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ZEROS OF ITERATED INTEGRALS OF POLYNOMIALS

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ABSTRACT. The operator I_m is defined as m-fold indefinite integration with zero constants of integration. The zero distribution of $I_m(p)$ for polynomials p is studied in general, and for two special classes of polynomials in detail. The main results are: (i) The zeros of $I_n(P_n)$, where $P_n(z)$ is the n-th Legendre polynomial, converge to a certain algebraic curve; (ii) the zeros of $\sum_{k=n+1}^{cn}(nz)^k/k!$ ($c \geq 2$ an integer) converge to pieces of a circle and of two "Szegö curves".

1. **Introduction.** The phenomena associated with the location of roots of polynomials are rich and varied. Consider, for example,

$$S_{n,m}(z) := \sum_{k=m+1}^{n} \frac{z^k}{k!}. \tag{1.1}$$

These are "parts" of the partial sums for the exponential function. Classic work of Szegö [15], followed by Dieudonné [4], Rosenbloom [12] and others, give that the zeros of the set of "normalized" partial sums to exp, that is

$$\{S_{n,-1}(nz)\} \tag{1.2}$$

are dense in the "Szegö curve" defined as

$$\{z : |ze^{1-z}| = 1 \text{ and } |z| \leq 1\}. \tag{1.3}$$

This is discussed in detail in Varga [16] (see Figure 1). However, a very minor change gives very different results. Suppose we consider the partial sums to $e^z - \alpha$ ($\alpha \neq 0$), namely

$$\{S_{n,-1}(nz) - \alpha\} \tag{1.4}$$

then, as Rosenbloom [12] shows, the zeros in $\{z : \mathrm{Re}(z) < 0\}$ now approach the half circle $\{z : |z| = 1/e, \mathrm{Re}(z) < 0\}$, while the zeros in the right half plane approach the piece of Szegö's curve, as before.

Now consider the polynomials

$$\{S_{2n,n}(nz)\} = \left\{ \sum_{k=n+1}^{2n} \frac{(nz)^k}{k!} \right\}. \tag{1.5}$$

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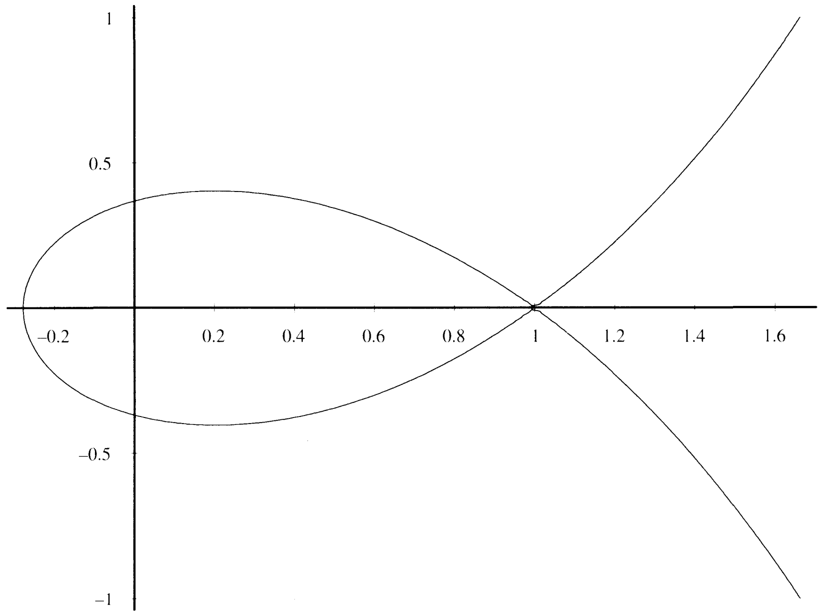

FIGURE 1: $|ze^{1-z}| = 1$

In Figure 2 we have plotted the zeros for $n = 20, 40$ and 80. From the case $n = 20$ it looks like the zeros might approach a circle $\{z : |z| = 2/e\}$ and indeed many of them do. But the higher computations show that this is unlikely to be the whole story. Theorem 4 describes the limit curve for these polynomials (see also Figure 3) and more generally for sequences

$$\{S_{cn,n}(nz)\}, \tag{1.6}$$

for a constant integer $c \geq 2$.

In Section 2 we introduce the n-th integration operator and derive some general properties. The second main result of this paper concerns the n-th integral of the n-th Legendre polynomial (Theorem 2). It is an easy consequence of Theorem 1 on the limit properties of the zeros of a certain class of polynomials. Theorem 3 gives further geometric properties of the zeros of these polynomials.

The theorems are proved in Sections 3–5. A few further properties of zeros of integrals of polynomials are discussed in Section 6, and Section 7 contains some additional remarks.

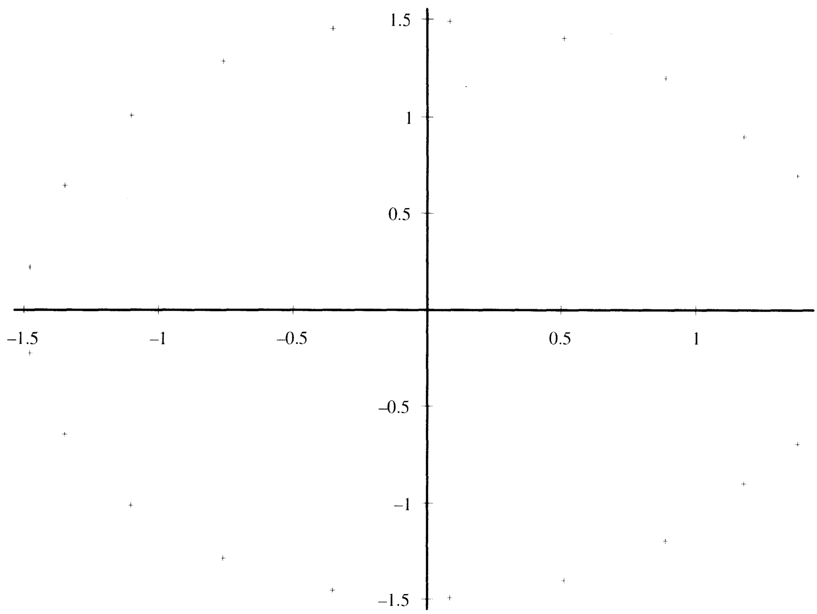

FIGURE 2A: ZEROS OF $\sum_{k=20}^{40} \frac{(20z)^k}{k!}$

2. **The n-th integration operator; summary of results.** The polynomial $S_{2n,n-1}(z)$ is the n-th integral of $S_{n,-1}(z)$ provided we make the assumption that the constants of integration are zero. (It is the problem of the location of the zeros of the n-th integral of a polynomial of degree n that interests us most.) We define the m-th integration operator

$$
\begin{aligned}
I_m\big(p(z)\big) &:= \int_0^z \int_0^{t_{m-1}} \cdots \int_0^{t_1} p(t_0)\,dt_0\,dt_1 \cdots dt_m \\
&= \frac{1}{(m-1)!} \int_0^z (z-t)^{m-1} p(t)\,dt
\end{aligned} \tag{2.1}
$$

for any p integrable in some neighbourhood of zero. (In fact, p will always be a polynomial in this paper.) Then $I_m(p)$ is an m-th antiderivative of p and has a zero of order m at zero.

Note that

$$
I_m\left(\sum_{k=0}^{n} \binom{n}{k} a_k z^k\right) = z^m \sum_{k=0}^{n} \binom{n}{k} \frac{a_k k!}{(m+k)!} z^k. \tag{2.2}
$$

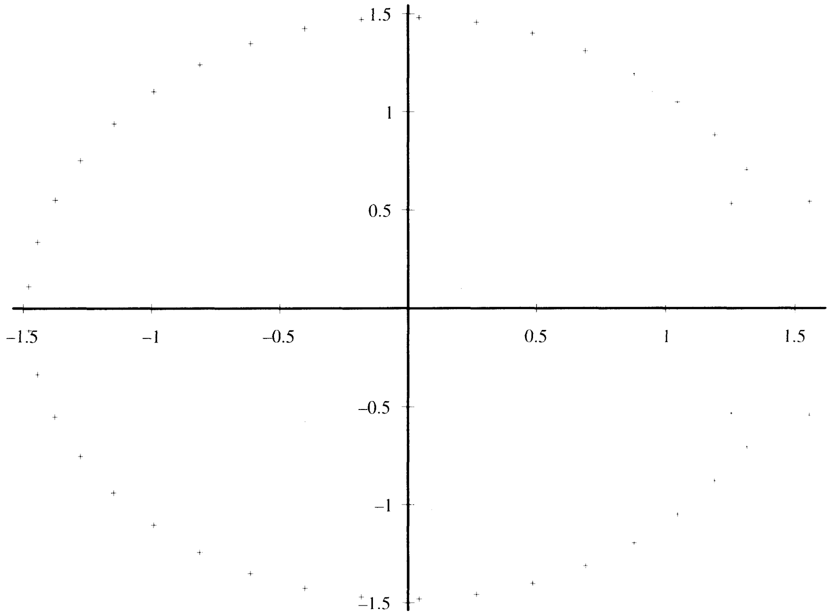

FIGURE 2B: ZEROS OF $\sum_{k=40}^{80} \frac{(40z)^k}{k!}$

So the n-th integral of an n-th degree polynomial is

$$I_n\left(\sum_{k=0}^{n}\binom{n}{k}a_k z^k\right) = \frac{n!z^n}{(2n)!}\sum_{k=0}^{n}\binom{2n}{n+k}a_k z^k \tag{2.3}$$

and the $(n+1)$-th integral is

$$I_{n+1}\left(\sum_{k=0}^{n}\binom{n}{k}a_k z^k\right) = \frac{n!z^{n+1}}{(2n+1)!}\sum_{k=0}^{n}\binom{2n+1}{n+k+1}a_k z^k. \tag{2.4}$$

The Gauss-Grace-Lucas Theorem (see, *e.g.*, Marden [8] or Borwein and Erdélyi [2]) says that the zeros of the derivative of a polynomial lie in the convex hull of the zeros of the polynomial. This is a theorem with many refinements (see, *e.g.*, Marden [8]). There are various, but far fewer, results on the location of the zeros of the integral. Some of these follow from the Schur-Szegö Theorem (see, *e.g.*, Pólya and Szegö [9, Vol. II, p. 60–61]).

THEOREM A (SCHUR, SZEGÖ). *Suppose $a_n b_n \neq 0$,*

$$f(z) = a_0 + \binom{n}{1}a_1 z + \binom{n}{2}a_2 z^2 + \cdots + \binom{n}{n-1}a_{n-1}z^{n-1} + a_n z^n,$$

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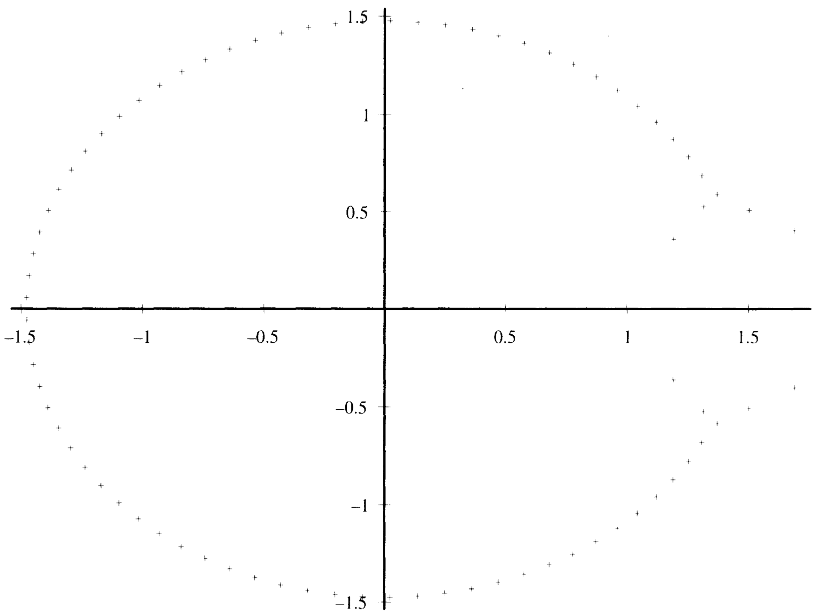

FIGURE 2C: ZEROS OF $\sum_{k=80}^{160} \frac{(80z)^k}{k!}$

$$g(z) = b_0 + \binom{n}{1} b_1 z + \binom{n}{2} b_2 z^2 + \cdots + \binom{n}{n-1} b_{n-1} z^{n-1} + b_n z^n,$$

and

$$h(z) = a_0 b_0 + \binom{n}{1} a_1 b_1 z + \binom{n}{2} a_2 b_2 z^2 + \cdots + \binom{n}{n-1} a_{n-1} b_{n-1} z^{n-1} + a_n b_n z^n.$$

a) *If the zeros of f lie in a disc of radius r and the zeros of g lie in a disc of radius s then the zeros of h lie in a disc of radius rs.*

b) *If the zeros of f lie in a convex set K and the zeros of g are real and lie in the interval* $[-1, 0]$ *then the zeros of h also lie in K.*

From this result and (2.2) we have the following.

COROLLARY 1. *Suppose p is a polynomial of exact degree n.*

a) *If p has all its zeros in a disc of radius* 1 *then $I_m(p)$ has all its zeros in a disc of radius r_m, where r_m is the modulus of the largest zero of*

$$Q_{n,m}(z) := \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(m+k)!} z^k.$$

b) *If p has all its zeros in $[-1,0]$ then all the zeros of $I_m(p)$ lie in the convex hull of the zeros of $Q_{n,m}$.*

PROOF. Let $Q_{m,n}$ take the role of g and p take the role of f in the Schur-Szegö Theorem. Then h is $I_m(p)$. ■

We see that bounds, which are in fact sharp, for the location of zeros of the integrals of polynomials can be derived from precise knowledge of the location of the zeros of the $Q_{m,n}$. With this in mind, we prove, for example,

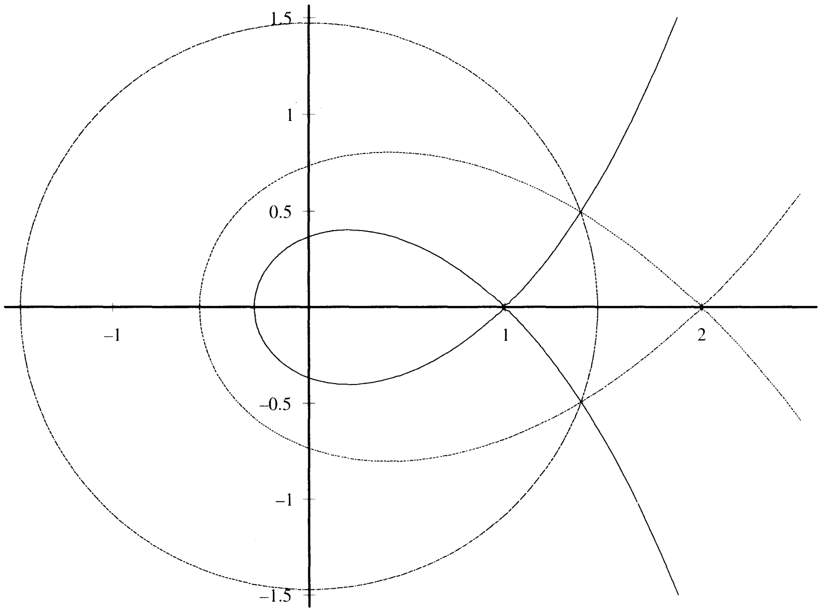

FIGURE 3: $|z| = 2/e$, $|ze^{1-z}| = 1$, $\left|\frac{z}{2}e^{1-z/2}\right| = 1$

THEOREM 1. *The zeros of*

$$\left\{Q_{n,n+1}(z) = \frac{n!}{(2n+1)!}\sum_{k=0}^{n}\binom{2n+1}{n+1+k}z^k\right\}$$

are dense in the curve

$$\Gamma := \left\{z : \left|\frac{(1+z)^2}{4z}\right| = 1 \text{ and } |z| \geq 1\right\},$$

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and these are the only limit points of the zeros (Figure 4a and 4b). Furthermore, all the zeros lie in the convex hull of Γ *so the largest zero of* $Q_{n,n+1}$ *is of modulus at most* $3 + 2\sqrt{2} = 5.8284\cdots$ *(see Figure 4a and 4b). (The same result holds for* $\{Q_{n,n}(z)\}$*.)*

COROLLARY 2. *Suppose p is a polynomial of exact degree n.*

a) *If p has all its zeros in a disc of radius r then* $I_n(p)$ *and* $I_{n+1}(p)$ *have all their zeros in a disc of radius* $(3+2\sqrt{2})r$.

b) *If p has all its zeros in* $[-1, 0]$ *then* $I_{n+1}(p)$ *has all its zeros in the interior of* Γ.

This corollary requires Theorem 3 below. We examine two special cases in detail. The first, as already described, concerns the partial sums of the exponential function and is treated in Theorem 4. The second follows easily from Theorem 1 and concerns the Legendre polynomials on $[-1, 1]$.

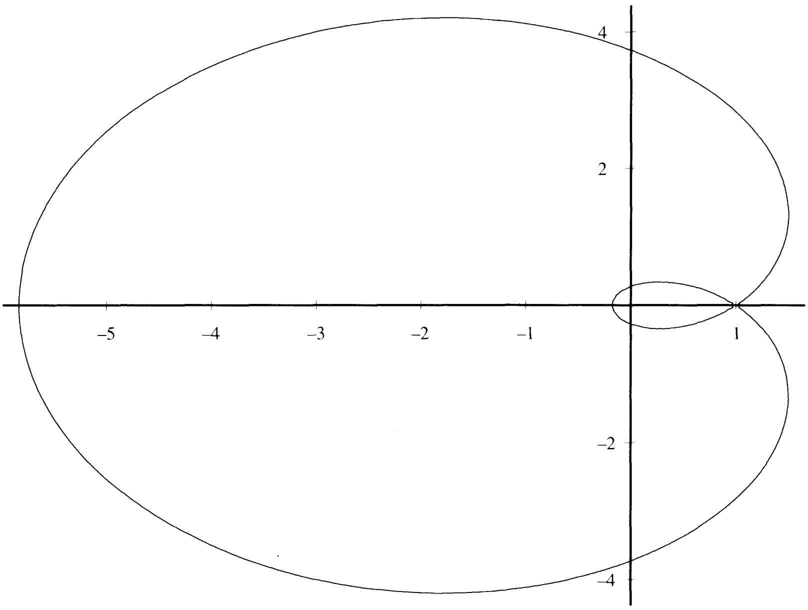

FIGURE 4A: $\left|\frac{(1+z)^2}{4z}\right| = 1$

THEOREM 2. *Let* P_n *be the n-th Legendre polynomial on* $[-1, 1]$. *Then the zeros of* $I_n(P_n)$ *are dense in the curve*

$$\Gamma_2 := \left\{ z : \left|\frac{1-z^2}{2z}\right| = 1 \text{ and } |z| \geq 1 \right\}$$

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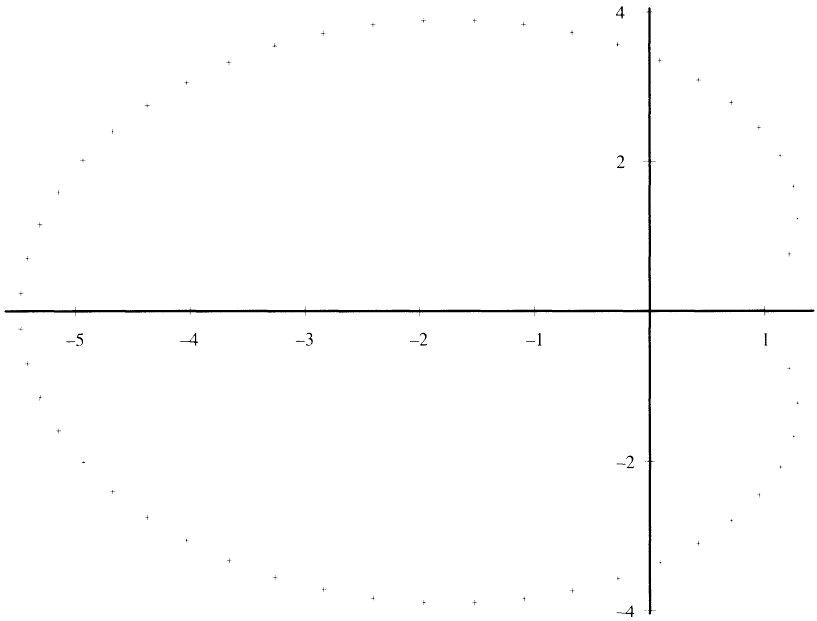

FIGURE 4B: ZEROS OF $\sum_{k=0}^{50} \binom{100}{50+k} z^k$

(and nowhere else). Furthermore, all the zeros lie inside Γ_2 (see Figure 5a and 5b).

PROOF. We use the Rodrigues formula for P_n, namely

$$P_n(z) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dz^n}\big((1 - z^2)^n\big) \tag{2.5}$$

(see, *e.g.*, [1, Ch. 22]) to deduce

$$I_{2n}(P_{2n}) \Big|_{\sqrt{z}} = \frac{(-z)^n}{4^n (2n)!} \sum_{k=0}^{n} \binom{2n}{n+k} (-z)^k, \tag{2.6}$$

and similarly for $2n + 1$. The result now follows from Theorem 1 on changing variables $(z \longrightarrow -z^2)$. ■

THEOREM 3. *Let*

$$p_n(z) = \sum_{k=0}^{n} \binom{2n+1}{n+k+1} z^k, \quad q_n(z) = \sum_{k=0}^{n} \binom{2n}{n+k} z^k,$$

and J_n, K_n be the convex hulls of the zeros of $p_n(z)$ and $q_n(z)$, respectively. Then

$$K_n \subset J_n \subset K_{n+1} \subset J_{n+1}$$

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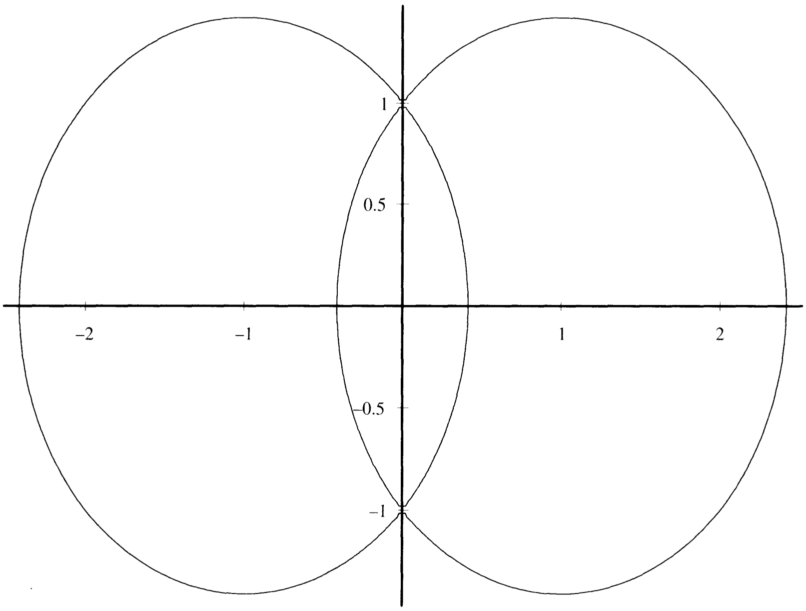

FIGURE 5A: $\left|\frac{1-z^2}{2z}\right| = 1$

for $n = 1, 2, \ldots$.

Before stating Theorem 4, we introduce some notations. Let $\{z : |ze^{1-z}| = 1\}$ and $\{z : |(z/c)e^{1-z/c}| = 1\}$ be two Szegö curves, where $c > 1$ is a real constant. With these curves in mind we denote

$D_1 := \{z : |ze^{1-z}| < 1, |z| < 1\}$,
$D_2 := \{z : |ze^{1-z}| > 1\}$,
$D_3 := \{z : |ze^{1-z}| < 1, |z| > 1\}$,
$E_1 := \{z : |\frac{z}{c}e^{1-z/c}| < 1, |z| < c\}$,
$E_2 := \{z : |\frac{z}{c}e^{1-z/c}| > 1\}$,
$E_3 := \{z : |\frac{z}{c}e^{1-z/c}| < 1, |z| > c\}$,
$D_{12} := \{z : |ze^{1-z}| \leq 1, |z| < 1\} \cup D_2 \setminus \{0\}$,
$E_{12} := \{z : |\frac{z}{c}e^{1-z/c}| \leq 1, |z| < c\} \cup E_2 \setminus \{0\}$.

Note that all the above sets are connected open sets. Moreover, let

$F_1 := \{z : |z| = \frac{1}{e}c^{c/(c-1)}\} \cap D_{12}$,
$F_2 := \{z : |ze^{1-z}| = 1, |z| \geq 1\} \cap \{z : |\frac{z}{c}e^{1-z/c}| \leq 1, |z| \leq c\}$,
$F_3 := \{z : |\frac{z}{c}e^{1-z/c}| = 1, |z| \leq c\} \cap \{z : |ze^{1-z}| \leq 1, |z| \geq 1\}$,

and

$$F := F_1 \cup F_2 \cup F_3 \cup \{0\}.$$

Then we have the following result.

THEOREM 4. *In the notation of (1.1), $\hat{z}$ is a limit point of zeros of the sequence of normalized sums $\{S_{cn,n}(nz)\}_{n=1}^{\infty}$ if and only if $\hat{z} \in F$.*

3. **Proof of Theorem 1.** With the curve Γ in mind (see Figure 4a and 4b) we denote

$$G_1 := \{z : |\tfrac{(1+z)^2}{4z}| > 1, |z| < 1\},$$
$$G_2 := \{z : |\tfrac{(1+z)^2}{4z}| < 1\},$$
$$G_3 := \{z : |\tfrac{(1+z)^2}{4z}| > 1, |z| > 1\}.$$

Next we observe that the zeros of $Q_{n,n+1}(z)$ must lie in the closed annulus

$$\frac{n+2}{n} \leq z \leq 2n+1,$$

by the well-known Eneström-Kakeya Theorem (see, *e.g.*, [9, Vol. I, p. 107]).

Since $G_1 \subset \{z : |z| < 1\}$, G_1 is free of zeros of $\{Q_{n,n+1}(z)\}_{n=1}^{\infty}$. It is easy to verify, by expanding and integrating, that $Q_{n,n+1}(z)$ has the integral representation

$$Q_{n,n+1}(z) = \frac{(2n+1)!}{(n!)^2} \int_0^1 [(1-t)(1+zt)]^n \, dt. \tag{3.1}$$

We proceed to analyse the zeros of $Q_{n,n+1}$ by an asymptotic analysis of the above integral. Now denote

$$f_z(t) := (1-t)(1+zt),$$

and for $\varepsilon > 0$ set

$$B_z(\varepsilon) := \{t : |f_z(t)| \leq \varepsilon\}.$$

Then

$$B_z(\varepsilon) = \left\{ t : t = \frac{z - 1 \mp \sqrt{(z-1)^2 + 4z(1 - \mathrm{r}e^{i\theta})}}{2z}, 0 \leq r \leq \varepsilon, 0 \leq \theta \leq 2\pi \right\}$$
$$=: B_z^+(\varepsilon) \cup B_z^-(\varepsilon),$$

where $B_z^+(\varepsilon)$, (resp. $B_z^-(\varepsilon)$) denote $B_z(\varepsilon)$ with positive, (resp. negative) principal square root. Note that $B_z^+(\varepsilon)$ and $B_z^-(\varepsilon)$ are connected. If

$$\varepsilon \geq \left| \frac{(1+z)^2}{4z} \right| = \left| f_z\Big(\frac{z-1}{2z}\Big) \right|,$$

let

$$\frac{(1+z)^2}{4z} = re^{i\theta};$$

then

$$t^+ := \frac{z - 1 + [(z-1)^2 + 4z - (z+1)^2]^{1/2}}{2z} = \frac{z-1}{2z} \in B_z^+(\varepsilon),$$

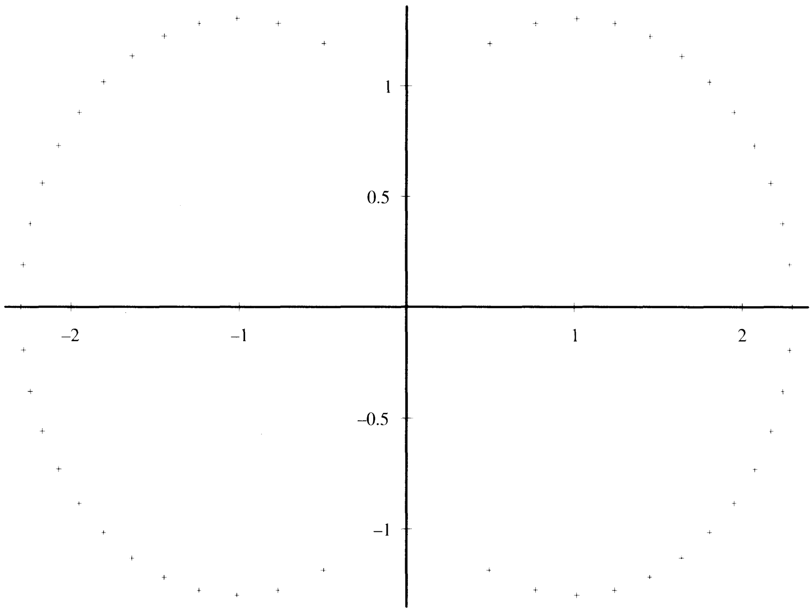

FIGURE 5B: ZEROS OF $I_{50}(L_{50}(z))$

and

$$t^- := \frac{z-1-[(z-1)^2+4z-(z+1)^2]^{1/2}}{2z} = \frac{z-1}{2z} \in B_z^-(\varepsilon).$$

Thus, $B_z(\varepsilon)$ is connected.

Now, for any $z \in G_2$ we have

$$d := \left|\frac{(1+z)^2}{4z}\right| = \left|f_z\Big(\frac{z-1}{2z}\Big)\right| < 1.$$

Choose $\varepsilon > 0$ such that $1-d > \varepsilon$; then $d < 1-\varepsilon$, and

$$\frac{z-1}{2z} \in B_z(1-\varepsilon).$$

Let now

$$t^* := \frac{z-1}{2z} - \frac{1}{2z}[(1+z)^2 - 4z(1-\varepsilon)]^{1/2};$$

then $t^* \in B_z(1-\varepsilon)$ and since $B_z(1-\varepsilon)$ is connected, we can find a curve λ in $B_z(1-\varepsilon)$ such that $\lambda(a) = t^*$ and $\lambda(b) = 1$. For $1-\varepsilon \le r \le 1$, let

$$t(r) := \frac{z-1}{2z} - \frac{1}{2z}[(1+z)^2 - 4zr]^{1/2};$$

then $t(1-\varepsilon)=t^*$, $t(1)=0$.

Now we estimate

$$\begin{aligned}
&\left|\int_0^1 [(1-t)(1+tz)]^n\,dt\right| \\
&\quad=\left|\left(\int_0^{t^*}+\int_\lambda\right)[(1-t)(1+tz)]^n\,dt\right| \\
&\quad\geq\left|\int_0^{t^*}[(1-t)(1+tz)]^n\,dt\right|-\int_\lambda\left|(1-t)(1+tz)\right|^n dt \\
&\quad\geq\left|\int_{1-\varepsilon}^1\left\{\left[1-\frac{z-1}{2z}+\frac{\left((1+z)^2-4zr\right)^{1/2}}{2z}\right]\left[1+z\left[\frac{z-1}{2z}-\right.\right.\right.\right. \\
&\qquad\left.\left.\left.\left.\frac{\left((1+z)^2-4zr\right)^{1/2}}{2z}\right]\right]\right\}^n t'(r)\,dr\right|-\operatorname{arc}\left(\lambda(t^*,1)\right)(1-\varepsilon)^n \\
&\quad=\left|\int_{1-\varepsilon}^1\left\{\left[\frac{z+1}{2z}+\frac{\left((1+z)^2-4zr\right)^{1/2}}{2z}\right]\left[\frac{z+1}{2}-\frac{\left((1+z)^2-4zr\right)^{1/2}}{2}\right]\right\}^n\right. \\
&\qquad\left.[(1+z)^2-4zr]^{-1/2}\,dr\right|-\operatorname{arc}\left(\lambda(t^*,1)\right)(1-\varepsilon)^n \\
&\quad=\left|\int_{1-\varepsilon}^1\left\{\frac{1}{4z}\left[(1+z)^2-\left((1+z)^2-4zr\right)\right]\right\}^n[(1+z)^2-4zr]^{-1/2}\,dr\right| \\
&\qquad-\operatorname{arc}\left(\lambda(t^*,1)\right)(1-\varepsilon)^n \\
&\quad=\left|\int_{1-\varepsilon}^1 r^n[(1+z)^2-4zr]^{-1/2}\,dr\right|-\operatorname{arc}\left(\lambda(t^*,1)\right)(1-\varepsilon)^n.
\end{aligned}$$

where $\operatorname{arc}(\gamma)$ denotes the length of the curve γ. Write $(1+z)^2/4z=de^{i\theta}$; then

$$\begin{aligned}
[(1+z)-4zr]^{-1/2}&=(4z)^{-1/2}[de^{i\theta}-r]^{-1/2} \\
&=(4z)^{-1/2}[r^2-2dr\cos\theta+d^2]^{-1/4}e^{-i\alpha/2},
\end{aligned}$$

where

$$\begin{aligned}
\cos\alpha&=(d\cos\theta-r)(r^2-2dr\cos\theta+d^2)^{-1/2}, \\
\sin\alpha&=d\sin\theta(r^2-2dr\cos\theta+d^2)^{-1/2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\left|\cos\frac{\alpha}{2}\right|&=\left(\frac{1}{2}\right)^{1/2}[(r^2-2dr\cos\theta+d^2)^{1/2}+d\cos\theta-r]^{1/2}(r^2-2dr\cos\theta+d^2)^{-1/4}, \\
\left|\sin\frac{\alpha}{2}\right|&=\left(\frac{1}{2}\right)^{1/2}[(r^2-2dr\cos\theta+d^2)^{1/2}-d\cos\theta+r]^{1/2}(r^2-2dr\cos\theta+d^2)^{-1/4}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\left|\int_{1-\varepsilon}^{1} r^n[(1+z)^2-4zr]^{-1/2}\,dr\right| \\
&\geq |8z|^{-1/2}\left|\int_{1-\varepsilon}^{1} r^n[(r^2-2dr\cos\theta+d^2)^{1/2}+r-d\cos\theta]^{1/2}\right. \\
&\qquad\qquad \left.\cdot(r^2-2dr\cos\theta+d^2)^{-1/4}\,dr\right| \\
&\geq |8z|^{-1/2}\int_{1-\varepsilon}^{1} r^n[(r^2-2dr+d^2)^{1/2}+r-d]^{1/2}(r^2+2dr+d^2)^{-1/2}\,dr \\
&\geq |8z|^{-1/2}\int_{1-\varepsilon}^{1} r^n 2^{1/2}(r-d)^{1/2}2^{-1/2}\,dr \\
&\geq |8z|^{-1/2}(1-\varepsilon-d)^{1/2}\int_{1-\varepsilon}^{1} r^n\,dr \\
&\geq |8z|^{-1/2}(1-\varepsilon-d)^{1/2}\Big(1-\frac{\varepsilon}{2}\Big)^n\frac{\varepsilon}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_0^1[(1-t)(1+tz)]^n\,dt\right|^{1/n} \\
&\qquad \geq \left\{|8z|^{-1/2}(1-\varepsilon-d)^{1/2}\Big(1-\frac{\varepsilon}{2}\Big)^n\cdot\frac{\varepsilon}{2}-\operatorname{arc}\big(\lambda(t^*,1)\big)(1-\varepsilon)^n\right\}^{1/n} \\
&\qquad = \Big(1-\frac{\varepsilon}{2}\Big)\left\{|8z|^{-1/2}(1-\varepsilon-d)^{1/2}\frac{\varepsilon}{2}-\operatorname{arc}\big(\lambda(t^*,1)\big)\Big(1-\frac{\varepsilon}{2-\varepsilon}\Big)^n\right\}^{1/n}
\end{aligned}
$$

Thus, we have with (3. 1),

$$
\begin{aligned}
\liminf_{n\to\infty}|Q_{n,n+1}(z)|^{1/n} &= \liminf_{n\to\infty}\left\{\frac{(2n+1)!}{(n!)^2}\left|\int_0^1[(1-t)(1+tz)]^n\,dt\right|\right\}^{1/n} \\
&\geq 4\Big(1-\frac{\varepsilon}{2}\Big).
\end{aligned}
\tag{3. 2}
$$

On the other hand, when $z\in G_3$, we have

$$
d:=\left|\frac{(1+z)^2}{4z}\right|=\left|f_z\Big(\frac{z-1}{2z}\Big)\right|>1.
$$

Choose $\varepsilon>0$ such that $\varepsilon<d-1$. Write

$$
\frac{(1+z)^2}{4z}=de^{i\theta},
$$

and here let

$$
\begin{aligned}
t^+ &:= \frac{z-1+[(1+z)^2-4z(d-\varepsilon)e^{i\theta}]^{1/2}}{2z}\in B_z^+(d-\varepsilon), \\
t^- &:= \frac{z-1-[(1+z)^2-4z(d-\varepsilon)e^{i\theta}]^{1/2}}{2z}\in B_z^-(d-\varepsilon).
\end{aligned}
$$

Let λ_1 be the curve in $B_z^+(d-\varepsilon)$ which connects t^+ and 1, and λ_2 be the curve in $B_z^-(d-\varepsilon)$ which connects 0 and t^-. Now we connect t^- and $(z-1)/2z$ by

$$
t^-(r):=\frac{z-1-[(1+z)^2-4zre^{i\theta}]^{1/2}}{2z},\quad d-\varepsilon\leq r\leq d,
$$

and $(z-1)/2z$ and t^+ are connected by

$$t^+(r) := \frac{z-1+[(1+z)^2-4zre^{i\theta}]^{1/2}}{2z}, \quad d-\varepsilon \le r \le d.$$

Then we can write

$$\begin{aligned}
&\left|\int_0^1 [(1-t)(1+tz)]^n\,dt\right| \\
&\quad = \left|\left(\int_{\lambda_2} + \int_{t^-}^{\frac{z-1}{2z}} + \int_{\frac{z-1}{2z}}^{t^+} + \int_{\lambda_1}\right)[(1-t)(1+tz)]^n\,dt\right| \\
&\quad \ge \left|\left(\int_{t^-}^{\frac{z-1}{2z}} + \int_{\frac{z-1}{2z}}^{t^+}\right)[(1-t)(1+tz)]^n\,dt\right| \\
&\qquad - \left|\int_{\lambda_2}[(1-t)(1+tz)]^n\,dt\right| - \left|\int_{\lambda_1}[(1-t)(1+tz)]^n\,dt\right| \\
&\quad \ge \left|\int_{d-\varepsilon}^{d} \{[1-t^-(r)][1+zt^-(r)]\}^n[(1+z)^2-4zre^{i\theta}]^{-1/2}e^{i\theta}\,dr\right. \\
&\qquad \left.+ \int_{d}^{d-\varepsilon} \{[1-t^+(r)][1+zt^+(r)]\}^n[(1+z)^2-4zre^{i\theta}]^{-1/2}(-e^{i\theta})\,dr\right| \\
&\qquad - [\operatorname{arc}(\lambda_1)+\operatorname{arc}(\lambda_2)](d-\varepsilon)^n \\
&\quad = 2\left|\int_{d-\varepsilon}^{d} \{re^{i\theta}\}^n e^{i\theta}[(1+z)^2-4zre^{i\theta}]^{-1/2}\,dr\right| \\
&\qquad - [\operatorname{arc}(\lambda_1)+\operatorname{arc}(\lambda_2)](d-\varepsilon)^n.
\end{aligned}$$

Since $4ze^{i\theta} = (1+z)^2/d$, we have

$$\begin{aligned}
\left|\int_{d-\varepsilon}^{d} r^n[(1+z)^2-4zre^{i\theta}]^{-1/2}\,dr\right| &= |1+z|^{-1}\int_{d-\varepsilon}^{d} r^n\left(1-\frac{r}{d}\right)^{-1/2} dr \\
&\ge |1+z|^{-1}\left(\frac{d}{\varepsilon}\right)^{1/2}\int_{d-\varepsilon}^{d} r^n\,dr \\
&\ge |1+z|^{-1}\left(\frac{d}{\varepsilon}\right)^{1/2}\frac{\varepsilon}{2}\left(d-\frac{\varepsilon}{2}\right)^n.
\end{aligned}$$

Thus, again with (3.1),

$$\begin{aligned}
&\liminf_{n\to\infty} |Q_{n,n+1}(z)|^{1/n} \\
&\quad = \liminf_{n\to\infty}\left\{\frac{(2n+1)!}{(n!)^2}\left|\int_0^1[(1-t)(1+tz)]^n\,dt\right|\right\}^{1/n} \\
&\quad \ge 4\left(d-\frac{\varepsilon}{2}\right)\liminf_{n\to\infty}\left\{|1+z|^{-1}\left(\frac{d}{\varepsilon}\right)^{1/2}\varepsilon - [\operatorname{arc}(\lambda_1)+\operatorname{arc}(\lambda_2)]\left(1-\frac{\varepsilon}{2d-\varepsilon}\right)^n\right\}^{1/n} \\
&\quad = 4\left(d-\frac{\varepsilon}{2}\right).
\end{aligned}$$

Therefore, with this and (3.2), all the zeros of the sequence $\{Q_{n,n+1}(z)\}_{n=1}^{\infty}$ must lie on the curve Γ.

On the other hand, suppose we have a point $\hat{z} \in \Gamma$ which is not a limit point of zeros of $\{Q_{n,n+1}(z)\}_{n=1}^{\infty}$. Then there is an open disk centered at $\hat{z}$ with radius $r > 0$, $D\,(\hat{z}, r)$, such that $D\,(\hat{z}, r)$ is free of zeros of $\{Q_{n,n+1}(z)\}_{n=1}^{\infty}$. We first prove that

$$[Q_{n,n+1}(z)]^{1/n} \longrightarrow 4 \quad \text{as } n \longrightarrow \infty \tag{3.3}$$

uniformly on any compact subset of

$$G := G_2 \cup \left\{ z : \left| \frac{(1+z)^2}{4z} \right| \geq 1, |z| < 1 \right\},$$

and

$$l[Q_{n,n+1}(z)]^{1/n} \longrightarrow 4\frac{(1+z)^2}{4z} \quad \text{as } n \longrightarrow \infty \tag{3.4}$$

uniformly on any compact subset of G_3.

Indeed, let $A := \{z : \operatorname{Im} z = 0, -4 \leq z \leq -2\} \subset G$. Using Problem #198 in [9, Vol. I. p. 96], one can easily deduce

$$\{Q_{n,n+1}(z)\}^{1/n} \longrightarrow 4 \quad \text{as } n \longrightarrow \infty,$$

uniformly on A. Similary, with $B := \{z : \operatorname{Im} z = 0, 2 \leq z \leq 4\} \subset G_3$ we find

$$\{Q_{n,n+1}(z)\}^{1/n} \longrightarrow 4\frac{(1+z)^2}{4z} \quad \text{as } n \longrightarrow \infty,$$

uniformly on B. With the uniqueness theorem for analytic functions and Montel's theorem on normal families, the last two limits imply (3.3) and (3.4), respectively.

Now, by the same argument we can prove that $\{Q_{n,n+1}(z)\}^{1/n}$ will converge to 4 uniformly on any compact subset of $D(\hat{z}, r) \cup G$. But on the other hand we can also prove, again using the above methed, that $\{Q_{n,n+1}(z)\}^{1/n}$ will converge to $4(1+z)^2/4z$ uniformly on any compact subset of G_3. Thus, 4 and $4(1+z)^2/4z$ must agree on $D\,(\hat{z}, r)$, which is a contradiction. The proof of Theorem 1 is now complete. ∎

4. **Proof of Theorem 3.** 1. We can rewrite $p_n(z)$ as

$$p_n(z) = \sum_{k=0}^{n} \binom{2n}{n+k} b_k z^k,$$

where

$$b_k := \binom{2n+1}{n+k+1} \Big/ \binom{2n}{n+k} = \frac{2n+1}{n+k+1}.$$

Using the Schur-Szegö Composition Theorem (Theorem A), we can see that it suffices to prove that

$$h_n(z) := \sum_{k=0}^{n} \binom{n}{k} \frac{1}{b_k} z^k$$

has all its zeros in $[-1,0]$. Now we have

$$\begin{aligned} h_n(z) &= \frac{1}{2n+1}\sum_{k=0}^{n}(n+k+1)\binom{n}{k}z^k \\ &= \frac{1}{2n+1}(1+z)^{n-1}[(2n+1)z+(n+1)], \end{aligned}$$

and therefore all the zeros of $h_n(z)$ lie in the interval $[-1,0]$, which implies $K_n \subset J_n$ by Theorem A.

2. Similarly, we write

$$q_{n+1}(z) = \sum_{k=0}^{n+1}\binom{2n+1}{n+k+1}\frac{2n+2}{n-k+1}z^k,$$

so we need to check the zeros of the polynomial

$$\frac{1}{2n+2}\sum_{k=0}^{n}\binom{n+1}{k}(n-k+1)z^k = \frac{1}{2}(1+z)^n,$$

and Theorem A implies $J_n \subset K_{n+1}$. This completes the proof of Theorem 3. ∎

5. **Proof of Theorem 4.** We use the notation preceding the statement of Theorem 4, and we recall that

$$S_{cn,n}(nz) = S_{cn}(nz) - S_n(nz), \tag{5.1}$$

where

$$S_n(z) = \sum_{k=0}^{n}\frac{z^k}{k!}.$$

Differentiation now establishes

$$\begin{aligned} S_n(z) &= e^z - \frac{1}{n!}\int_0^z t^n e^{z-t}\,dt \\ &= e^z - \frac{z}{n!}\int_0^1 (tz)^n e^{(1-t)z}\,dt, \end{aligned}$$

and we get

$$S_n(nz) = e^{nz} - \frac{n^{n+1}z}{n!}\int_0^1 [tze^{(1-t)z}]^n\,dt.$$

Similarly, we have

$$S_{cn}(nz) = e^{nz} - \frac{n^{cn+1}z}{(cn)!}\int_0^1 [(tz)^c e^{(1-t)z}]^n\,dt.$$

These last two identities together with (5.1) give

$$\begin{aligned} S_{cn,n}(nz) &= \frac{n^{n+1}z}{n!}\int_0^1 [tze^{(1-t)z}]^n\,dt - \frac{n^{cn+1}z}{(cn)!}\int_0^1 [(tz)^c e^{(1-t)z}]^n\,dt \\ &=: U_n(z) - V_n(z). \end{aligned} \tag{5.2}$$

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If $z \in A := \{z : \operatorname{Im} z = 0, \varepsilon \leq z \leq 1 - \varepsilon\}$ for some $0 < \varepsilon < 1/3$, then it is easy to see that

$$\{U_n(z)\}^{1/n} \longrightarrow ez \quad \text{as } n \longrightarrow \infty;$$

(see [9, Vol. I, p. 96, # 198] and note that $\{n^{n+1}/n!\}^{1/n} \longrightarrow e$ as $n \longrightarrow \infty$). Similarly, for $z \in B := \{z : \operatorname{Im} z = 0, 1 + \varepsilon \leq z \leq 4\}$ we have

$$\{U_n(z)\}^{1/n} \longrightarrow e^z \quad \text{as } n \longrightarrow \infty.$$

Now, since $\left\{\left(U_n(z)\right)^{1/n}\right\}_{n=1}^{\infty}$ is a sequence of bounded analytic functions on D_{12}, it will converge uniformly on compact subsets of D_{12} to ez. For, if $\{U_n(z)\}^{1/n}$ were to fail to converge uniformly on a compact subset $K \subset D_{12}$, there would be two subsequences of $\{U_n(z)\}^{1/n}$ converging to distinct limit functions f and g. But f and g must agree on A, by the uniqueness theorem for analytic functions, and we would have a contradiction. Using the same argument, we prove that $\{U_n(z)\}^{1/n} \longrightarrow e^z$ as $n \longrightarrow \infty$, uniformly on any compact subset of D_3.

Similarly, we have

$$\{V_n(z)\}^{1/n} \longrightarrow \left(\frac{e}{c}\right)^c z^c \quad \text{as } n \longrightarrow \infty$$

uniformly on any compact subset of E_{12}, and

$$\{V_n(z)\}^{1/n} \longrightarrow e^z \quad \text{as } n \longrightarrow \infty$$

uniformly on any compact subset of E_3.

Let K be any compact subset of $\{z : |z| < c^{c/(c-1)}/e\} \cap D_{12}$; then

$$\{U_n(z)\}^{1/n} \longrightarrow ez, \quad \{V_n(z)\}^{1/n} \longrightarrow \left(\frac{e}{c}\right)^c z^c$$

as $n \longrightarrow \infty$, uniformly on K, and

$$\left|\frac{V_n(z)}{U_n(z)}\right|^{1/n} \longrightarrow \frac{e^{c-1}}{c^c}|z|^{c-1} < 1.$$

Therefore, with (5.2) we have

$$\{S_{cn,n}(nz)\}^{1/n} = \{U_n(z)\}^{1/n}\left\{1 - \frac{V_n(z)}{U_n(z)}\right\}^{1/n} \longrightarrow ez \tag{5.3}$$

as $n \longrightarrow \infty$, uniformly on K.

Now if K is a compact subset of $D_3 \cap E_1$ then we have

$$\{U_n(z)\}^{1/n} \longrightarrow e^z \quad \text{and} \quad \{V_n(z)\}^{1/n} \longrightarrow \left(\frac{e}{c}\right)^c z^c$$

as $n \longrightarrow \infty$, uniformly on K. Furthermore, since $|(z/c)e^{1-z/c}| < 1$,

$$\left(\frac{e}{c}\right)^c |z|^c < |e^z|.$$

Thus,

$$\left|\frac{V_n(z)}{U_n(z)}\right|^{1/n} \longrightarrow \frac{(e/c)^c|z|^c}{|e^z|} < 1,$$

and, again with (5.2),

$$\{S_{cn,n}(nz)\}^{1/n} = \{U_n(z)\}^{1/n}\left\{1 - \frac{V_n(z)}{U_n(z)}\right\}^{1/n} \longrightarrow e^z \tag{5.4}$$

as $n \longrightarrow \infty$, uniformly on K.

If K is any compact subset of $D_3 \cap E_2$ then we have again

$$\{U_n(z)\}^{1/n} \longrightarrow e^z, \quad \{V_n(z)\}^{1/n} \longrightarrow \left(\frac{e}{c}\right)^c z^c$$

as $n \longrightarrow \infty$, uniformly on K. Since $|(z/c)e^{1-z/c}| > 1$, we have

$$\left(\frac{e}{c}\right)^c |z|^c > |e^z|,$$

and therefore

$$\{S_{cn,n}(nz)\}^{1/n} = \{V_n(z)\}^{1/n}\left\{1 - \frac{U_n(z)}{V_n(z)}\right\}^{1/n} \longrightarrow \left(\frac{e}{c}\right)^c z^c, \tag{5.5}$$

as $n \longrightarrow \infty$, uniformly on K.

Finally, if K is a compact subset of $\{z : |z| > c^{c/(c-1)}/e\} \cap D_{12}$, then

$$\{U_n(z)\}^{1/n} \longrightarrow ez, \quad \{V_n(z)\}^{1/n} \longrightarrow \left(\frac{e}{c}\right)^c z^c$$

as $n \longrightarrow \infty$, uniformly on K. Since

$$\left|\frac{U_n(z)}{V_n(z)}\right|^{1/n} \longrightarrow \frac{c^c}{e^{c-1}|z|^{c-1}} > 1,$$

we have

$$\{S_{cn,n}(nz)\}^{1/n} = \{V_n(z)\}^{1/n}\left\{1 - \frac{U_n(z)}{V_n(z)}\right\}^{1/n} \longrightarrow \left(\frac{e}{c}\right)^c z^c \tag{5.6}$$

as $n \longrightarrow \infty$, uniformly on K. Therefore, by (5.3)–(5.6), all the limit points of the zeros of the sequence $\{S_{cn,n}(nz)\}_{n=1}^{\infty}$ must belong to F.

On the other hand, if we have a $\hat{z} \in F$ which is not a limit point of zeros of the sequence $\{S_{cn,n}(nz)\}_{n=1}^{\infty}$ then there is an open disk $D(\hat{z}, r)$ centered at $\hat{z}$ with radius $r > 0$ such that $D(\hat{z}, r)$ is free of zeros of any $S_{cn,n}(nz)$. Suppose $\hat{z} \in F_1$; then by (5.3) $\{S_{cn,n}(nz)\}^{1/n}$ will converges to ez uniformly on any compact subset of $D(\hat{z}, r) \cup \{D(0, c^{c/(c-1)}/e) \cap D_{12}\}$. But on the other hand, by (5.6) $\{S_{cn,n}(nz)\}^{1/n}$ converges to $(e/c)^c z^c$ uniformly on any compact subset of $D(\hat{z}, r) \cup \left\{\{z : |z| > c^{c/(c-1)}/e)\} \cap D_{12}\right\}$. Thus, ez and $(e/c)^c z^c$ must agree on $D(\hat{z}, r)$, a contradiction.

This completes the proof of Theorem 4. ∎

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6. **Some general properties of iterated integrals.** The main emphasis of this paper is on n-th integrals of n-th degree polynomials, with zero constants of integration. In this section we will make a few easy remarks on some more general questions.

1. It is clear that the constant of integration plays an essential role in the distribution of the zeros of the integral of a polynomial. If this constant is left to be arbitrary, no reasonable result on the sizes of the zeros will be possible: On the one hand, the zeros can clearly be made arbitrarily large; on the other hand, they may not grow at all upon repeated integration. (This can be illustrated by the example $p(z) = (z+1)^n$; the polynomial becomes $\big(n!/(n+m)!\big)(z+1)^{n+m}$ upon integrating m times, with appropriate constants.)

In this last example the constant in the k-th integration is $n!/(n+k)!$; hence the sequence of constants is rapidly decreasing with k. We will now show that this kind of decrease is necessary for the zeros of the iterated integrals to remain bounded.

PROPOSITION 6.1. *Let $p(z) = a_n z^n + \cdots + a_1 z + a_0$ be a polynomial of degree n. Suppose that there are constants $\varepsilon > 0$ and $\delta > 0$ and an infinite sequence of positive integers $k_1 < k_2 < \cdots$ such that the constant of integration c_j after the k_j th integration satisfies*

$$|c_j| \geq \frac{\delta}{[(n+k_j)!]^{1-\varepsilon}}. \tag{6.1}$$

Then the modulus of the largest zero of the k-th iterated integral is unbounded as k grows.

PROOF. We will show in fact that at least one zero of the k_j-th iterated integral satisfies

$$|z| > \left(\frac{\delta}{|a_n|n!}\right)^{1/(n+k_j)} \left(\frac{n+k_j}{e}\right)^{\varepsilon}. \tag{6.2}$$

To do this, we note that the k_j-th iterated integral of $p(z)$ is

$$P(z) := a_n \frac{n!}{(n+k_j)!} z^{n+k_j} + a_{n-1} \frac{(n-1)!}{(n+k_j-1)!} z^{n+k_j-1} + \cdots + a_0 \frac{z^{k_j}}{k_j!} + \cdots + c_j.$$

After normalizing this to a monic polynomial, we see that the product of all zeros of $P(z)$ has modulus

$$\left|\frac{c_j}{a_n}\right| \frac{(n+k_j)!}{n!} \geq \frac{\delta}{|a_n|n!}[(n+k_j)!]^{\varepsilon} > \frac{\delta}{|a_n|n!}\left(\frac{n+k_j}{e}\right)^{(n+k_j)\varepsilon};$$

here we used (6.1) and Stirling's formula. Now the modulus of one of the $n+k_j$ zeros has to be at least the $(n+k_j)$-th root of this expression. Thus we obtain (6.2). ∎

PROPOSITION 6.2. *If the sequence of constants of integration is eventually constant, then the modulus of the largest zeros of the k times iterated integral of a polynomial $p(z)$ is unbounded as $k \longrightarrow \infty$, unless p is a monomial and all constants of integration are zero.*

PROOF. If the sequence of constants of integration stabilizes to a nonzero constant then it obviously satisfies (6.1), and the result follows from Proposition 6.1.

If $p(z)$ has at least two nonzero coefficients, or if there is at least one nonzero constant of integration with the other constants eventually vanishing, we may assume without loss of generality that $p(z)$ is of the form

$$p(z) = a_n z^n + \cdots + a_0, \quad a_0 a_n \neq 0,\ n \geq 1.$$

Then the k-th iterated integral is

$$P(z) = a_n \frac{n!}{(n+k)!} z^{n+k} + \cdots + \frac{a_0}{k!} z^k.$$

Now the product of the n nontrivial zeros of $P(z)$ has modulus

$$\left|\frac{a_0}{a_n}\right| \frac{(n+k)!}{n!k!} = \frac{|a_0|}{|a_n|n!}(k+1)(k+2)\cdots(k+n) \geq \frac{|a_0|}{|a_n|n!}(k+1)^n. \tag{6.3}$$

Hence one zero has modulus of at least the n-th root of this last expression; but this is an unbounded function of k. ∎

The next proposition shows that the zeros move (mostly "outwards") under integration in a balanced way. This follows directly from the well-known fact that the zeros of a polynomial and of its derivative have the same centroid (or center of mass) if we imagine a unit mass attached to each zero, counting multiplicities (see, *e.g.*, [17, p. 7]).

PROPOSITION 6.3. *The zeros of a polynomial and those of each iterated integral with arbitrary constants of integration have the same centroid.*

2. We return to the operator I_m of m-times iterated intergration with zero constants. In particular, we will now examine the behaviour of $I_m\big(p(z)\big)$ as $m \longrightarrow \infty$, for a fixed polynomial $p(z)$. It is clear from (6.3) that at least one zero of $I_m\big(p(z)\big)$ has order of magnitude m. This suggests to normalize the polynomials $I_m\big(p(z)\big)$ by dividing their zeros by m. Let the polynomial $p(z) := a_n z^n + \cdots + a_1 z + a_0$ be given.

To simplify notation, we introduce

$$\tilde{I}_m\big(p(z)\big) := m! z^{-m} I_m(p)\Big|_{mz}. \tag{6.4}$$

PROPOSITION 6.4. *The sequence $\{\tilde{I}_m(p)\}$ converges uniformly on compact subset of $\mathbb{C}$ to the polynomial*

$$Lp(z) := \sum_{j=0}^{n} a_j j! z^j. \tag{6.5}$$

Consequently the nonvanishing zeros of $I_m(p)$, divided by m, converge to the zeros of $Lp(z)$, as $m \longrightarrow \infty$.

PROOF. With (2.2) and (6.4) it is clear that

$$\tilde{I}_m(p) = \sum_{j=0}^{n} a_j \frac{m!j!}{(m+j)!} m^j z^j.$$

Using the fact that for all $j = 0, 1, \ldots, n$ we get

$$\frac{m!m^j}{(m+j)!} = \frac{m \cdots m}{(m+1)\cdots(m+j)} \longrightarrow 1 \quad \text{as } m \longrightarrow \infty,$$

we obtain the first statement of the proposition. The second statement follows from Hurwitz's Theorem (see, *e.g.*, [8, p. 4]). ∎

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COROLLARY. *The nonvanishing zeros of $I_m(p)$ lie in the annulus*

$$c(p)m \leq |z| \leq C(p)m,$$

where $c(p)$ and $C(p)$ are constants depending only on p.

We note that L, as defined in (6.5), is a linear transformation on the vector space of n-th degree polynomials. Operators of this kind have been studied in greater generality; see [10]. We also note that it follows from the results surveyed in [10] that L is a "zero diminishing linear transformation", *i.e.*, $Lp(z)$ cannot have more real zeros than $p(z)$. Here we will use the special structure of L to give a few particular examples.

EXAMPLES. 1. Let $a_j = 1/j!$, *i.e.*, $p(z) = \sum_{j=0}^{n} z^j/j!$ (see Section 1). Then the zeros of $I_m(p)$, divided by m, converge to the zeros of $(z^{n+1} - 1)/(z-1)$, *i.e.*, to the $(n+1)$-th roots of unity with the exception of $z = 1$.

2. For the n-th degree Laguerre polynomial

$$p(z) = L_n(z) = \sum_{j=0}^{n} (-1)^j \binom{n}{n-j} \frac{z^j}{j!}$$

we get

$$Lp(z) = \sum_{j=0}^{n} (-1)^j \binom{n}{n-j} z^j = (1-z)^n.$$

Note that the zeros of $L_n(z)$ are all real and positive (see, *e.g.*, [1, Ch. 22]); they are mapped under L to an n-fold zero at $z = 1$.

3. If $p(z)$ is the n-th Legendre polynomial, shifted by 1,

$$p(z) = P_n(z+1) = \sum_{j=0}^{n} \binom{n}{j} \binom{n+j}{j} \left(\frac{z}{2}\right)^j,$$

then $Lp(z)$ is the n-th Bessel polynomial (see, *e.g.*, [7])

$$Lp(z) = y_n(z) = \sum_{j=0}^{n} \binom{n}{j} \binom{n+j}{j} j! \left(\frac{z}{2}\right)^j.$$

The zeros of $p(z)$ are all real and located in the interval $[-2, 0]$. The zeros of $y_n(z)$ are all simple, have negative real part and lie inside the unit circle; at most one zero is real (see [7, p. 75 ff.]).

7. **Further remarks.** 1. The integration operator I_m can be considered as an operator on the sequence of coefficients of a polynomial or power series. Such operators have been studied in great generality; results on the distribution of zeros have also been obtained (see [11]). However, the results in [11, p. 213] concerning the operator I_m as a special case are considerably weaker than those obtained here.

2. For some special polynomials there is a close relationship between the integration operator I_m and the operation of truncating a polynomial. Indeed, it is clear from the

Rodrigues formula (2.5) for the Legendre polynomials that $I_n\big(P_n(z)\big)$ is just the polynomial $(1-z^2)^n$, with the "lower half" removed (see also the explicit formula (2.6)). The polynomials $S_{n,m}(z)$ in (1.1) can also be obtained both by truncating and by integrating repeatedly. The zero distribution of truncated polynomials is an interesting question in its own right; for some other special cases, see [5].

3. The proof of Theorem 4 can easily be modified to yield analogous results for truncations

$$S_{cn,dn}(z) = \sum_{k=dn+1}^{cn} \frac{z^k}{k!},$$

with integers $d < c$. If we then normalize by the factor dn, rather than n, we get the statement of Theorem 4, with $c > 1$ a rational number.

4. The Legendre polynomials are special cases of the Gegenbauer (or Ultraspherical) polynomials. Other important special cases are the Chebyshev polynomials of the first and second kind. It would be interesting to obtain results, analogous to Theorem 2, for these polynomials. Numerical experiments indicate that we may expect the same limit curve Γ_2 at least for the Chebyshev polynomials $T_n(z)$.

5. Finally we note that the operation of truncating a power series has been generalized in various ways. Already Szegö [14] studied the zeros of sequences of polynomials which converge uniformly in a region to some function. Another generalization are series of functions and their truncations; see [6].

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