ANOTHER REMARK ON A RESULT OF K. GOLDBERG

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In [3] K. Goldberg showed that if A is a 0-1 matrix that satisfies

$$AA^* = sA \tag{1}$$

then for some permutation matrix P, PAP^* is a direct sum of matrices each of which is either zero or consists only of ones. More recently J. L. Brenner [1] proved that if $A \geq 0$ (i.e. A has non-negative entries) and satisfies (1) then there exists a permutation matrix P such that $PAP^* = A_1 \oplus \dots \oplus A_n$ in which each A_i is either 0 or all positive, $A_i > 0$, and satisfies (1) as well.

In this note we exhibit an argument that is somewhat different from those used by the above authors and which yields a generalization of both results. We then specialize sufficiently to obtain Brenner's theorem.

Observe first that if (1) is satisfied for $A \geq 0$ then in fact A is symmetric and (1) becomes $p(A) = 0$ where $p(\lambda) = \lambda(\lambda - s)$. Notice that in this simple case the only root of $p(\lambda)$ of maximum modulus s is s itself. It is this property of $p(\lambda)$ that is significant here.

We recall that a primitive non-negative matrix B is one for which $B^k > 0$ for some positive integer k.

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THEOREM.[1] *Suppose* A *is a non-negative normal matrix satisfying*

$$p(A) = 0 \tag{2}$$

in which $p(\lambda)$ *is a monic polynomial no two of whose non-zero roots have the same modulus. Then there exists a permutation matrix* P *such that* PAP^* *is a direct sum,*

$$PAP^* = A_1 \oplus \ldots \oplus A_m ,$$

in which each A_i *is either* 0 *or primitive.*

Proof. Since A^* is a polynomial in A it follows that if P is unitary and PAP^* is a subdirect sum it must in fact be a direct sum. Now either A is irreducible [2: p. 75] or there exists a permutation matrix P such that

$$PAP^* = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \cdot & \cdot & \cdot & \\ A_{m1} & \cdot & A_{m-1,m} & A_{mm} \end{pmatrix}$$

where each A_{ii} is either 0 or irreducible. By the above remark $A_{ij} = 0$ for $i > j$ and if we set $A_{ii} = A_i$, $i = 1, \ldots, m$, we have

$$PAP^* = A_1 \oplus \ldots \oplus A_m$$

Now $p(A) = 0$ clearly implies that $p(A_i) = 0$, $i = 1, \ldots, m$, and moreover each A_i is normal. The distinct characteristic roots of A_i are then roots of the polynomial $p(\lambda) = 0$ (not counting multiplicities, of course). If $A_i \neq 0$ then it is

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irreducible and has a simple positive root r. Moreover the conditions on $p(\lambda)$ ensure that r is the only root of $p(\lambda)$ of modulus r.

It follows [2: p. 80] that A_i is primitive and the proof is complete.

Now let $p(\lambda) = \lambda^k(\lambda - s)$ where $s > 0$ and k is a positive integer. Then s is the only root of $p(\lambda)$ of modulus s. But we know more: $p(A) = 0$ implies that each non-zero A_i has only s as a simple root and 0 as a possible multiple root. Hence A_i has rank 1 (since it is normal and in fact symmetric) and is thus of the form $A_i = (u_\alpha u_\beta)$. Now, A_i is irreducible so no $u_\alpha = 0$, otherwise A_i would have a zero row and column. Thus no element of $A_i \geq 0$ is 0 and hence $A_i > 0$ and has rank 1.

Brenner's case is $k = 1$.

REFERENCES

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