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COMMUTATIVITY OF (2×2) SELFADJOINT MATRICES

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An elementary proof is given of the fact that an n-tuple $A = (A_1, \ldots, A_n)$ of selfadjoint matrices in a 2-dimensional Hilbert space consists of mutually commuting matrices A_j, $1 \leqslant j \leqslant n$, if and only if $\gamma(A)$ is non-empty. Here $\gamma(A) \subseteq \mathbb{R}^n$ is the joint spectrum of A (in the sense of McIntosh and Pryde) consisting of those points $\beta \in \mathbb{R}^n$ for which the matrix $\sum_{j=1}^{n} (A_j - \beta_j)^2$ is not invertible.

Let $A = (A_1, \ldots, A_n)$ be an n-tuple of bounded linear operators in a Banach space. A notion of joint spectrum $\gamma(A)$ was introduced in [**2, 3**], namely

$$\gamma(A) = \Big\{\beta \in \mathbb{R}^n; 0 \in \sigma\Big(\sum_{j=1}^{n} (A_j - \beta_j I)^2\Big)\Big\}, \tag{1}$$

where I is the identity operator and $\sigma(B)$ denotes the usual spectrum of an operator B. For *pairwise commuting* operators A_j, $j = 1, \ldots, n$, it turns out (for $n \geqslant 2$) that $\gamma(A)$ is always a non-empty, compact subset of $\mathbb{R}^n$, [**2, 6**]. In some recent work of Pryde [**4, 5**] the spectral set $\gamma(A)$ has proved to be useful in the consideration of certain classes of *non-commuting* n-tuples A. In [**1**] a detailed study is made of the sets $\gamma(A)$ for non-commuting, selfadjoint operators A_j, $j = 1, \ldots, n$, in Hilbert spaces. An application of the general results developed there (especially the notion of the maximal joint abelian subspace of A) is the following commutativity criterion; see [**1**, Proposition 7].

PROPOSITION. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of selfadjoint operators in a 2-dimensional Hilbert space. Then $\gamma(A) \neq \emptyset$ if and only if the operators A_j, $j = 1, \ldots, n$, pairwise commute.*

The aim of this short note is to give an elementary proof of this result independent of the more abstract techniques developed in [**1**]. The proof proceeds in two simple stages. Firstly, the result is established for two operators by a direct calculation based only on some very *elementary* properties of the sets $\gamma(A)$. The second step consists of reducing the case of n operataors to that of a pair of operators.

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Step 1. *Let $A = (A_1, A_2)$ be a pair of selfadjoint operators in a 2-dimensional Hilbert space H such that $A_1A_2 \neq A_2A_1$. Then $\gamma(A) = \emptyset$.*

Proof: It is a direct consequence of the definition of joint spectral set given in (1) that

(i) $\gamma(tA) = t\gamma(A)$ for any real number $t > 0$, and

(ii) $\gamma(UAU^{-1}) = \gamma(A)$ whenever $U: H \to H$ is a linear isomorphism.

In fact, properties (i) and (ii) are valid for any n-tuple of bounded linear operators A in an arbitrary Banach space.

So, in (ii) choose for U an orthogonal transformation such that the matrix of UA_1U^{-1}, with respect to the basis of H consisting of the orthonormal eigenvectors of A_1, is diagonal, say $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$. Then the matrix of UA_2U^{-1} with respect to this basis is of the form $\begin{bmatrix} b_1 & w \\ \overline{w} & b_2 \end{bmatrix}$ for some $w \in \mathbb{C}$ and $b_1, b_2 \in \mathbb{R}$. Since $A_1A_2 \neq A_2A_1$ we have $a_1 \neq a_2$ (with $a_1, a_2 \in \mathbb{R}$) and $w \neq 0$. Choosing $t = |w|^{-1}$ in (i) allows us to reduce the proof to a consideration of the special case when $A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ with $a_1 \neq a_2$ real numbers and $A_2 = \begin{bmatrix} b_1 & w \\ \overline{w} & b_2 \end{bmatrix}$ with $b_1, b_2 \in \mathbb{R}$ and w a *unimodular* complex number. But then, for any $\beta \in \mathbb{R}^2$, a direct calculation (using $|w| = 1$) shows that $\det[(A_1 - \beta_1 I)^2 + (A_2 - \beta_2 I)^2]$ is equal to

$$(\beta_1 - a_1)^2[(\beta_1 - a_2)^2 + (\beta_2 - b_2)^2] + (\beta_2 - b_1)^2(\beta_1 - a_2)^2$$
$$+[1 - (\beta_2 - b_1)(\beta_2 - b_2)]^2 + \sum_{j=1}^{2}(\beta_1 - a_j)^2.$$

It is easy to check that this expression cannot equal zero, and hence it must be strictly positive. Since $\beta \in \mathbb{R}^2$ was arbitrary it follows that $\gamma(A) = \emptyset$. ∎

Step 2. *Suppose there exist indices $j < k$ in $\{1, 2, \dots, n\}$ such that $A_jA_k \neq A_kA_j$. Then $\gamma(A) = \emptyset$.*

Proof: Let $\beta \in \mathbb{R}^n$ be arbitrary. Since $\gamma((A_j, A_k)) = \emptyset$ (by Step 1) the operator $T = (A_j - \beta_j I)^2 + (A_k - \beta_k I)^2$ is invertible. Of course, T is also selfadjoint and positive (in the usual sense, that is, $\langle Th, h\rangle \geqslant 0$ for each $h \in H$). Moreover,

$$S = \sum_{r \neq j,k} (A_r - \beta_r I)^2$$

is also positive and selfadjoint. But, whenever V is any invertible, positive, selfadjoint operator in any Hilbert space H (in which case $\sigma(V) \subseteq (0, \infty)$) and W is any positive, selfadjoint operator in H, then $V + W$ is necessarily invertible. Accordingly,

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$\sum_{r=1}^{n}(A_r - \beta_r I)^2 = S + T$ is invertible. Since $\beta \in \mathbb{R}^n$ was arbitrary we conclude that $\gamma(A) = \emptyset$. ∎

Proof of Proposition: If the operators A_j, $1 \leqslant j \leqslant n$, pairwise commute, then we have already noted that $\gamma(A) \neq \emptyset$; see [**2**, **6**]. Conversely, if the operators A_j, $1 \leqslant j \leqslant n$, do not pairwise commute, then Step 2 shows that $\gamma(A) = \emptyset$. ∎

Remark. The above Proposition is particular to 2-dimensional Hilbert spaces. For, let B_1, B_2 be selfadjoint matrices in $\mathbb{C}^2$ such that $B_1 B_2 \neq B_2 B_1$. Let $u = (u_1, u_2) \in \mathbb{R}^2$, with both u_1 and u_2 non-zero, and define selfadjoint operators in $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$ by $A_j = B_j \oplus u_j I$, $j \in \{1, 2\}$. Then $A_1 A_2 \neq A_2 A_1$. However, since the non-zero vector $h = (0, 0, 1)$ satisfies $A_j h = u_j h$ for $j \in \{1, 2\}$, it follows that $u \in \gamma(A)$.

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