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PAPER

Modelling of the fluid flow in a thin domain with injection through permeable boundary

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Abstract

In this paper, we derive the effective model describing a thin-domain flow with permeable boundary through which the fluid is injected into the domain. We start with incompressible Stokes system and perform the rigorous asymptotic analysis. Choosing the appropriate scaling for the injection leads to a compressible effective model. In this paper, we derive the effective model describing a thin-domain flow with permeable boundary through which the fluid is injected into the domain. We start with incompressible Stokes system and perform the rigorous asymptotic analysis. Choosing the appropriate scaling for the injection leads to a compressible effective model.

1. Introduction

Incompressible fluid flows through thin domains (i.e. domains whose size in some directions is much larger than the size in others) appear naturally in various applications. Typical examples of such domains are thin channels, pipes and fractures. Due to its two-scale geometry, numerical studies of partial differential equations in such domains are difficult. Typically, thin domains have impermeable, immobile upper and lower boundaries, in which case their flow is governed by the Hagen-Poiseuille flow [11], [27]. Hagen-Poiseuille type approximations have been rigorously derived for steady flows through a single tube (see e.g. [8], [9], [13], [23], [18]) and employed for analysing the flows in more complex thin structures (see e.g. [2], [3], [14]) and in time-dependent regime as well (see e.g. [24], [25], [26]). Introducing the boundary roughness leads to the Darcy-Weisbach law [29] and its improvements (see e.g. [15], [21], [22]). In the lubrication theory, upper and lower boundaries are in relative motion, leading to the non-homogeneous Dirichlet condition. However, the prescribed non-zero velocity on the boundary is tangential to the boundary leading to the Reynolds law [28] and its variants (see e.g. [1], [4], [10]).

In the present paper, we study the case when the lower boundary is permeable (for example, porous) so that the prescribed velocity on the boundary is non-zero and perpendicular to the boundary. Even though the fluid was originally incompressible, as in [16], the obtained model is compressible and that represents the main novelty of this paper. Due to the injection of fluid through the lower boundary, the weak rescaled limit of the boundary velocity appears as the source term in the mass conservation equation, see Theorem 2. It is important to emphasise that the velocity on the permeable boundary is given, and not described by the Darcy boundary law, Beavers-Joseph law or pressure boundary law like in [5], [6], [7], [19] and [20].

To derive the effective model, we start from the stationary Stokes system and perform the rigorous asymptotic analysis, using the two-scale convergence for thin domains introduced in [12] (see also [17]).

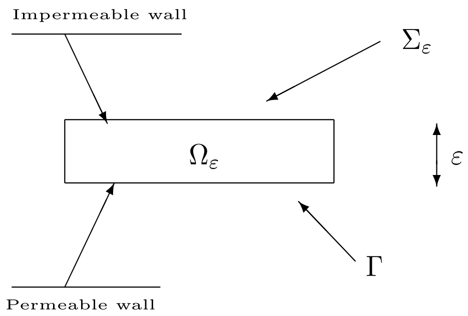

Figure 1. *Domain Ω_ε with permeable boundary Γ.*

We neglect the inertial term, assuming that the Reynolds number is not large, which is, in most cases, reasonable for thin domain. The inertial term causes problem with existence and uniqueness of the solution, due to the pressure boundary condition, unless we assume that the Reynolds number is small.

2. Setting of the problem

For simplicity, we assume that the domain Ω_ε is the rectangle with thickness $\varepsilon \ll 1$ that has impermeable upper and permeable lower boundary (see Figure 1):

$$\Omega_\varepsilon = \langle 0, L\rangle \times \langle 0, \varepsilon\rangle, \tag{2.1}$$
$$\Sigma^\varepsilon = \langle 0, L\rangle \times \{\varepsilon\}, \tag{2.2}$$
$$\Gamma = \langle 0, L\rangle \times \{0\}. \tag{2.3}$$

As indicated in the Introduction, the flow of incompressible viscous fluid in the domain Ω_ε is described by the Stokes system. We impose a no-slip condition on the upper boundary Σ^ε. The flow is governed by the pressure drop between the left and the right end of the domain, and the injection of the fluid through the lower boundary Γ, which is porous having periodically distributed holes (see Remark 1). As the system is linear, without losing generality, we can choose the viscosity $\mu = 1$. The injection through the porous boundary Γ occurs with some given velocity and, thus, on each hole, we prescribe the injection velocity $\mathbf{g}^\varepsilon$. In view of that, we study the following system:

$$-\Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{0} \;\; , \;\; \operatorname{div} \mathbf{u}^\varepsilon = 0 \;\text{ in }\; \Omega_\varepsilon \, , \tag{2.4}$$

$$\mathbf{u}^\varepsilon = \mathbf{g}^\varepsilon \quad \text{on} \;\; \Gamma, \tag{2.5}$$

$$\mathbf{u}^\varepsilon = \mathbf{0} \quad \text{on} \;\; \Sigma_\varepsilon, \tag{2.6}$$

$$u_2^\varepsilon = 0, \;\; p^\varepsilon(x, y) = \frac{1}{\varepsilon^2} P_x \;\; \text{for } x = 0, L, \tag{2.7}$$

where $P_0, P_L \in \mathbf{R}$.

3. The effective model

Since the domain is thin, we aim to find a simpler lower dimensional model approximating the solution $(\mathbf{u}^\varepsilon, p^\varepsilon)$ of the system (2.4), via rigorous asymptotic analysis with respect to ε. Before we proceed, we announce the main result. At this point, we skip technical assumptions on g^ε, concerning its asymptotic behaviour, regularity and boundary values. Roughly speaking, if the mean value of the boundary injection velocity asymptotically behaves like

$$\overline{g}_\varepsilon = \left[g_0(x) + \varepsilon g_1(x) \right] \, \mathbf{j},$$

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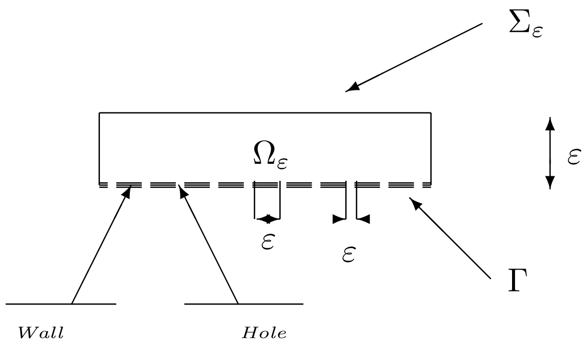

Figure 2. *The domain Ω_ε with perforated boundary.*

then we find an approximation of the form

$$\mathbf{v} = \left\{ \frac{1}{\varepsilon} \left[\frac{1}{L} \int_0^L t\, \overline{g}_\varepsilon(t)\, dt - \int_x^L \overline{g}_\varepsilon(t)\, dt \right] - \frac{1}{12} \frac{P_L - P_0}{L} \right\} \mathbf{i} + \frac{1}{2} \overline{g}_\varepsilon \mathbf{j}$$

$$p = \frac{1}{\varepsilon^3} \left\{ 12 \left[\int_x^L (x-t)\overline{g}_\varepsilon(t)dt + \left(1 - \frac{x}{L}\right) \int_0^L t\overline{g}_\varepsilon(t)dt \right] \right\} + \tag{3.1}$$
$$+ \frac{1}{\varepsilon^2} \left\{ P_0 + (P_L - P_0)\frac{x}{L} \right\}.$$

The above effective model is justified in the sequel through two steps.

4. Injection of order ε

In this section, we study the case of the weaker injection through the permeable boundary. More precisely, denoting the standard Cartesian basis by $(\mathbf{i}, \mathbf{j})$, we assume that $\mathbf{g}^\varepsilon = g^\varepsilon\, \mathbf{j}$, where $g^\varepsilon \in H_0^1(0, L)$ is such that

$$|g^\varepsilon|_{L^\infty(0,L)} \le C, \quad \left| \frac{dg^\varepsilon}{dx} \right|_{L^2(0,L)} \le C\varepsilon^{-1}, \tag{4.1}$$

with constant $C > 0$, independent of ε. Furthermore, we assume that

$$\varepsilon^{-1}\, \mathbf{g}^\varepsilon \rightharpoonup g \ \text{ weak}^* \text{ in } \mathcal{M}(0, L). \tag{4.2}$$

Remark 1. Let us give three examples of such functions g^ε.

The simplest example is a single function, independent on ε, multiplied by ε, i.e.

$$g^\varepsilon = \varepsilon\, g,$$

with $g \in H_0^1(0, L)$ independent of ε.

The second example is given by $g^\varepsilon(x) = \varepsilon g(x, x/\varepsilon)$, where $t \mapsto g(x, t)$ is a smooth periodic function, with period 1, such that $g(0, 0) = g(1, 1/\varepsilon) = 0$. For example, the lower boundary could be porous, with periodically distributed holes, as in the Figure 2 below.

The aim of this paper is to rigorously derive the effective model describing the fluid flow in Ω_ε described by (2.4)–(2.7). To begin with, we introduce the modified pressure as

$$q^\varepsilon = p^\varepsilon - \frac{1}{\varepsilon^2} \left(P^0 + \frac{x}{L}(P_L - P_0) \right). \tag{4.3}$$

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Now $(\mathbf{u}^\varepsilon, q^\varepsilon)$ satisfy the system:

$$-\Delta \mathbf{u}^\varepsilon + \nabla q^\varepsilon = \frac{P_0 - P_L}{\varepsilon^2 L}\,\mathbf{i} \;\;, \;\; \text{div } \mathbf{u}^\varepsilon = 0 \;\text{ in }\; \Omega_\varepsilon \,, \tag{4.4}$$

$$\mathbf{u}^\varepsilon = \mathbf{g}^\varepsilon \;\text{ on }\; \Gamma, \tag{4.5}$$

$$\mathbf{u}^\varepsilon = \mathbf{0} \;\text{ on }\; \Sigma_\varepsilon, \tag{4.6}$$

$$u_2^\varepsilon = 0, \;\; q^\varepsilon = 0 \;\text{ for } x = 0, L. \tag{4.7}$$

4.1 A priori estimates

Before we proceed, we recall that the constants in the Poincare, Sobolev and Nečas inequalities in thin domain depend on its thickness in the following way:

Lemma 1. *There exists a constant $C > 0$ independent of ε, such that for any $\phi \in H^1(\Omega_\varepsilon)$ satisfying $\phi(1, y) = 0$ and any $\varphi \in L^2_0(\Omega_\varepsilon)$ the following estimate hold:*

$$|\phi|_{L^2(\Omega_\varepsilon)} \le C\,\varepsilon \;\; |\nabla \phi|_{L^2(\Omega_\varepsilon)} \,, \tag{4.8}$$

$$|\phi|_{L^4(\Omega_\varepsilon)} \le C\,\sqrt{\varepsilon} \;\; |\nabla \phi|_{L^2(\Omega_\varepsilon)} \,, \tag{4.9}$$

$$|\varphi|_{L^2(\Omega_\varepsilon)} \le C\,\varepsilon^{-1} \;\; |\nabla \varphi|_{H^{-1}(\Omega_\varepsilon)} \,. \tag{4.10}$$

For the proofs, we refer the reader to [12], Lemmas 8, 9 and 11.

We continue by deriving the a priori estimates:

Theorem 1. *Let $(\mathbf{u}^\varepsilon, q^\varepsilon)$ be the solution to the Navier-Stokes system (4.4)–(4.7). There exists a constant $C > 0$, independent of ε, such that*

$$|\nabla \mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} \le \frac{C}{\sqrt{\varepsilon}}, \tag{4.11}$$

$$|\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} \le C\,\sqrt{\varepsilon}, \tag{4.12}$$

$$|\nabla q^\varepsilon|_{L^2(\Omega_\varepsilon)} \le \frac{C}{\sqrt{\varepsilon}}, \tag{4.13}$$

$$|q^\varepsilon|_{L^2(\Omega_\varepsilon)} \le \frac{C}{\varepsilon\sqrt{\varepsilon}}. \tag{4.14}$$

Proof. First, we need to lift the non-homogeneous boundary condition $\mathbf{g}^\varepsilon$. For that purpose, we construct the function

$$\mathbf{G}^\varepsilon\left(x, \frac{y}{\varepsilon}\right) = \left(-\frac{1}{\varepsilon}\, z'\left(\frac{y}{\varepsilon}\right) \int_0^x g^\varepsilon(s)\, ds \,,\; z\left(\frac{y}{\varepsilon}\right)\; g^\varepsilon(x)\right),$$

where the function z is chosen as

$$z(\xi) = \frac{1}{2}\left(\cos\frac{\pi}{2}\xi + \cos\pi\xi + \sin\frac{\pi}{2}\xi - \frac{1}{2}\sin\pi\xi\right)$$

such that

$$z(1) = z'(1) = z'(0) = 0\,,\; z(0) = 1.$$

Furthermore,

$$|z|_{L^\infty(\mathbf{R})} \le \frac{6}{5}.$$

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We denote by $\xi = \frac{y}{\varepsilon}$ the dilated variable. Now,

$$\mathbf{G}^\varepsilon(0,\xi) = \mathbf{G}^\varepsilon(x,1) = 0\,,\ \mathbf{G}^\varepsilon(1,\xi)\,\times\,\mathbf{i} = 0\,,\ \mathbf{G}^\varepsilon(x,0) = g^\varepsilon(x)\,\mathbf{j}. \tag{4.15}$$

Obviously,

$$\mathbf{G}^\varepsilon = \mathbf{g}^\varepsilon \text{ on } \Gamma^I_\varepsilon, \tag{4.16}$$

$$\mathbf{G}^\varepsilon = 0 \text{ on } \Gamma^N_\varepsilon, \tag{4.17}$$

$$\mathbf{G}^\varepsilon\,\times\,\mathbf{i} = 0 \text{ for } x = 0, L, \tag{4.18}$$

$$\operatorname{div}\mathbf{G}^\varepsilon = 0 \text{ in } \Omega_\varepsilon \tag{4.19}$$

$$|\mathbf{G}^\varepsilon|_{L^\infty(\Omega_\varepsilon)} \le C, \tag{4.20}$$

$$|\nabla\mathbf{G}^\varepsilon|_{L^2(\Omega_\varepsilon)} \le \frac{C}{\varepsilon}. \tag{4.21}$$

Testing the Equation (4.4) with $\mathbf{u}^\varepsilon - \mathbf{G}^\varepsilon$ gives

$$\int_{\Omega_\varepsilon} |\nabla\mathbf{u}^\varepsilon|^2 = \int_{\Omega_\varepsilon} \nabla\mathbf{u}^\varepsilon\,\nabla\mathbf{G}^\varepsilon + \frac{P_0 - P_L}{\varepsilon^2\,L}\int_{\Omega_\varepsilon} (\mathbf{u}^\varepsilon - \mathbf{G}^\varepsilon)\,\cdot\,\mathbf{i}. \tag{4.22}$$

Using the estimates (4.20), (4.21) and the Poincaré and Sobolev inequalities (4.8), (4.9), we get

$$\frac{P_0 - P_L}{\varepsilon^2\,L}\int_{\Omega_\varepsilon} \mathbf{u}^\varepsilon \le \frac{C}{\sqrt{\varepsilon}}\,|\nabla\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)}\,,$$
$$\frac{P_0 - P_L}{\varepsilon^2\,L}\int_{\Omega_\varepsilon} \mathbf{G}^\varepsilon \le \frac{C}{\varepsilon},$$
$$\int_{\Omega_\varepsilon} \nabla\mathbf{u}^\varepsilon\,\nabla\mathbf{G}^\varepsilon \le \frac{C}{\sqrt{\varepsilon}}\,|\nabla\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)}\,,$$

implying

$$|\nabla\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} \le \frac{C}{\sqrt{\varepsilon}},$$
$$|\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)} \le C\,\sqrt{\varepsilon}.$$

Next we take $\mathbf{w} \in H^1_0(\Omega)^2$ and use $\mathbf{w}^\varepsilon(x,y) = \mathbf{w}\left(x, \frac{y}{\varepsilon}\right)$ as a test function in (4.4). Using (4.8), we obtain

$$\int_{\Omega_\varepsilon} p^\varepsilon\,\operatorname{div}\mathbf{w}^\varepsilon = \int_{\Omega_\varepsilon} \nabla\mathbf{u}^\varepsilon\;\;\nabla\mathbf{w}^\varepsilon - \frac{P_0 - P_L}{\varepsilon^2\,L}\int_{\Omega_\varepsilon} \mathbf{w}^\varepsilon\;\cdot\,\mathbf{i} \le \frac{C}{\sqrt{\varepsilon}}\,|\nabla\mathbf{w}^\varepsilon|_{L^2(\Omega)} \tag{4.23}$$

leading to (4.13). Finally, the Nečas inequality (4.10) implies

$$|q^\varepsilon|_{L^2(\Omega_\varepsilon)} \le \frac{C}{\varepsilon\sqrt{\varepsilon}}.$$

□

4.2 Convergence

We recall the definition of the two-scale convergence for thin domains from [12]:

We say that the sequence $\{v^\varepsilon\}$, $v^\varepsilon \in L^2(\Omega_\varepsilon)$ converges two-scale in $L^2(\Omega_\varepsilon)$ to some $V \in L^2(\Omega)$ if for any $\phi \in L^2(\Omega)$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} v^\varepsilon(x,y)\,\phi\left(x, \frac{y}{\varepsilon}\right)\,dxdy = \int_\Omega V(x,\xi)\,\phi(x,\xi)\,dx\,d\xi.$$

We also need the two-scale convergence in $W'(\Omega_\varepsilon)$, the dual space of

$$W(\Omega_\varepsilon) = \{\psi \in H^1(\Omega_\varepsilon)\ ;\ \psi(x,0) = \psi(x,\varepsilon) = 0\},$$

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for the pressure gradient. Denoting

$$W(\Omega)=\{\psi \in H^1(\Omega)\ ;\ \psi(x,0)=\psi(x,1)=0\}$$

and by $W'(\Omega)$ its dual, following again [12], we say that the sequence $\{\varphi^\varepsilon\}$, $\varphi^\varepsilon \in W'(\Omega_\varepsilon)$ converges two-scale in $W'(\Omega_\varepsilon)$ to some $\Phi \in W'(\Omega)$ if for any $\psi \in W(\Omega)$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\langle \varphi^\varepsilon(x,y)\,\middle|\, \psi\left(x,\frac{y}{\varepsilon}\right)\right\rangle_{\Omega_\varepsilon} = \langle \Phi(x,\xi)\,|\,\psi(x,\xi)\,\rangle_\Omega .$$

Here, the brackets $\langle\cdot|\cdot\rangle_{\Omega_\varepsilon}$ and $\langle\cdot|\cdot\rangle_\Omega$ denote the duality between $W'(\Omega_\varepsilon)$ and $W(\Omega_\varepsilon)$ i.e. $W'(\Omega)$ and $W(\Omega)$, respectively.

The main result of this chapter can be formulated as follows:

Theorem 2. *Let* $(\mathbf{u}^\varepsilon, p^\varepsilon)$ *be the solution to the problem (2.4)–(2.7), then*

$$\mathbf{u}^\varepsilon \to \mathbf{U} \text{ two-scale in } L^2(\Omega_\varepsilon), \tag{4.24}$$

$$\varepsilon^2 p^\varepsilon \to P \text{ two-scale in } L^2(\Omega_\varepsilon), \text{ as } \varepsilon \to 0, \tag{4.25}$$

where $\mathbf{U}=U_1\,\mathbf{i}$, P *satisfy the Hagen- Poiseuille law*

$$U_1(x,\xi)=\frac{\xi}{2}(\xi-1)\,P'(x). \tag{4.26}$$

The mean velocity

$$\mathbf{v}=v\,\mathbf{i} \tag{4.27}$$

$$v(x)=\int_0^1 U_1(x,\xi)\,d\xi=-\frac{1}{12}P'(x) \tag{4.28}$$

is not divergence free, but

$$\frac{d}{dx}\left(\int_0^1 U_1(x,\xi)\,d\xi\right)=g(x). \tag{4.29}$$

The effective pressure P *is not linear, like in the standard Hagen-Poiseuille case, but satisfies the boundary-value problem*

$$P''=-12\,g,\ \ P(0)=P_0\,,\ P(L)=P_L, \tag{4.30}$$

which has a unique solution of the form

$$P=P_0+(P_L-P_0)\frac{x}{L}+12\left[\int_x^L (x-t)g(t)dt+\left(1-\frac{x}{L}\right)\int_0^L tg(t)dt\right]. \tag{4.31}$$

Proof. Using the a priori estimates from Theorem 1, we deduce from the two-scale compactness theorem (see [12], Theorem 1) that there exists $\mathbf{U}\in(Y^2)^2$, $Y^2=\left\{\phi\in L^2(\Omega)\ ;\ \frac{\partial\phi}{\partial\xi}\in L^2(\Omega)\right\}$ and $Q\in L^2(\Omega)$, such that (up to a subsequence)

$$\mathbf{u}^\varepsilon \to \mathbf{U} \text{ two-scale in } L^2(\Omega_\varepsilon), \tag{4.32}$$

$$\varepsilon\,\nabla\mathbf{u}^\varepsilon \to \frac{\partial \mathbf{U}}{\partial\xi}\,\mathbf{j} \text{ two-scale in } L^2(\Omega_\varepsilon), \tag{4.33}$$

$$\varepsilon^2 q^\varepsilon \to Q \text{ two-scale in } L^2(\Omega_\varepsilon), \tag{4.34}$$

$$\varepsilon^2\,\frac{\partial q^\varepsilon}{\partial x} \to \frac{\partial Q}{\partial x} \text{ two-scale in } W'(\Omega_\varepsilon)\,. \tag{4.35}$$

Our goal is to identify the limits $(\mathbf{U}, Q)$. For $\mathbf{w}\in H^1(\Omega)$, we put $\mathbf{w}^\varepsilon(x,y)=\mathbf{w}\left(x,\frac{y}{\varepsilon}\right)$ and then

$$\int_{\Omega_\varepsilon}\frac{\partial\mathbf{u}^\varepsilon}{\partial y}\,\mathbf{w}^\varepsilon=-\int_{\Omega_\varepsilon}\mathbf{u}^\varepsilon\,\frac{\partial\mathbf{w}^\varepsilon}{\partial y}-\int_\Gamma g^\varepsilon\mathbf{w}(x,0).$$

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For the left-hand side, we have

$$\int_{\Omega_\varepsilon} \frac{\partial \mathbf{u}^\varepsilon}{\partial y}\, \mathbf{w}^\varepsilon = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \varepsilon\, \frac{\partial \mathbf{u}^\varepsilon}{\partial y}\, \mathbf{w}^\varepsilon \to \int_\Omega \frac{\partial \mathbf{U}}{\partial \xi}(x,\xi)\, \mathbf{w}(x,\xi)\, dxd\xi .$$

For the right-hand side, we deduce

$$\int_{\Omega_\varepsilon} \frac{\partial \mathbf{w}^\varepsilon}{\partial y}\, \mathbf{u}^\varepsilon = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \frac{\partial \mathbf{w}}{\partial \xi}\left(x, \frac{y}{\varepsilon}\right) \mathbf{u}^\varepsilon \to \int_\Omega \frac{\partial \mathbf{w}}{\partial \xi}(x,\xi)\, \mathbf{U}(x,\xi)\, dxd\xi ,$$
$$\int_\Gamma g^\varepsilon\, \mathbf{w}(x,0) \to 0.$$

Thus,

$$\int_\Omega \frac{\partial \mathbf{U}}{\partial \xi}(x,\xi)\, \mathbf{w}(x,\xi)\, dxd\xi = -\int_\Omega \frac{\partial \mathbf{w}}{\partial \xi}(x,\xi)\, \mathbf{U}(x,\xi)\, dxd\xi$$

implying that

$$\mathbf{U}(x,0) = \mathbf{U}(x,1) = 0. \tag{4.36}$$

Next, since $\mathbf{u}^\varepsilon$ is divergence free, we get for $\phi \in H_0^1(\Omega)$ and $\phi^\varepsilon(x,y) = \phi\left(x, \frac{y}{\varepsilon}\right)$

$$0 = \int_{\Omega_\varepsilon} \operatorname{div} \mathbf{u}^\varepsilon\, \phi^\varepsilon .$$

Now,

$$\int_{\Omega_\varepsilon} \operatorname{div} \mathbf{u}^\varepsilon\, \phi^\varepsilon = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \varepsilon \operatorname{div} \mathbf{u}^\varepsilon\, \phi^\varepsilon \to \int_\Omega \frac{\partial U_2}{\partial \xi}\, \phi$$

implying that $\frac{\partial U_2}{\partial \xi} = 0$. Combined with (4.36), it leads to conclusion that $U_2 = 0$. Taking, instead, the test function $\phi = \phi(x)$, such that $\phi(0) = \phi(L) = 0$, gives

$$0 = \int_{\Omega_\varepsilon} \operatorname{div} \mathbf{u}^\varepsilon\, \phi^\varepsilon = -\int_{\Omega_\varepsilon} \mathbf{u}_1^\varepsilon \cdot \frac{d\phi}{dx} - \int_\Gamma g^\varepsilon\, \phi$$

leading to

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} u_1^\varepsilon \cdot \frac{d\phi}{dx} = -\int_\Gamma \frac{1}{\varepsilon}\, g^\varepsilon\, \phi \to \int_0^L g\, \phi .$$

Since

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} u_1^\varepsilon \cdot \frac{d\phi}{dx} \to \int_\Omega U_1 \cdot \frac{d\phi}{dx} = \int_0^L \left(\int_0^1 U_1(x,\xi)\, d\xi\right) \phi'(x)\, dx,$$

we conclude that

$$\frac{d}{dx}\left(\int_0^1 U_1(x,\xi)\, d\xi\right) = g(x). \tag{4.37}$$

So far, we did not use the momentum equation. Let $\phi \in C_0^1(\Omega)^2$ and let ϕ^ε be defined as above. Testing (4.4) by $\varepsilon^2\, \phi^\varepsilon\, \mathbf{j}$ gives

$$\varepsilon^2 \int_{\Omega_\varepsilon} \nabla u_2^\varepsilon\, \nabla \phi^\varepsilon = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \varepsilon^2\, q^\varepsilon\, \frac{\partial \phi}{\partial \xi}\left(x, \frac{y}{\varepsilon}\right) \to \int_\Omega Q\, \frac{\partial \phi}{\partial \xi} \tag{4.38}$$

At the same time,

$$\varepsilon^2 \int_{\Omega_\varepsilon} \nabla u_2^\varepsilon\, \nabla \phi^\varepsilon \le C\varepsilon \to 0.$$

Therefore, $\frac{\partial Q}{\partial \xi} = 0$ so that $Q = Q(x)$.

At this point, we use (4.35) and take the test function $\phi \in W(\Omega)$. Then,

$$\left\langle \frac{\partial Q}{\partial x} \middle| \psi \right\rangle_{\Omega} \leftarrow \frac{1}{\varepsilon} \left\langle \varepsilon^2 \frac{\partial q^\varepsilon}{\partial x} \middle| \psi \right\rangle_{\Omega_\varepsilon} = -\frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \varepsilon^2 q^\varepsilon \frac{\partial \phi}{\partial x} \to - \int_{\Omega} Q \frac{\partial \phi}{\partial x}$$

implying that

$$Q(0) = Q(1) = 0 \tag{4.39}$$

in the weak sense. Testing (4.4) by $\varepsilon\, \phi^\varepsilon\, \mathbf{i}$ gives

$$\varepsilon \int_{\Omega_\varepsilon} \nabla u_1^\varepsilon \, \nabla \phi^\varepsilon = \int_{\Omega_\varepsilon} \varepsilon\, q^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x} + \frac{P_0 - P_L}{\varepsilon\, L} \int_{\Omega_\varepsilon} \phi^\varepsilon \to$$
$$\to \int_{\Omega} Q \frac{\partial \phi}{\partial x} + \frac{P_0 - P_L}{L} \int_{\Omega} \phi$$

On the other hand,

$$\varepsilon \int_{\Omega_\varepsilon} \nabla u_1^\varepsilon \, \nabla \phi^\varepsilon = \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \varepsilon \frac{\partial u_1^\varepsilon}{\partial y}(x,y) \frac{\partial \phi}{\partial \xi} \left(x, \frac{x}{\varepsilon}\right) + \varepsilon \int_{\Omega_\varepsilon} \frac{\partial u_1^\varepsilon}{\partial x} \frac{\partial \phi}{\partial x} \to$$
$$\to \int_{\Omega} \frac{\partial U_1}{\partial \xi} \frac{\partial \phi}{\partial \xi} .$$

Combining the above and defining

$$P(x) = Q(x) - \left(P_0 + \frac{P_L - P_0}{L} x\right)$$

leads to

$$\frac{\partial^2 U_1}{\partial \xi^2} = \frac{dP}{dx}(x), \quad U_1(x,0) = U_1(x,1) = 0.$$

That is a boundary-value problem for $\xi \mapsto U_1(x,\xi)$, with x being just a parameter. Since the equation is linear, it has a unique solution

$$U_1(x,\xi) = \frac{\xi}{2}(\xi - 1)\, P'(x). \tag{4.40}$$

Thus, the mean velocity

$$v(x) = \int_0^1 U_1(x,\xi)\, d\xi = -\frac{1}{12} P'(x) \tag{4.41}$$

and (4.37), combined with (4.39), gives

$$P'' = -12\, g, \quad P(0) = P_0\, , \quad P(L) = P_L. \tag{4.42}$$

It has a unique solution of the form

$$P = P_0 + (P_L - P_0)\frac{x}{L} + 12 \left[\int_x^L (x - t) g(t) dt + \left(1 - \frac{x}{L}\right) \int_0^L t g(t) dt \right] \tag{4.43}$$

concluding the proof. □

5. Injection of order 1

If we assume that the boundary injection g^ε is stronger, the weak and the two-scale convergence appear to be insufficient. We still get the effective model, but using the asymptotic expansions and the appropriate error estimates.

Now, we do not make any assumption on the weaker magnitude of g^ε as we did in the previous section, where the convergence (4.2) was assumed. We start with the standard technique of changing the variable to have the domain independent on ε and deriving the asymptotic expansion for the solution (see e.g. [10], [13] or [18] for an introduction to such approach). To start with, we assume that g^ε is continuous and that it verifies the compatibility condition

$$g^\varepsilon(0) = g^\varepsilon(L) = 0. \tag{5.1}$$

5.1 Formal asymptotic expansion

By introducing the dilated variable $\xi = \frac{y}{\varepsilon}$ and denoting

$$\mathbf{U}^\varepsilon(x,\xi) = \mathbf{u}^\varepsilon(x,y), \;\; P^\varepsilon(x,\xi) = p^\varepsilon(x,y), \tag{5.2}$$

that are now defined on

$$\Omega = \langle 0, L\rangle \times \langle 0, 1\rangle,$$

we transform our problem to

$$-\left(\frac{\partial^2 \mathbf{U}^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \mathbf{U}^\varepsilon}{\partial \xi^2}\right) + \frac{\partial P^\varepsilon}{\partial x}\mathbf{i} + \frac{1}{\varepsilon}\frac{\partial P^\varepsilon}{\partial \xi}\mathbf{j} = \mathbf{0} \;\text{ in }\; \Omega, \tag{5.3}$$

$$\frac{\partial U_1^\varepsilon}{\partial x} + \frac{1}{\varepsilon}\frac{\partial U_2^\varepsilon}{\partial \xi} = 0 \;\text{ in }\; \Omega, \tag{5.4}$$

$$\mathbf{U}^\varepsilon = \mathbf{g}^\varepsilon \quad \text{on} \;\; \Gamma, \tag{5.5}$$

$$\mathbf{U}^\varepsilon = \mathbf{0} \quad \text{on} \quad \Sigma = \langle 0, L\rangle \times \{1\}, \tag{5.6}$$

$$U_2^\varepsilon = 0, \;\; P^\varepsilon(x,\xi) = \frac{1}{\varepsilon^2} P_x \;\text{ for }\; x = 0, L, \tag{5.7}$$

Since now, integrating (5.4) and using (5.5), we get

$$\frac{d}{dx}\int_0^1 U_1^\varepsilon = \frac{1}{\varepsilon} g^\varepsilon(x),$$

we look for asymptotic expansions of the solution in the form

$$\mathbf{U}^\varepsilon(x,\xi) = \frac{1}{\varepsilon}U^{-1}(x,\xi)\,\mathbf{i} + \mathbf{U}^0(x,\xi) + \varepsilon\mathbf{U}^1(x,\xi) + \cdots \tag{5.8}$$

$$P^\varepsilon(x,\xi) = \frac{1}{\varepsilon^3}P^{-1}(x) + \frac{1}{\varepsilon^2}P^0(x) + \frac{1}{\varepsilon}P^1(x,\xi) + \cdots . \tag{5.9}$$

Plugging the expansion (5.8), (5.9) in (5.3), (5.4) and collecting equal powers of ε, we obtain the recursive equations

$$\frac{1}{\varepsilon^3} : \frac{\partial^2 U_1^{-1}}{\partial \xi^2} = \frac{\partial P^{-1}}{\partial x}, \; U_1^{-1}(x,0) = U_1^{-1}(x,1) = 0 \tag{5.10}$$

$$\frac{1}{\varepsilon^2} : \frac{\partial^2 U_1^{0}}{\partial \xi^2} = \frac{\partial P^{0}}{\partial x};, \tag{5.11}$$

$$U_1^0(x,0) = U_1^0(x,1) = 0$$

$$\frac{1}{\varepsilon^2} : \frac{\partial^2 U_2^{0}}{\partial \xi^2} = \frac{\partial P^{1}}{\partial \xi}, \; U_2^0(x,0) = g^\varepsilon(x), \; U_2^0(x,1) = 0 \tag{5.12}$$

$$\frac{1}{\varepsilon} : \frac{\partial^2 U_1^{1}}{\partial \xi^2} = \frac{\partial P^{1}}{\partial x} - \frac{\partial^2 U_1^{-1}}{\partial x^2}, \; U_1^1(x,0) = U_1^1(x,1) = 0 \tag{5.13}$$

$$\frac{1}{\varepsilon} : \frac{\partial^2 U_2^1}{\partial \xi^2} = \frac{\partial P^2}{\partial \xi}, \; U_2^1(x,0) = U_2^1(x,1) = 0 \tag{5.14}$$

$$\frac{1}{\varepsilon} : \frac{\partial U_1^{-1}}{\partial x} + \frac{\partial U_2^0}{\partial \xi} = 0 \tag{5.15}$$

$$1 : \frac{\partial^2 U_1^2}{\partial \xi^2} = \frac{\partial P^2}{\partial x} - \frac{\partial^2 U_1^0}{\partial x^2}, \; U_1^2(x,0) = U_1^2(x,1) = 0 \tag{5.16}$$

$$1 : \frac{\partial^2 U_2^2}{\partial \xi^2} = \frac{\partial P^3}{\partial \xi} - \frac{\partial^2 U_2^0}{\partial x^2}, \; U_2^2(x,0) = U_2^2(x,1) = 0 \tag{5.17}$$

$$1 : \frac{\partial U_1^0}{\partial x} + \frac{\partial U_2^1}{\partial \xi} = 0 \;\Rightarrow\; \frac{d}{dx}\int_0^1 U_1^0(x,\xi)\,d\xi = 0 \tag{5.18}$$

$$\varepsilon : \frac{\partial U_1^1}{\partial x} + \frac{\partial U_2^2}{\partial \xi} = 0 \;\Rightarrow\; \frac{d}{dx}\int_0^1 U_1^1(x,\xi)\,d\xi = 0 \tag{5.19}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

In general, for $k = 0, 1, 2, \cdots$

$$\varepsilon^k : \frac{\partial^2 U_1^{k+2}}{\partial \xi^2} = \frac{\partial P^{k+2}}{\partial x} - \frac{\partial^2 U_1^k}{\partial x^2}, \; U_1^{k+2}(x,0) = U_1^{k+2}(x,1) = 0 \tag{5.20}$$

$$\varepsilon^k : \frac{\partial^2 U_2^{k+2}}{\partial \xi^2} = \frac{\partial P^{k+3}}{\partial \xi} - \frac{\partial^2 U_2^k}{\partial x^2}, \; U_2^{k+2}(x,0) = U_2^{k+2}(x,1) = 0 \tag{5.21}$$

$$\varepsilon^k : \frac{\partial U_1^k}{\partial x} + \frac{\partial U_2^{k+1}}{\partial \xi} = 0 \;\Rightarrow\; \frac{d}{dx}\int_0^1 U_1^k(x,\xi)\,d\xi = 0, \; k \ge 0. \tag{5.22}$$

From (5.10) and (5.11), we deduce that

$$U_1^{-1}(x,\xi) = \frac{\xi}{2}(\xi - 1)\frac{dP^{-1}}{dx}(x). \tag{5.23}$$

Integrating (5.15), with respect to ξ from 0 to 1, gives

$$\frac{d}{dx}\int_0^1 U_1^{-1}(x,\xi)\,d\xi = g^\varepsilon(x).$$

Combining with (5.23) gives

$$\frac{1}{12}\frac{d^2P^{-1}}{dx^2}(x) = -g^\varepsilon(x), \;\; P^{-1}(0) = P^{-1}(L) = 0, \tag{5.24}$$

leading to

$$P^{-1}(x) = 12\left[\frac{x}{L}\int_0^L (L-t)g^\varepsilon(t)\,dt - \int_0^x (x-t)g^\varepsilon(t)\,dt\right] \tag{5.25}$$

$$U_1^{-1}(x,\xi) = 6\xi\,(\xi - 1)\left[\frac{1}{L}\int_0^L (L-t)g^\varepsilon(t)\,dt - \int_0^x g^\varepsilon(t)\,dt\right]. \tag{5.26}$$

Going back to (5.15) gives, by simple integration with respect to ξ,

$$U_2^0 = 6\left(\frac{\xi^3}{3} - \frac{\xi^2}{2}\right) g^\varepsilon(x) + A(x).$$

Using the boundary conditions, $U_2^0(x,0) = 0$, $U_2^0(x,1) = g^\varepsilon$ leads to $A(x) = g^\varepsilon(x)$ and thus

$$U_2^0(x,\xi) = \left(2\,\xi^3 - 3\,\xi^2 + 1\right)\, g^\varepsilon(x). \tag{5.27}$$

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Now that we have computed U_1^{-1} and U_2^0, we are in position to solve the Equation (5.11) for U_1^0. We have

$$\frac{\partial^2 U_1^0}{\partial \xi^2} = \frac{\partial P^0}{\partial x} \ , \tag{5.28}$$

$$U_1^0(x,\xi) = \frac{\xi}{2}(\xi - 1)\frac{d\,P^0}{d\,x}(x). \tag{5.29}$$

Similarly, integrating (5.18), with respect to ξ from 0 to 1, gives

$$\frac{d}{dx}\int_0^1 U_1^0(x,\xi)\,d\xi = 0.$$

Combining with (5.29) gives

$$\begin{aligned} &\frac{d^2P^0}{dx^2}(x) = 0 \\ &P^0(0) = P_0\ ,\ P^0(L) = P_L. \end{aligned} \tag{5.30}$$

The problem (5.30) has a unique solution

$$P^0(x) = P_0 + \frac{x}{L}\,(P_L - P_0) \tag{5.31}$$

$$U_1^0(x,\xi) = \frac{\xi}{2}(\xi - 1)\,\frac{P_L - P_0}{L}.$$

It remains to satisfy (5.12) by picking an appropriate pressure P^1. Since

$$\frac{\partial^2 U_2^0}{\partial \xi^2} = 6(2\xi - 1)g^\varepsilon(x)$$

the Equation (5.12) gives

$$P^1(x,\xi) = 6\left(\xi^2 - \xi\right)\ g^\varepsilon(x) + b(x)_1. \tag{5.32}$$

At this point, b_1 is an arbitrary function satisfying $b_1(0) = b_1(L) = 0$ as we do not want is to spoil the boundary value of the pressure. If we want to determine it, we need to proceed with next recurrence equation for $\mathbf{U}^1$ and P^1 (5.13). It is easy to see that

$$\frac{\partial^2 U_1^1}{\partial \xi^2} = \frac{\partial P^1}{\partial x} - \frac{\partial^2 U_1^{-1}}{\partial x^2} = b_1{}'(x). \tag{5.33}$$

and

$$\frac{d}{dx}\int_0^1 U_1^1(x,\xi)\,d\xi = 0. \tag{5.34}$$

Thus $b_1 = 0$ and $U_1^1 = 0$.

Furthermore, to satisfy (5.14), it is sufficient to take

$$U_2^1 = 0,\ \ P^2 = P^2(x). \tag{5.35}$$

Next, (5.19) implies that

$$U_2^2 = 0$$

and then, from (5.17),

$$\frac{\partial P^3}{\partial \xi} = \frac{\partial^2 U_2^0}{\partial x^2} = (2\xi^3 - 3\xi^2 + 1)\,\frac{d^2 g^\varepsilon}{dx^2}.$$

Thus,

$$P^3 = \left(\frac{1}{2}\xi^4 - \xi^3 + \xi\right)\,\frac{d^2 g^\varepsilon}{dx^2} + b_3(x). \tag{5.36}$$

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On the other hand, (5.16) implies

$$\frac{\partial^2 U_1^2}{\partial \xi^2} = \frac{dP^2}{dx}.$$

Due to (5.19), we have

$$P^2 = 0 \text{ and } U_1^2 = 0. \tag{5.37}$$

Computation of higher-order terms is straightforward but tedious. It is easy to see by induction that $(\mathbf{U}^k, P^k)$ have the form

$$\mathbf{U}^k(x,\xi) = \mathbf{W}^k(\xi)\,\frac{d^k g^\varepsilon}{dx^k}(x), \quad P^k(x,\xi) = S_k(\xi)\,\frac{d^{k-1} g^\varepsilon}{dx^{k-1}}(x),$$

but the expressions for S_k and $\mathbf{W}^k$ are complex. So we decide to stop here and take $b_3 = 0$.

Also, we should remember that g^ε still depends on ε in some way that has not been precised yet. We will come back to that question later. At this point, we try to leave as much freedom in choice of g^ε as possible.

5.2 Convergence

Recalling that

$$\mathbf{U}^1 = \mathbf{U}^2 = 0, \;\; P^2 = 0,$$

our approximation now reads

$$\mathbf{A}^\varepsilon(x,\xi) = \frac{1}{\varepsilon} U^{-1}(x,\xi)\,\mathbf{i} + \mathbf{U}^0(x,\xi) \tag{5.38}$$

$$a^\varepsilon(x,\xi) = \frac{1}{\varepsilon^3} P^{-1}(x) + \frac{1}{\varepsilon^2} P^0(x) + \frac{1}{\varepsilon} P^1(x,\xi) + \varepsilon P^3(x,\xi). \tag{5.39}$$

The choice of $\mathbf{U}^j$, $j = -1, 0$ and P^k, $k = -1, 0, 1, 3$ leads to

$$-\mu\left(\frac{\partial^2 \mathbf{A}^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \mathbf{A}^\varepsilon}{\partial \xi^2}\right) + \frac{\partial a^\varepsilon}{\partial x}\mathbf{i} + \frac{1}{\varepsilon}\frac{\partial a^\varepsilon}{\partial \xi}\mathbf{j} = \mathbf{R}^\varepsilon \text{ in } \Omega, \tag{5.40}$$

$$\frac{\partial A_1^\varepsilon}{\partial x} + \frac{1}{\varepsilon}\frac{\partial A_2^\varepsilon}{\partial \xi} = 0 \text{ in } \Omega, \tag{5.41}$$

$$\mathbf{A}^\varepsilon = \mathbf{g}^\varepsilon \text{ on } \Gamma, \tag{5.42}$$

$$\mathbf{A}^\varepsilon = \mathbf{0} \text{ on } \Sigma, \tag{5.43}$$

$$A_2^\varepsilon = 0, \;\; a^\varepsilon(x,\xi) = \frac{1}{\varepsilon^2} P_x \text{ for } x = 0, L, \tag{5.44}$$

The reminder $\mathbf{R}^\varepsilon$ has the form

$$\mathbf{R}^\varepsilon = \varepsilon \frac{\partial P^3}{\partial x} = \varepsilon\left(\frac{1}{2}\xi^4 - \xi^3 + \xi\right)\frac{d^3 g^\varepsilon}{dx^3}. \tag{5.45}$$

At this point, we need to impose some conditions on the dependence of g^ε on ε. We assume that g^ε is of the class C^2 and that

$$g^\varepsilon(0) = \frac{dg^\varepsilon}{dx}(0) = \frac{d^2 g^\varepsilon}{dx^2}(0) = g^\varepsilon(L) = \frac{dg^\varepsilon}{dx}(L) = \frac{d^2 g^\varepsilon}{dx^2}(0) = 0. \tag{5.46}$$

Remark 2. Such situation appears if, for example:

1. If g^ε is a single function, independent of ε, i.e. $g^\varepsilon(x) = g(x)$ belonging to $C_0^2(0, L)$

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2. If g^ε is produced by function $g = g(x, t)$, periodic in the second variable by

$$g^\varepsilon(x) = g(x, x/\varepsilon^\alpha), \ \ \alpha < 1,$$

with

$$g(0, 0) = g(L, L/\varepsilon^\alpha) = \nabla g(0, 0) = \nabla g(L, L/\varepsilon^\alpha) = 0\,.$$

Case $\alpha = 1$ is different. In that case, the reminder in (5.40) satisfies $|\mathbf{R}^\varepsilon|_{L^2(\Omega)} = O(\varepsilon^{-2})$ and our approximation is not good enough to get some convergence. That is not a surprise as for $\alpha = 1$, we have the classical homogenisation case that requires different asymptotic expansion depending on the dilated variable $\xi = y/\varepsilon$ and the fast variable $t = x/\varepsilon$. We will get back to that case later.

The main result of this chapter can be formulated as follows:

Theorem 3. *Let* $(\mathbf{U}^\varepsilon, P^\varepsilon)$ *be the solution to the problem (5.3)-(5.7).*
If

$$\lim_{\varepsilon\to 0} \varepsilon^2 \left|\frac{d^2 g^\varepsilon}{dx^2}\right|_{L^2(0,L)} = 0,$$

then

$$\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon \to 0 \ \ in \ \ L^2(\Omega), \tag{5.47}$$

$$\frac{\partial}{\partial\xi}\,(\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon) \to 0 \ \ in \ \ L^2(\Omega), \tag{5.48}$$

$$\varepsilon^2\,(\,P^\varepsilon - a^\varepsilon\,) \to 0 \ \ in \ \ L^2(\Omega), \ as \ \ \varepsilon \to 0, \tag{5.49}$$

where $\mathbf{A}^\varepsilon$, a^ε *are given by (5.38) and (5.39). If, in addition*

$$\lim_{\varepsilon\to 0} \varepsilon \left|\frac{d^2 g^\varepsilon}{dx^2}\right|_{L^2(0,L)} = 0,$$

then

$$\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon \to 0 \ \ in \ \ H^1(\Omega)\,. \tag{5.50}$$

Furthermore, if, for some $s \in \langle 1, +\infty\rangle$,

$$g^\varepsilon \rightharpoonup g \ \ weakly \ in \ \ L^s(0, L),$$

then, denoting $\langle\phi\rangle = \frac{1}{\varepsilon}\int_0^\varepsilon \phi(x, y)\, dy$, *the mean value of function* ϕ *over the cross-section of the domain* Ω_ε, *we have the following pointwise convergences on* $[0, L]$

$$\left\langle U_1^{-1}\left(x, \frac{y}{\varepsilon}\right)\right\rangle \to -\left[\frac{1}{L}\int_0^L (L - t)g(t)\, dt - \int_0^x g(t)\, dt\right] \tag{5.51}$$

$$P^{-1}(x) \to 12\left[\frac{x}{L}\int_0^L (L - t)g(t)\, dt - \int_0^x (x - t)g(t)\, dt\right] \tag{5.52}$$

$$P^0 \to P_0 + x\,\frac{P_L - P_0}{L}. \tag{5.53}$$

Furthermore, we have the following weak convergences in $L^s(0, L)$

$$\left\langle U_1^0\left(x, \frac{y}{\varepsilon}\right)\right\rangle \rightharpoonup -\frac{1}{12}\,\frac{P_L - P_0}{L} \tag{5.54}$$

$$\left\langle U_2^0\left(x, \frac{y}{\varepsilon}\right)\right\rangle \rightharpoonup \frac{1}{2}\, g(x)\ . \tag{5.55}$$

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Proof. We start by subtracting the Equation (5.40) from (5.3) and testing it by $\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon$. Now,

$$\left|\frac{\partial}{\partial x}\left(\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon\right)\right|^2_{L^2(\Omega)} + \frac{1}{\varepsilon^2}\left|\frac{\partial}{\partial \xi}\left(\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon\right)\right|^2_{L^2(\Omega)} =$$

$$= -\varepsilon\int_\Omega P^3\,\frac{\partial}{\partial x}\left(\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon\right) \leq \frac{1}{2}\left|\frac{\partial}{\partial x}\left(\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon\right)\right|^2_{L^2(\Omega)} + C\,\varepsilon^2\left|\frac{d^2 g^\varepsilon}{dx^2}\right|^2_{L^2(\Omega)}.$$

Thus,

$$\left|\frac{\partial}{\partial x}\left(\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon\right)\right|_{L^2(\Omega)} \leq C\varepsilon\left|\frac{d^2 g^\varepsilon}{dx^2}\right|_{L^2(\Omega)} \tag{5.56}$$

$$\left|\frac{\partial}{\partial \xi}\left(\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon\right)\right|_{L^2(\Omega)} \leq C\,\varepsilon^2\left|\frac{d^2 g^\varepsilon}{dx^2}\right|_{L^2(\Omega)} \tag{5.57}$$

$$\left|\mathbf{U}^\varepsilon - \mathbf{A}^\varepsilon\right|_{L^2(\Omega)} \leq C\varepsilon^2\left|\frac{d^2 g^\varepsilon}{dx^2}\right|_{L^2(\Omega)}. \tag{5.58}$$

The rest of the proof is straightforward.

Remark 3 (*On the Navier-Stokes case*). If an inertial term is added to the Stokes system of the form

$$Re\,(\mathbf{u}^\varepsilon\cdot\nabla)\mathbf{u}^\varepsilon$$

then inertial terms appear in the approximation. More precisely, U_1^{-1} and P^{-1} remain the same, but

$$P^0(x) = \boxed{P_0 + \frac{x}{L}\left(P_L - P_0\right)} +$$

$$+\frac{297}{35}\,Re\left\{\int_0^x g^\varepsilon(s)\left[\frac{1}{L}\int_0^L (L-t)g^\varepsilon(t)\,dt - \int_0^s g^\varepsilon(t)\,dt\right]ds -\right.$$

$$\left.-\frac{x}{L}\int_0^L g^\varepsilon(s)\left[\frac{1}{L}\int_0^L (L-t)g^\varepsilon(t)\,dt - \int_0^s g^\varepsilon(t)\,dt\right]ds\right\}$$

$$U_1^0(x,\xi) = \boxed{\frac{\xi}{2}(\xi-1)\,\frac{P_L-P_0}{L}} - Re\left\{g^\varepsilon(x)\left[\frac{1}{L}\int_0^L (L-t)g^\varepsilon(t)dt -\right.\right.$$

$$\left.-\int_0^x g^\varepsilon(t)\,dt\right]\left(\xi^6 - 6\xi^5 + \frac{9}{2}\xi^4 + 2\xi^3 - \frac{507\xi^2}{70} + \frac{131}{35}\xi\right) +$$

$$\left.+\frac{297}{70\,L}\xi(\xi-1)\int_0^L g^\varepsilon(s)\left[\frac{1}{L}\int_0^L (L-t)g^\varepsilon(t)\,dt - \int_0^s g^\varepsilon(t)\,dt\right]ds\right\}.$$

The rigorous justification of such asymptotic expansion is, however, another matter, and it seems that some assumptions on the magnitude of the Reynolds number *Re* are needed, just as they are for the existence and uniqueness of the solution for such Navier-Stokes system.

6. Homogenization case

In the last part, we treat the case of the injection function g^ε of the form

$$g^\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}\right) \tag{6.1}$$

where $g(x,t)$ is a smooth $C^2(\,[0,L]\times[0,1]\,)$ function, such that

$$g(0,t) = g(L,t) = 0 \tag{6.2}$$

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$$t \mapsto g(x,t) \text{ is 1-periodic} \tag{6.3}$$

$$\varepsilon = L/m, \quad m \in \mathbf{N}. \tag{6.4}$$

To apply the classical homogenisation approach using the two-scale expansions, we need an injection with zero mean value, and, in our case

$$\overline{g}(x) = \int_0^1 g(x,t)\, dt$$

is not assumed to be zero. We can decompose g^ε as

$$g^\varepsilon(x) = \overline{g}(x) + \left[g\left(x, \frac{x}{\varepsilon}\right) - \overline{g}(x) \right].$$

The first part is independent on ε, and the results from the previous chapter apply. The second part has zero mean value.

Thus,

$$\mathbf{U}^\varepsilon = \mathbf{V}^\varepsilon + \mathbf{W}^\varepsilon, \;\; P^\varepsilon = H^\varepsilon + M^\varepsilon,$$

where

$$-\left(\frac{\partial^2 \mathbf{V}^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon^2} \frac{\partial^2 \mathbf{V}^\varepsilon}{\partial \xi^2} \right) + \frac{\partial H^\varepsilon}{\partial x} \mathbf{i} + \frac{1}{\varepsilon} \frac{\partial H^\varepsilon}{\partial \xi} \mathbf{j} = \mathbf{0} \;\text{ in }\; \Omega, \tag{6.5}$$

$$\frac{\partial V_1^\varepsilon}{\partial x} + \frac{1}{\varepsilon} \frac{\partial V_2^\varepsilon}{\partial \xi} = 0 \;\text{ in }\; \Omega, \tag{6.6}$$

$$\mathbf{V}^\varepsilon = \overline{g}\, \mathbf{j} \;\text{ on }\; \Gamma, \tag{6.7}$$

$$\mathbf{V}^\varepsilon = \mathbf{0} \;\text{ on }\; \Sigma = \langle 0, L \rangle \times \{1\}, \tag{6.8}$$

$$V_2^\varepsilon = 0, \;\; P^\varepsilon(x,\xi) = \frac{1}{\varepsilon^2} P_x \;\text{ for }\; x = 0, L, \tag{6.9}$$

and

$$-\left(\frac{\partial^2 \mathbf{W}^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon^2} \frac{\partial^2 \mathbf{W}^\varepsilon}{\partial \xi^2} \right) + \frac{\partial M^\varepsilon}{\partial x} \mathbf{i} + \frac{1}{\varepsilon} \frac{\partial M^\varepsilon}{\partial \xi} \mathbf{j} = \mathbf{0} \;\text{ in }\; \Omega, \tag{6.10}$$

$$\frac{\partial W_1^\varepsilon}{\partial x} + \frac{1}{\varepsilon} \frac{\partial W_2^\varepsilon}{\partial \xi} = 0 \;\text{ in }\; \Omega, \tag{6.11}$$

$$\mathbf{W}^\varepsilon = (g^\varepsilon - \overline{g})\, \mathbf{j} \;\text{ on }\; \Gamma, \tag{6.12}$$

$$\mathbf{W}^\varepsilon = \mathbf{0} \;\text{ on }\; \Sigma, \tag{6.13}$$

$$W_2^\varepsilon = 0, \;\; M^\varepsilon(x,\xi) = 0 \;\text{ for }\; x = 0, L, \tag{6.14}$$

The results from the previous chapter apply to (6.5)-(6.9) and we can conclude that

$$\mathbf{V}^\varepsilon - \mathbf{A}^\varepsilon \to 0 \text{ in } H^1(\Omega) \tag{6.15}$$

$$\varepsilon^2 \left(H^\varepsilon - a^\varepsilon \right) \to 0 \text{ in } L^2(\Omega), \text{ as } \varepsilon \to 0, \tag{6.16}$$

where $(\mathbf{A}^\varepsilon, a^\varepsilon)$ are given by (5.38) and (5.39) and (5.25), (5.26), (5.31), (5.32) and (5.36), with g^ε replaced by $\overline{g}$. It remains to study (6.10)-(6.14).

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6.1 Two-scale expansion

In this section, we study the asymptotic analysis of the problem (6.10)-(6.14) using the two-scale asymptotic expansion of the form

$$\mathbf{W}^\varepsilon \approx \mathbf{W}^0\left(x, \xi, \frac{x}{\varepsilon}\right) + \varepsilon \mathbf{W}^1\left(x, \xi, \frac{x}{\varepsilon}\right) + \cdots \tag{6.17}$$

$$M^\varepsilon = \frac{1}{\varepsilon} M^0\left(x, \xi, \frac{x}{\varepsilon}\right) + M^1\left(x, \xi, \frac{x}{\varepsilon}\right) + \cdots \tag{6.18}$$

Denoting $t = x/\varepsilon$, substituting in (6.10)-(6.14) and collecting equal powers of ε, leads to a sequence of auxiliary boundary-value problems posed on a unit square $Y = \langle 0, 1\rangle^2$:

$$\tfrac{1}{\varepsilon^2} \quad -\frac{\partial^2 \mathbf{W}^0}{\partial t^2} - \frac{\partial^2 \mathbf{W}^0}{\partial \xi^2} + \frac{\partial M^0}{\partial t}\mathbf{i} + \frac{\partial M^0}{\partial \xi}\mathbf{j} = 0 \text{ in } Y \tag{6.19}$$

$$\tfrac{1}{\varepsilon} \quad \frac{\partial W_1^0}{\partial t} + \frac{\partial W_2^0}{\partial \xi} = 0 \text{ in } Y \tag{6.20}$$

$$(\mathbf{W}^0, M^0) \text{ is 1-periodic in } t \tag{6.21}$$

$$\mathbf{W}^0 = \left[g(x,t) - \overline{g}(x)\right]\mathbf{j} \text{ for } \xi = 0 \tag{6.22}$$

$$\mathbf{W}^0 = 0 \text{ for } \xi = 1 \tag{6.23}$$

That is a Stokes system and, due to the fact that

$$\int_0^1 \left[g(x,t) - \overline{g}(x)\right]\, dt = 0,\ 0 < x < L\,,$$

we have:

Proposition 1. *The problem (6.19)–(6.23) has a unique solution*

$$(\mathbf{W}^0, M^0) \in V \times L^2(Y)\backslash \mathbf{R},$$

with

$$V = \{\mathbf{Z} \in H^1(Y)^2\ ; \mathbf{Z} \text{ is 1-periodic in } t\,,\ \mathbf{Z} = 0 \text{ for } \xi = 1\}.$$

Furthermore,

$$\int_0^1 W_1^0(x,t,\xi)\, dt = 0 \ \Rightarrow\ \int_Y W_1^0(x,t,\xi)\, dt\, d\xi = 0,\ x \in [0, L]\ . \tag{6.24}$$

and

$$\int_0^1 W_2^0(x,t,\xi)\, dt = 0 \ \Rightarrow\ \int_Y W_2^0(x,t,\xi)\, d\xi\ dt = 0,\ x \in [0, L]\ . \tag{6.25}$$

Finally, assuming that

$$\int_Y M^0\, dt\, d\xi = 0 \tag{6.26}$$

implies

$$\int_Y M^0(x,t,\xi)\, dt = 0,\ x \in [0, L]\ ,\ \xi \in [0, 1]\,. \tag{6.27}$$

Finally, for $x = 0, L$, the solution of (6.19)–(6.23) is trivial, i.e.

$$\mathbf{W}^0(0,t,\xi) = \mathbf{W}^0(L,t,\xi) = 0, \quad M^0(0,t,\xi) = M^0(L,t,\xi) = 0. \tag{6.28}$$

Proof. It is a linear Stokes system, and its existence is straightforward consequence of the Lax and Milgram theorem. The solution is, in fact, smooth, i.e. classical, due to the standard regularity theory

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for the Stokes system. The variables in the system are t and ξ, while x is just a parameter and the regularity with respect to x is the same as the smoothness imposed on g.

Integrating the first component of (6.19) with respect to t leads to

$$\frac{d^2}{d\xi^2}\int_0^1 W_1^0(x,t,\xi)\,dt = 0 \;\Rightarrow\; \int_0^1 W_1^0(x,t,\xi)\,dt = A\xi + B\,,$$

with A and B independent on ξ. For $\xi = 0$ and $\xi = 1$ we have $W_1^0 = 0$ so that $A = B = 0$. Thus we have (6.24).

Integrating (6.20) with respect to t implies

$$\frac{d}{d\xi}\int_0^1 W_2^0(x,t,\xi)\,dt = 0 \;\Rightarrow\; \int_0^1 W_2^0(x,t,\xi)\,dt = C.$$

Again, C is independent on ξ. For $\xi = 0, 1$, the value of the above integral is zero, so that we have (6.25).

If we integrate the second component of (6.19) with respect to t we get

$$\frac{d^2}{d\xi^2}\int_0^1 W_2^0(x,t,\xi)\,dt = \frac{d}{d\xi}\int_0^1 M^0(x,t,\xi)\,dt \;\Rightarrow\; \int_0^1 M^0(x,t,\xi)\,dt = C\,.$$

Since the pressure M^0 is determined up to a constant, the assumption (6.26) implies (6.27).

For $x = 0, L$, the boundary values $g(x,t)$ and $\overline{g}(x)$ equal zero, and the solution $(\mathbf{W}^0, M^0)$ is trivial.

We go one step forward and compute

$$\tfrac{1}{\varepsilon}\quad -\frac{\partial^2 \mathbf{W}^1}{\partial t^2} - \frac{\partial^2 \mathbf{W}^1}{\partial \xi^2} + \frac{\partial M^1}{\partial t}\,\mathbf{i} + \frac{\partial M^1}{\partial \xi}\,\mathbf{j} = -2\frac{\partial \mathbf{W}^0}{\partial x \partial t} - \frac{\partial M^0}{\partial x}\,\mathbf{i} \tag{6.29}$$

$$1\quad \frac{\partial W_1^1}{\partial t} + \frac{\partial W_2^1}{\partial \xi} = -\frac{\partial W_1^0}{\partial x} \;\text{ in } Y \tag{6.30}$$

$$(\mathbf{W}^1, M^1) \text{ is 1-periodic in } t \tag{6.31}$$

$$\mathbf{W}^1 = 0 \text{ for } \xi = 0\,,\ 1. \tag{6.32}$$

Proposition 2. *Due to (6.24), the problem is well posed and has a unique solution*

$$(\mathbf{W}^1, M^1) \in V \times L^2(Y)\backslash \mathbf{R}.$$

Furthermore,

$$\int_0^1 W_1^1(x,t,\xi)\,dt = 0,\ x \in [0,L]\ ,\ \xi \in [0,1]\,. \tag{6.33}$$

Finally, for $x = 0, L$

$$\mathbf{W}^1(0,t,\xi) = \mathbf{W}^1(L,t,\xi) = 0, \quad M^1(0,t,\xi) = M^1(L,t,\xi) = 0. \tag{6.34}$$

Proof. We skip the existence and uniqueness proof due to its simplicity. To prove (6.33), we integrate the first component of (6.29) with respect to t. It gives (using (6.27))

$$\frac{d^2}{d\xi^2}\int_0^1 W_1^1(x,t,\xi)\,dt = -\frac{\partial}{\partial x}\int_0^1 M^0\,dt = 0.$$

Thus,

$$\int_0^1 W_1^1(x,t,\xi)\,dt = A\xi + B.$$

Like for W_1^0, the choice $\xi = 0, 1$ implies $A = B = 0$ and thus (6.33).

For $x = 0, L$, the right-hand side of (6.29) equal zero, and the solution $(\mathbf{W}^1, M^1)$ is trivial.

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To finish, we look at the problem for $(\mathbf{W}^2, M^2)$ that reads

$$1 \quad -\frac{\partial^2 \mathbf{W}^2}{\partial t^2} - \frac{\partial^2 \mathbf{W}^2}{\partial \xi^2} + \frac{\partial M^2}{\partial t}\,\mathbf{i} + \frac{\partial M^2}{\partial \xi}\,\mathbf{j} = \tag{6.35}$$

$$= -\frac{\partial \mathbf{W}^0}{\partial x^2} - 2\frac{\partial \mathbf{W}^1}{\partial x \partial t} - \frac{\partial M^1}{\partial x}\,\mathbf{i} \text{ in } Y \tag{6.36}$$

$$\varepsilon \quad \frac{\partial W_1^2}{\partial t} + \frac{\partial W_2^2}{\partial \xi} = -\frac{\partial W_1^1}{\partial x} \text{ in } Y \tag{6.37}$$

$$(\mathbf{W}^2, M^2) \text{ is 1-periodic in } t \tag{6.38}$$

$$\mathbf{W}^2 = 0 \text{ for } \xi = 0,\ 1. \tag{6.39}$$

We can now prove the error estimate. Let

$$\mathbf{E}^\varepsilon = \mathbf{W}^0 + \varepsilon\,\mathbf{W}^1 + \varepsilon^2\,\mathbf{W}^2$$
$$e^\varepsilon = \frac{1}{\varepsilon}\,M^0 + M^1 + \varepsilon\,M^2.$$

Now,

$$-\frac{\partial^2 \mathbf{E}^\varepsilon}{\partial t^2} - \frac{\partial^2 \mathbf{E}^\varepsilon}{\partial \xi^2} + \frac{\partial e^\varepsilon}{\partial t}\,\mathbf{i} + \frac{\partial e^\varepsilon}{\partial \xi}\,\mathbf{j} = \mathbf{R}^\varepsilon \tag{6.40}$$

$$\frac{\partial E_1^\varepsilon}{\partial t} + \frac{\partial E_2^\varepsilon}{\partial \xi} = -\varepsilon^2\,\frac{\partial W_1^2}{\partial x} \text{ in } Y \tag{6.41}$$

$$(\mathbf{E}^\varepsilon, e^\varepsilon) \text{ is 1-periodic in } t \tag{6.42}$$

$$\mathbf{E}^\varepsilon = g - \overline{g} \text{ for } \xi = 0 \tag{6.43}$$

$$\mathbf{E}^\varepsilon = 0 \text{ for } \xi = 1, \tag{6.44}$$

with

$$\mathbf{R}^\varepsilon = -\varepsilon\left(\frac{\partial \mathbf{W}^1}{\partial x^2} - 2\frac{\partial \mathbf{W}^2}{\partial x \partial t} - \frac{\partial M^2}{\partial x}\,\mathbf{i}\right) - \varepsilon^2\,\frac{\partial^2 \mathbf{W}^2}{\partial x^2}\,.$$

Thus,

$$|\mathbf{R}^\varepsilon|_{L^2(\Omega)} \le C\,\varepsilon. \tag{6.45}$$

Standard a priori estimate for the Stokes system leads to

$$|\mathbf{W}^\varepsilon - \mathbf{E}^\varepsilon|_{H^1(\Omega)} \le C\,\varepsilon \tag{6.46}$$

$$|M^\varepsilon - e^\varepsilon|_{L^2(\Omega)} \le C\,\varepsilon. \tag{6.47}$$

If we put this all together, we end up with:

Theorem 4. *Let* $(\mathbf{U}^\varepsilon, P^\varepsilon)$ *be the solution to the problem (5.3)–(5.7) with* g^ε *satisfying (6.1)–(6.4). Let*

$$\overline{g}(x) = \int_0^1 g(x,t)\,dt$$

and let

$$\mathbf{V}^{-1} = 6\xi\,(\xi - 1)\left[\frac{1}{L}\int_0^L (L-t)\overline{g}(t)\,dt - \int_0^x \overline{g}(t)\,dt\right]\mathbf{i} \tag{6.48}$$

$$\mathbf{V}^0 = \frac{\xi}{2}(\xi - 1)\,\frac{P_L - P_0}{L}\,\mathbf{i} + \left(2\,\xi^3 - 3\,\xi^2 + 1\right)\,\overline{g}(x)\,\mathbf{j} \tag{6.49}$$

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$$H^{-1} = 12\left[\frac{x}{L}\int_0^L (L-t)\overline{g}(t)\,dt - \int_0^x (x-t)\overline{g}(t)\,dt\right] \tag{6.50}$$

$$H^0 = P_0 + \frac{x}{L}(P_L - P_0). \tag{6.51}$$

For $(\mathbf{W}^0, M^0)$*, the solution to the auxiliary problem (6.19)–(6.23) and* $(\mathbf{W}^1, M^1)$*, the solution to the auxiliary problem (6.29)–(6.36), denoting*

$$\mathbf{W}^k_\varepsilon(x,\xi) = \mathbf{W}^k(x,\xi,\frac{x}{\varepsilon}),\ k=0,1\,,$$

the following convergence holds

$$\lim_{\varepsilon\to 0}\left|\mathbf{U}^\varepsilon - \left(\frac{1}{\varepsilon}\mathbf{V}^{-1} + \mathbf{V}^0 + \mathbf{W}^0_\varepsilon + \varepsilon\,\mathbf{W}^1_\varepsilon\right)\right|_{H^1(\Omega)} = 0 \tag{6.52}$$

$$\lim_{\varepsilon\to 0}\varepsilon^2\left|P^\varepsilon - \left(\frac{1}{\varepsilon^3}H^{-1} + \frac{1}{\varepsilon^2}H^0\right)\right|_{L^2(\Omega)} = 0. \tag{6.53}$$

Furthermore,

$$\mathbf{U}^\varepsilon - \left(\frac{1}{\varepsilon}\mathbf{V}^{-1} + \mathbf{V}^0\right) \rightharpoonup 0 \text{ weakly in } L^2(\Omega).$$

7. Conclusion

We have studied the asymptotic behaviour of the viscous fluid flow through a thin domain, with thickness ε, governed by the pressure drop between the ends of the domain and the injection of the fluid through the permeable bottom of the domain. The effective mean velocity and pressure of the fluid do not obey the standard Hagen-Poiseuille law. There is an additional term coming from the boundary injection and the mean flow is not incompressible, due to the boundary source.

As in the case of the classical Hagen-Poiseuille flow, we have assumed that the pressures on the sides of the domain are of the order ε^{-2} and are given by two constants P_0 and P_L.

In the first chapter of the paper, we have assumed that the injection velocity through the permeable boundary has the magnitude of order ε. Denoting by g the mean boundary injection velocity, we have obtained an effective velocity of the form

$$\mathbf{v} = \left[-\frac{1}{12}\frac{P_L - P_0}{L} + \frac{1}{L}\int_0^L t\,g(t)\,dt - \int_x^L g(t)\,dt\right]\mathbf{i}\,,$$

while the effective pressure is of the form

$$p = \frac{1}{\varepsilon^2}\left\{P_0 + (P_L - P_0)\frac{x}{L} + 12\left[\int_x^L (x-t)g(t)dt + \left(1-\frac{x}{L}\right)\int_0^L tg(t)dt\right]\right\}.$$

In the second part of the paper, the boundary injection velocity is of order 1 and the effective velocity has the form

$$\mathbf{v} = \left\{\frac{1}{\varepsilon}\left[\frac{1}{L}\int_0^L t\,g(t)\,dt - \int_x^L g(t)\,dt\right] - \frac{1}{12}\frac{P_L - P_0}{L}\right\}\mathbf{i} + \frac{1}{2}\,g\,\mathbf{j}.$$

If the inertial effects are taken into account, we get in addition

$$\mathbf{v}_{inert} = \frac{297}{420}Re\,g(x)\left[\frac{1}{L}\int_0^L (L-t)g(t)\,dt - \int_0^x g(t)\,dt\right]\Bigg\}\,\mathbf{i}.$$

The effective pressure is

$$p = \frac{1}{\varepsilon^3}\left\{12\left[\int_x^L (x-t)g(t)dt + \left(1-\frac{x}{L}\right)\int_0^L tg(t)dt\right]\right\} + $$
$$+\frac{1}{\varepsilon^2}\left\{P_0 + (P_L - P_0)\frac{x}{L}\right\}.$$

Again, if the inertial term is taken into account (significant Reynolds number), then the pressure correction appears of the form

$$p_{inert} = \frac{297}{35} Re \left\{ \frac{1-x}{L} \int_0^L (L-t) g(t) dt - \int_0^x g(t) dt + \right.$$
$$\left. + \frac{1}{2} \left[\left(\int_0^x g(t) dt \right)^2 - \frac{x}{L} \left(\int_0^L g(t) dt \right)^2 \right] \right\}$$

Thus, in the second case, the boundary injection dominates the flow. The Hagen-Poiseuille part remains there, but it has smaller magnitude. Apart from the standard flow along the pipe, a transversal flow appears and equals half of the mean boundary injection velocity.

Finally, if the boundary injection is oscillating with the period having the same order as the domain thickness, $g^\varepsilon(x) = g\,(x, x/\varepsilon)$, we are in the homogenisation case. Basically, the result is the same as in the previous case, with correctors for those small oscillations of the flow. However, the mean values of those correctors are zero and they disappear in the weak limit.

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