of f are minimal.*
(iii) *f is horizontally homothetic,* i.e., $\mathrm{grad}_H(\lambda^{-2}) = 0$.

Also, by using Proposition 4.5 we have the following.

Proposition 5.5 *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion with dilation λ, with semi-Kähler total space M. Then, any two of the following conditions imply the third:*

(i) *The fibres are semi-Kähler.*
(ii) $\mathrm{trac}\, B = 0$.
(iii) $\mathrm{grad}_V(\lambda^{-2}) = 0$.

In a similar way, from Proposition 4.8 we deduce the following.

Proposition 5.6 *Let $f\colon (M, J, g) \to (N, J', g')$ be a horizontally conformal holomorphic submersion with dilation λ, with base space N and semi-Kähler fibres. Then, the total space M is semi-Kähler if and only if f is harmonic and*

$$\frac{1}{2}\,\mathrm{trac}\, B = -n\lambda^2 J(\mathrm{grad}_V(\lambda^{-2})).$$

Now, if $f\colon (M, J, g) \rightarrow (N, J', g')$ is an almost Hermitian submersion then $\operatorname{grad}(\lambda^{-2}) = 0$, ($\lambda = 1$). Thus, from above propositions we have,

Proposition 5.7 [10] *Let $f\colon (M, J, g) \rightarrow (N, J', g')$ be an almost Hermitian submersion with semi-Kähler total space M. Then N is semi-Kähler if and only if the fibres of f are minimal.*

Proposition 5.8 [11] *Let $f\colon (M, J, g) \rightarrow (N, J', g')$ be an almost Hermitian submersion with semi-Kähler total space M. Then the fibres are semi-Kähler if and only if* $\operatorname{trac} B = 0$.

Proposition 5.9 [11] *Let $f\colon (M, J, g) \rightarrow (N, J', g')$ be an almost Hermitian submersion with base space N and semi-Kähler fibres. Then the total space M is semi-Kähler if and only if f is harmonic and* $\operatorname{trac} B = 0$.

We note that on a Riemannian submersion, the harmonicity is equivalent to the minimality of the fibres.

References

[1] P. Baird and J. Eells, *A conservation law for harmonic maps.* Geometry Symposium Utrecht 1980, In: Lecture Notes in Math. 894, Springer, Berlin-New York, 1981, pp. 1–25.

[2] B. Fuglede, *Harmonic morphisms between Riemannian manifolds.* Ann. Inst. Fourier **28**(1978), no. 2, 107–144.

[3] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. **86**(1964), 109–160.

[4] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariant.* Ann. Mat. Pura Appl. **123**(1980), 35–58.

[5] S. Gudmundsson *The geometry of harmonic morphisms.* Ph.D. Thesis, University of Leeds (1992).

[6] S. Gudmundsson and J. C. Wood, *Harmonic morphisms between almost Hermitian manifolds.* Boll. Un. Mat. Ital. B(7) **11**(1997), no. 2, suppl., 185–197.

[7] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions.* J. Math. Kyoto Univ. **19** (1979), 215–229.

[8] A. Lichnerowicz, *Applications harmoniques et variétés kähleriennes.* In: 1968/1969 Symposia Mathematica, Vol. III, Academic Press, London, 1970, pp. 341–402.

[9] B. O'Neill, *The fundamental equations of a submersion.* Michigan Math. J. **13**(1966), 459–469.

[10] B. Watson, *Almost Hermitian submersions.* J. Differential Geometry **11**(1976), no. 1, 147–165.

[11] B. Watson and L. Vanhecke, *The structure equation of an almost semi-Kähler submersion.* Houston J. Math. **5**(1979), no. 2, 295–305.

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