UNBOUNDED POSITIVE DEFINITE FUNCTIONS

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1. Introduction. Let G be an abelian group, written additively. A complex-valued function f, defined on G, is said to be *positive definite* if the inequality

$$\sum_{i,j=1}^{n} f(s_i - s_j)c_i\bar{c}_j \geqq 0 \tag{1}$$

holds for every choice of complex numbers $c_1, \ldots, c_n$ and $s_1, \ldots, s_n$ in G. It follows directly from (1) that every positive definite function is bounded. Weil (**9**, p. 122) and Raĭkov (**5**) proved that every continuous positive definite function on a locally compact abelian group is the Fourier-Stieltjes transform of a bounded positive measure, thus generalizing theorems of Herglotz (**4**) ($G = Z$, the integers) and Bochner (**1**) ($G = R$, the real numbers).

If f is a continuous function, then condition (1) is equivalent to the condition that

$$\int_G\int_G f(s-t)\varphi(s)\overline{\varphi(t)}\, dsdt \geqq 0 \tag{2}$$

for every $\varphi \in C_c(G)$, the continuous functions with compact support on G, where ds denotes integration with respect to Haar measure. Any essentially bounded function f satisfying (2) is equal to a continuous positive definite function locally almost everywhere. However, if f is not assumed to be continuous or bounded, but merely locally summable, then condition (2) gives rise to a much larger class of functions.

For the case $G = R$, Cooper (**2**) called f *positive definite for* F, where F is a set of functions on R, if the integral in (2) exists and is non-negative for every $\varphi \in F$. His principal result is that every function which is positive definite for C_c is the Fourier-Stieltjes transform of a positive measure, possibly unbounded, in the sense of Cesàro summability almost everywhere.

Our aim in this paper is to study the functions which are positive definite for various function classes on a locally compact abelian group G. For the case where f is in some $L^p(G)$, $1 \leqq p \leqq \infty$, Weil (**9**) has given a representation for f as the Fourier-Stieltjes transform in a summability sense of a positive measure μ, possibly unbounded, on the dual group $\hat{G}$. In § 3 his result is described and restrictions are placed on the measure μ. A similar representation theorem for f when it is not in any L^p class was proved in (**8**, p. 77, Theorem 15) for the case where G is compactly generated. In § 4 we remove the restriction that G be compactly generated.

Received May 6, 1968.

https://doi.org/10.4153/CJM-1969-143-4 Published online by Cambridge University Press

Throughout this paper G will denote a locally compact abelian group and $\hat{G}$ its dual group. G is topologically isomorphic to $R^a \times G_1$, where a is a uniquely determined integer and G_1 contains a compact open subgroup H. We shall assume that the Haar measure on G has been normalized so that H has measure 1.

If $s \in G$ and $\hat{s} \in \hat{G}$, we shall write $[s, \hat{s}]$ for the value of the character $\hat{s}$ at the point s, and $[\hat{s}, s] = [-s, \hat{s}]$. The Fourier transform of a function φ will be denoted by $\hat{\varphi}$ or Φ, and $\tilde{\varphi}$ is the function defined by $\tilde{\varphi}(s) = \overline{\varphi(-s)}$. Theorems invoked without a reference can be found in (**6**).

2. The class $P(F)$**.** Let F be a set of complex-valued functions on G. A complex-valued function f on G is called *positive definite for* F if the integral

$$\iint f(s-t)\varphi(s)\overline{\varphi(t)}\, dsdt$$

exists as a Lebesgue integral over the product set $G \times G$ and is non-negative for every $\varphi \in F$. The class of all functions which are positive definite for F will be denoted by $P(F)$.

The theorems stated in this section are for the most part straightforward generalizations of theorems given in (**2**). They are proved in (**8**, Chapter II, § 1, Theorems 17–19).

THEOREM 2.1. *Let $f \in P(F)$, where F has the property that for any compact set C of G, F contains a function with compact support which is strictly positive on C. Then f is locally summable and $f = \tilde{f}$ locally almost everywhere.*

Information as to how the class $P(F)$ varies with F is provided by the following two theorems and the obvious fact that $F_1 \subset F_2$ implies that

$$P(F_1) \supset P(F_2).$$

THEOREM 2.2. *$P(C_c) = P(L_c^p)$ for every $p \geqq 2$, where L_c^p is the set of functions in L^p with compact support.*

THEOREM 2.3. *Let $1 \leqq p \leqq 2$ and $q = p/2(p-1)$. If $f \in P(L_c^2)$ and f is locally in L^q, then $f \in P(L_c^p)$.*

The converse of Theorem 2.3, which may be true in general, can be proved for $p = 1$. In fact, for this case a strengthened version of the converse is given by Theorem 2.4 which follows from two applications of the uniform boundedness theorem.

THEOREM 2.4. *If $f \in P(L_c^1)$, then there is a constant A such that $|f(s)| \leqq A$ locally almost everywhere.*

It follows from this theorem that $P(L_c^1) = P(L^1)$ and that any function in $P(L_c^1)$ is equal to a continuous positive definite function locally almost everywhere.

https://doi.org/10.4153/CJM-1969-143-4 Published online by Cambridge University Press

3. The case $f \in L^p(G)$. If U is a compact neighbourhood of 0 in G, let φ_U be a positive function with support in U, $\int \varphi_U(s)\,ds = 1$, $\Phi_U = \hat{}_U \geqq 0$, and $\Phi_U \in L^1(\hat{G})$. Weil (**9**) has shown that for every $f \in L^p \cap P(C_c)$, $1 \leqq p < \infty$, there is a positive measure μ on $\hat{G}$ such that

$$(3) \qquad f(s) = \lim_{U \to 0} \int_{\hat{G}} \Phi_U(\hat{s})[s, \hat{s}]\,d\mu(\hat{s}),$$

where the limit is taken in $L^p(G)$. This result remains true if the summability function Φ_U is replaced by a function Φ_α with the following properties: $\Phi_\alpha \geqq 0$, $\Phi_\alpha \in L^1(\hat{G})$, $\Phi_\alpha = \hat{\varphi}_\alpha$, where $\varphi_\alpha \in L^1(G)$ and φ_α is an approximate identity in $L^p(G)$, i.e., $||g * \varphi_\alpha - g||_p \to 0$ for every $g \in L^p(G)$. In particular, Simon's generalization of Cesàro summability (**7**) could be used here.

The summability function Φ_U used by Weil and the generalized Cesàro summability function both have the property that $\Phi_U(\hat{s}) \to 1$ as $U \to 0$ uniformly on compact sets in $\hat{G}$. This is enough to ensure the uniqueness of the measure μ.

THEOREM 3.1. *Let μ_1 and μ_2 be positive regular Borel measures on $\hat{G}$ such that for any directed set of functions Φ_α such that $\Phi_\alpha(\hat{s}) \to 1$ uniformly on compact sets,*

$$\lim \int \Phi_\alpha(\hat{s})[s, \hat{s}]\,d\mu_1(\hat{s}) = \lim \int \Phi_\alpha(\hat{s})[s, \hat{s}]\,d\mu_2(\hat{s}),$$

where both limits are in the sense of $L^p(G)$, $1 \leqq p < \infty$. Then $\mu_1 = \mu_2$.

Proof. Let g be any function in $L^1(G)$ such that $\hat{g}$ has compact support. Then, using the Fubini theorem and the fact that $g \in L^{p'}(G)$, where $1/p + 1/p' = 1$, we have:

$$\begin{aligned} \lim \int \hat{g}(\hat{s})\Phi_\alpha(\hat{s})\,d\mu_1(\hat{s}) &= \lim \int g(s)\{\int \Phi_\alpha(\hat{s})[\hat{s}, s]\,d\mu_1(\hat{s})\}\,ds \\ &= \lim \int g(s)\{\int \Phi_\alpha(\hat{s})[\hat{s}, s]\,d\mu_2(\hat{s})\}\,ds \\ &= \lim \int \hat{g}(\hat{s})\Phi_\alpha(\hat{s})\,d\mu_2(\hat{s}). \end{aligned}$$

Since $\Phi_\alpha \to 1$ uniformly on the support of $\hat{g}$ we obtain, by taking the limit,

$$\int \hat{g}(\hat{s})\,d\mu_1(\hat{s}) = \int \hat{g}(\hat{s})\,d\mu_2(\hat{s}).$$

The set of such functions $\hat{g}$ is dense in $C_c(\hat{G})$, and therefore $\mu_1 = \mu_2$.

Not every positive measure μ on $\hat{G}$ gives rise to a function $f \in P(L_c^2)$ in the manner of (3) since the limit need not exist. Necessary and sufficient conditions for μ to generate $f \in L^p \cap P(L_c^2)$ are not known, but the following theorem does impose restrictions on such a measure.

THEOREM 3.2. *Suppose that $\Phi_\alpha(\hat{s})$ converges to 1 uniformly on compact sets in $\hat{G}$ and that $\int \Phi_\alpha(\hat{s})[s, \hat{s}]\,d\mu(\hat{s})$ converges in $L^p(G)$ to $f(s)$, where μ is a positive measure on $\hat{G}$. Let C be a compact set in $\hat{G}$, and let $1/p + 1/p' = 1$. Then*

(i) *$\mu(\hat{s} + C) \to 0$ as $\hat{s} \to \infty$,*

(ii) *$\mu(\hat{s} + C) \in L^{p'}(\hat{G})$ if $1 \leqq p \leqq 2$ and $\mu(\hat{s} + C) \in L^2(\hat{G})$ if $p \geqq 2$.*

Proof. Let φ be any function in $L^1(G)$ such that $\Phi = \hat{\varphi}$ has compact support. Then $\varphi \in L^{p'}(G)$ and hence

$$\begin{aligned}\int \varphi(s)\overline{f(s)}ds &= \lim \int \varphi(s)\{\int \overline{\Phi_\alpha(\hat{s})}[\hat{s}, s]\, d\mu(\hat{s})\}\, ds \\ &= \lim \int \overline{\Phi_\alpha(\hat{s})}\Phi(\hat{s})\, d\mu(\hat{s}) \\ &= \int \Phi(\hat{s})\, d\mu(\hat{s}).\end{aligned}$$

In particular, choose $\varphi \in L^1(G)$ so that $\Phi(\hat{s}) = 1$ for $\hat{s} \in C$, $0 \leqq \Phi(\hat{s}) \leqq 1$, and Φ has compact support. Then

$$0 \leqq \mu(\hat{s} + C) \leqq \int \Phi(\hat{s} + \hat{t})\, d\mu(\hat{t}) = \int \overline{f(s)}\varphi(s)[\hat{s}, s]\, ds.$$

The right-hand side is the Fourier transform of a function in $L^1(G)$ and thus converges to 0 as $\hat{s} \to \infty$. Hence $\mu(\hat{s} + C) \to 0$. The function $\bar{f}\varphi$ is also in L^p since $\varphi \in L^\infty$, so that its Fourier transform is in $L^{p'}$ if $1 \leqq p \leqq 2$ and in L^2 if $p \geqq 2$.

If $G = R$, then the condition $\mu(\hat{s} + C) \to 0$ as $\hat{s} \to \infty$ is equivalent to the condition $\rho(u) = o(u)$ given in (**2**), where $\rho(u)$ is an increasing function such that

$$f(x) = \int_{-\infty}^{\infty} e^{iux}\, d\rho(u) \quad \text{a.e. } (C, 1).$$

4. The main representation theorem (Theorem 4.2). Let μ be a positive measure on a space X and $X = \bigcup\{X_\alpha; \alpha \in I\}$ a given decomposition of X with $\mu(X_\alpha) < \infty$. Define

$$||\psi||_{2,\alpha} = [\int_{X_\alpha} |\psi(s)|^2\, d\mu(s)]^{1/2}$$

and let $S^2(X)$ be the set of functions ψ on X for which

$$|||\psi|||_2 = \sum_{\alpha \in I} ||\psi||_{2,\alpha} < \infty.$$

If $\psi \in S^2(X)$, then the set $\{s \in X; \psi(s) \neq 0\}$ is σ-finite, and in fact

$$S^2(X) \subset L^2(X).$$

If $S^2(Y)$ has been similarly defined by means of the measure ν and the decomposition $Y = \bigcup\{Y_\beta; \beta \in J\}$, we define $S^2(X \times Y)$ in terms of the product measure $\mu \times \nu$ and the decomposition $X \times Y = \bigcup\{X_\alpha \times Y_\beta; \alpha \in I, \beta \in J\}$. If $\psi \in S^2(X)$ and $\lambda \in S^2(Y)$, then the function $\varphi(s, t) = \psi(s)\lambda(t)$ is in $S^2(X \times Y)$ and in fact $|||\varphi|||_2 = |||\psi|||_2|||\lambda|||_2$.

By the structure theorem for locally compact abelian groups, G is of the form $R^a \times G_1$, where G_1 contains an open compact subgroup H. Define $S^2(G_1)$ by means of Haar measure and the coset decomposition of G_1 with respect to H. If $S^2(R)$ is defined from the equation $R = \bigcup_{-\infty}^{\infty} [n, n + 1]$, then $S^2(G)$ is defined by the product decomposition.

Given $\varphi, \psi \in S^2(G)$, the union of their supports is a σ-compact set which we can write as $\bigcup_1^\infty K_n$, where $K_n = g_n + K$ and $K = [0, 1]^a \times H$. The proof of the inequality

$$(4) \qquad |\int_{K_m}\int_{K_n} f(s-t)\varphi(s)\overline{\psi(t)}\, dsdt| \leqq M||\varphi||_{2,n}||\psi||_{2,m},$$

where

$$M = \int_{K-K}|f(s)|\, ds,^{(\dagger)} \quad ||\varphi||_{2,n} = [\int_{K_n}|\varphi(s)|^2 ds]^{1/2},$$

for $f \in P(L_c^2)$ is similar to the proof of (4.7) in (**2**). It follows from (4) that, whenever $f \in P(L_c^2)$ and $\varphi, \psi \in S^2(G)$, the integral $\int\int f(s-t)\varphi(s)\overline{\psi(t)}\, dsdt$ exists, not necessarily as a Lebesgue integral, but as the limit of the integral

$$\int_{G_m}\int_{G_n} f(s-t)\varphi(s)\overline{\psi(t)}\, dsdt$$

as $m, n \to \infty$, where $G_n = \bigcup_{i=1}^n K_i$, i.e., as the sum of the absolutely convergent series

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \int_{K_m}\int_{K_n} f(s-t)\varphi(s)\overline{\psi(t)}\, dsdt.$$

It is this interpretation of the integral that is meant in Theorem 4.1.

LEMMA 1. *Given a compact set C in $\hat{G}$, there is a function $\psi \in S^2(G) \cap L^1(G)$ such that $\Psi = \hat{\psi}$ has compact support and is equal to* 1 *on C.*

Proof. We have $C \subset C_1 \times C_2$, where C_1 is compact in R^a and C_2 is compact in G_1. Let A be the annihilator group of H, $A = \{\hat{s} \in \hat{G}_1;\ [s, \hat{s}] = 1$ for every $s \in H\}$. Since H is open, G_1/H is discrete, and thus A, which is the dual group of G_1/H, is compact. Since H is compact, its dual group $\hat{G}_1/A$ is discrete, and therefore A is open. C_2 is thus covered by a finite number of cosets of A: $\hat{s}_1 + A, \ldots, \hat{s}_k + A$. If χ_H denotes the characteristic function of H, let

$$\psi_2(s) = \chi_H(s)\sum_{j=1}^k [s, \hat{s}_j].$$

Since $\hat{\chi}_H = \chi_A$, we have

$$\Psi_2(\hat{s}) = \sum_{j=1}^k \chi_A(\hat{s} - \hat{s}_j),$$

and thus Ψ_2 has compact support and is equal to 1 on C_2. Since the support of ψ_2 is H, it is in $S^2(G_1)$. Let Ψ_1 on R^a be an a-fold product of twice differentiable functions of a real variable which is equal to 1 on C_1 and 0 outside a larger compact set. Then ψ_1 is an a-fold product of functions which are $O(x^{-2})$ as $x \to \pm\infty$ and are therefore in $S^2(R)$. Hence $\psi_1 \in S^2(R^a)$. Define ψ on $G = R^a \times G_1$ by $\psi(s, t) = \psi_1(s)\psi_2(t)$. Then ψ and Ψ have the required properties.

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†As usual, for any subset A of an abelian group, $A - A = \{a - b;\ a, b \in A\}$.

LEMMA 2. *There is a net $\{\alpha_U\}$ of continuous functions with compact support on G such that $|\alpha_U(s)| \leqq 1$, $\alpha_U(s) \to 1$ uniformly on any compact set, and*

$$\alpha_U = \beta_U * \tilde{\beta}_U.$$

Proof. Simon (**7**) constructed a net $\{\alpha_U\} \subset C_c(G)$ such that $|\alpha_U(s)| \leqq 1$, $\alpha_U(s) \to 1$ uniformly on compacts, $\hat{\alpha}_U \geqq 0$, and $\hat{\alpha}_U \in L^1(\hat{G})$. Write $\hat{\alpha}_U = \gamma_U{}^2$, where $\gamma_U \in L^2(\hat{G})$. If $\hat{\beta}_U = \gamma_U$, then $\beta_U * \beta_U = \alpha_U$ and $\tilde{\beta}_U = \beta_U$.

LEMMA 3. *Let X be a linear space of complex-valued functions which is closed under complex conjugation and let Y be a subspace of X. Suppose that for each $\varphi \in X$ there exists $\psi \in Y$ such that $|\varphi(s)| \leqq \psi(s)$. Then any positive linear functional on Y can be extended to a positive linear functional on X.*

For the case where X is a space of real-valued functions, see (**3**, p. 219, Theorem 3) and since X is assumed to be closed under complex conjugation, we can extend the functional from the real-valued functions in X to all of X.

THEOREM 4.1. *If $f \in P(L_c{}^2)$, there is a positive measure μ such that*

$$\iint \overline{f(s-t)}\varphi(s)\overline{\psi(t)}\, dsdt = \int \Phi(\hat{s})\overline{\Psi(\hat{s})}\, d\mu(\hat{s})$$

whenever $\varphi, \psi \in S^2(G)$ and $\Phi, \Psi \in C_c(\hat{G})$.

Proof. Define $P_U(\hat{s}) = \int \alpha_U(s) f(s)[\hat{s}, s]\, ds$, where α_U has the properties listed in Lemma 2. Then

$$P_U(\hat{s}) = \int \beta_U * \tilde{\beta}_U(s) f(s)[\hat{s}, s]\, ds = \iint f(s-t)\beta_U(s)[\hat{s}, s]\overline{\beta_U(t)[\hat{s}, t]}\, dsdt \geqq 0.$$

Since the Fourier transforms of the functions $\alpha_U f$ and $\varphi * \tilde{\psi}$ are P_U and $\Phi\overline{\Psi}$, respectively, the Parseval formula yields:

$$\int P_U(\hat{s})\Phi(\hat{s})\overline{\Psi(\hat{s})}\, d\hat{s} = \iint \overline{\alpha_U(s-t) f(s-t)}\varphi(s)\overline{\psi(t)}\, dsdt$$

and the properties of α_U ensure that the integral on the right-hand side converges to $\iint \overline{f(s-t)}\varphi(s)\overline{\psi(t)}\, dsdt$. Now define

$$T(\Phi) = \lim \int P_U(\hat{s})\Phi(\hat{s})\, d\hat{s}$$

and let Y be the set of functions $\Phi \in C_c(\hat{G})$ for which the limit exists. T is a positive functional on Y since $P_U(\hat{s}) \geqq 0$. Y contains those functions $\Phi \in C_c(\hat{G})$ whose inverse transform φ is in S^2. For by Lemma 1 there exists $\Psi \in C_c(\hat{G})$ such that Ψ is equal to 1 on the support of Φ and $\psi \in S^2$, and thus

$$T(\Phi) = \lim \int P_U(\hat{s})\Phi(\hat{s})\overline{\Psi(\hat{s})}\, d\hat{s}$$

exists by the arguments above.

We can extend T to be a positive functional on $C_c(\hat{G})$ by virtue of Lemma 3 since any function in $C_c(\hat{G})$ can be majorized by a suitable constant multiple of the function constructed in Lemma 1. Hence there is a positive measure μ on $\hat{G}$ such that

$$T(\Phi) = \int \Phi(\hat{s})\, d\mu(\hat{s}) \qquad (\Phi \in Y).$$

In particular, if $\Phi, \Psi \in C_c(\hat{G})$ and $\varphi, \psi \in S^2(G)$, we have:

$$\iint \overline{f(s-t)}\varphi(s)\overline{\psi(t)}\, dsdt = \int \Phi(\hat{s})\overline{\Psi(\hat{s})}\, d\mu(\hat{s}).$$

If U is a neighbourhood of 0 in G, let V be a compact neighbourhood of 0 such that $V - V \subset U$ and $V = V_1 \times V_2$, where $V_1 \subset R^a$, $V_2 \subset G_1$. For $i = 1, 2$, let ψ_i be a positive L^2 function having support in V_i and integral equal to 1. Set $\psi_U(s, t) = \psi_1(s)\psi_2(t)$ and $\varphi_U = \psi_U * \tilde{\psi}_U$. Then φ_U is a positive function with support in U and $\int \varphi_U(s)\, ds = 1$. $\Phi_U = \hat{\varphi}_U$ will be the summability function for the integral representation of f.

THEOREM 4.2. *If $f \in P(L_c^2)$, there is a positive measure μ on $\hat{G}$ such that*

$$f(s) = \lim_{U\to 0} \int_{\hat{G}} [s, \hat{s}]\Phi_U(\hat{s})\, d\mu(\hat{s}),$$

where the limit exists uniformly on any compact set on which f is continuous, and exists in L^1 over any compact subset of G.

Proof. We know from Theorem 4.1 that

$$\iint \overline{f(s-t)}\varphi(s)\overline{\varphi(t)}\, dsdt = \int |\Phi(\hat{s})|^2\, d\mu(\hat{s}) \tag{5}$$

holds whenever $\varphi \in S^2$ and $\Phi \in C_c$. We show first that (5) continues to hold if

$$\varphi \in J = \{\varphi \in S^2(G);\ \varphi(s, t) = \psi(s)\lambda(t),\ \lambda \in L_c^2(G_1),\ \Lambda \in L^1(\hat{G}_1),$$
$$\psi \in L^1(R^a),\ \Psi \in L^1(R^a),\ \partial^{2a}\Psi/\partial x_1^2 \ldots \partial x_a^2 \in C \cap L^1(R^a)\}.$$

Since $\Lambda \in L^1(\hat{G}_1)$, its support is σ-compact and thus contained in the union of a countable number of cosets of the annihilator A: $\hat{s}_1 + A$, $\hat{s}_2 + A, \ldots$. The Fourier transform of the function

$$\zeta_n(s) = \chi_H(s) \sum_{j=1}^{n} [s, \hat{s}_j]$$

is the characteristic function of the set $\bigcup_{j=1}^{n} (\hat{s}_j + A)$, so that if $\lambda_n = \lambda * \zeta_n$, then $|\Lambda_n(\hat{s})|$ increases to $|\Lambda(\hat{s})|$ and $||\Lambda - \Lambda_n||_1 \to 0$ as $n \to \infty$. Since the support of λ is compact and the support of each ζ_n is H, each λ_n has support in a common compact set which can be covered by a finite number, say r, of cosets of H. Thus for every n we have $|||\lambda_n|||_2 \leqq r||\lambda_n||_2 \leqq r||\lambda||_2$.

Since $\partial^{2a}\Psi/\partial x_1^2 \ldots \partial x_a^2 \in L^1(R^a)$, we can find Ψ_n with compact support such that $||\Psi - \Psi_n||_1 \to 0$, $|\Psi_n|$ increases to $|\Psi|$, and the functions

$$\Omega_n = \partial^{2a}\Psi_n/\partial x_1^2 \ldots \partial x_a^2$$

have uniformly bounded L^1 norms: $||\Omega_n||_1 \leqq B$. Thus we have

$$|\psi_n(s_1, \ldots, s_a)| \leqq B|s_1|^{-2} \ldots |s_a|^{-2}$$

for every n. This inequality shows not only that $\psi_n \in S^2(R^a)$ but also that the sequence $\{|||\psi_n|||_2\}$ is bounded and the convergence of the series $\sum_{m=1}^{\infty} ||\psi_n||_{2,m}$ is uniform in n.

Set $\varphi_n(s, t) = \psi_n(s)\lambda_n(t)$ so that $\Phi_n(\hat{s}, \hat{t}) = \Psi_n(\hat{s})\Lambda_n(\hat{t})$ is in $C_c(\hat{G})$, $||\Phi - \Phi_n||_1 \to 0$, and $|\Phi_n|$ increases to $|\Phi|$. Since $|||\varphi_n|||_2 = |||\psi_n|||_2|||\lambda_n|||_2$, the sequence $\{|||\varphi_n|||_2\}$ is bounded. The fact that the convergence of the series $\sum_{m=1}^{\infty} ||\varphi_n||_{2,m}$ is uniform in n follows from the corresponding fact for ψ_n and the fact that each λ_n has support in a common compact set. Thus, given $\epsilon > 0$, we can choose N so that

$$M|||\varphi_n|||_2 \sum_{m=N+1}^{\infty} ||\varphi_n||_{2,m} < \epsilon/8$$

for every n and

$$M|||\varphi|||_2 \sum_{m=N+1}^{\infty} ||\varphi||_{2,m} < \epsilon/8.$$

Writing

$$\iint \overline{f(s-t)}[\varphi(s)\overline{\varphi(t)} - \varphi_n(s)\overline{\varphi_n(t)}]\, dsdt = I_1 + I_2,$$

where I_1 and I_2 are the integrals over $G_N \times G_N$ and its complement, respectively, we see from (4) that

$$\begin{aligned}|I_2| &\leqq \left(\sum_{m=1}^{\infty}\sum_{p=N+1}^{\infty} + \sum_{m=N+1}^{\infty}\sum_{p=1}^{\infty}\right)\left\{\left|\int_{K_m}\int_{K_p} \overline{f(s-t)}\varphi(s)\overline{\varphi(t)}\, dsdt\right|\right. \\ &\qquad\qquad \left. + \left|\int_{K_m}\int_{K_p} \overline{f(s-t)}\varphi_n(s)\overline{\varphi_n(t)}\, dsdt\right|\right\} \\ &\leqq 2M|||\varphi|||_2 \sum_{m=N+1}^{\infty} ||\varphi||_{2,m} + 2M|||\varphi_n|||_2 \sum_{m=N+1}^{\infty} ||\varphi_n||_{2,m} \\ &< \epsilon/2.\end{aligned}$$

Since $||\Phi - \Phi_n||_1 \to 0$ we have

$$||\varphi(s)\overline{\varphi(t)} - \varphi_n(s)\overline{\varphi_n(t)}||_\infty \leqq ||\Phi(\hat{s})\overline{\Phi(-\hat{t})} - \Phi_n(\hat{s})\overline{\Phi_n(-\hat{t})}||_1 \to 0,$$

and thus

$$||\varphi(s)\overline{\varphi(t)} - \varphi_n(s)\overline{\varphi_n(t)}||_\infty < (\epsilon/2)\int_{G_N - G_N} |f(s)|\, ds$$

for $n > n_0$, say. Thus $|I_1| < \epsilon/2$ for $n > n_0$. It follows that

$$\iint \overline{f(s-t)}\varphi_n(s)\overline{\varphi_n(t)}\, dsdt \to \iint \overline{f(s-t)}\varphi(s)\overline{\varphi(t)}\, dsdt.$$

On the other hand,

$$\int |\Phi_n(\hat{s})|^2\, d\mu(\hat{s}) \to \int |\Phi(\hat{s})|^2\, d\mu(\hat{s})$$

by the monotone convergence theorem. Since (5) holds for φ_n, it must therefore hold for $\varphi \in J$.

We now extend the range of validity of (5) further to the case where $\varphi(s, t) = \psi(s)\lambda(t)$, $\psi \in L_c^2(R^a)$, $\lambda \in L_c^2(G_1)$. Let $\{\psi_n\}$ be a sequence of infinitely differentiable functions with supports in a common compact set and such that $||\psi - \psi_n||_2 \to 0$ and $|\Psi_n| \leqq |\Psi|$. Since $\Lambda \in L^2(\hat{G}_1)$, it has σ-compact support and thus, by the same construction as in the first part of the proof,

https://doi.org/10.4153/CJM-1969-143-4 Published online by Cambridge University Press

there exist functions $\lambda_n \in L_c^2(G_1)$ with supports in a common compact set such that $||\lambda - \lambda_n||_2 \to 0$, $\Lambda_n \in L^1(\hat{G}_1)$, and $|\Lambda_n| \leqq |\Lambda|$. Then $\varphi_n(s, t) = \psi_n(s)\lambda_n(t)$ is in J, $||\varphi - \varphi_n||_2 \to 0$, and $|\Phi_n| \leqq |\Phi|$. The functions $\varphi_n * \tilde{\varphi}_n$ have supports in a common compact set and

$$||\varphi * \tilde{\varphi} - \varphi_n * \tilde{\varphi}_n||_\infty \leqq ||\varphi||_2||\tilde{\varphi} - \tilde{\varphi}_n||_2 + ||\tilde{\varphi}_n||_2||\varphi - \varphi_n||_2 \to 0.$$

Consequently,

$$\int \overline{f(s)}\varphi_n * \tilde{\varphi}_n(s)\, ds \to \int \overline{f(s)}\varphi * \tilde{\varphi}(s)\, ds.$$

Since $||\Phi - \Phi_n||_2 = ||\varphi - \varphi_n||_2 \to 0$ we may assume, by passing to a subsequence if necessary, that $\Phi_n \to \Phi$ a.e. Then, by Fatou's lemma,

$$\int |\Phi(\hat{s})|^2\, d\mu(\hat{s}) \leqq \liminf_{n\to\infty} \int |\Phi_n(\hat{s})|^2\, d\mu(\hat{s}) = \lim_{n\to\infty} \int \overline{f(s)}\varphi_n * \tilde{\varphi}_n(s)\, ds$$

so that the integral $\int |\Phi(\hat{s})|^2\, d\mu(\hat{s})$ is finite. Thus an application of the dominated convergence theorem proves that

$$\int \overline{f(s)}\varphi * \tilde{\varphi}(s)\, ds = \int |\Phi(\hat{s})|^2\, d\mu(\hat{s}).$$

Applying this equation to the function $\psi_U(t + s)$ we obtain

$$\int \varphi_U(t + s)\overline{f(s)}\, ds = \int [t, \hat{s}]\Phi_U(\hat{s})\, d\mu(\hat{s}).$$

The left-hand side is $\varphi_U * f(t)$ (using Theorem 2.1) which converges to $f(t)$ in the manner of the statement of the theorem.

THEOREM 4.3. *Let $f \in P(L_c^2)$ and let μ be the positive measure on $\hat{G}$ associated with f by Theorem 4.2. Then for any compact set C in $\hat{G}$, $\mu(\hat{s} + C) \to 0$ as $\hat{s} \to \infty$. If f is locally in L^p, then $\mu(\hat{s} + C) \in L^{p'}(\hat{G})$ if $1 \leqq p \leqq 2$ and $\mu(\hat{s} + C) \in L^2(\hat{G})$ if $p \geqq 2$.*

Proof. From the proof of the preceding theorem we have

$$\int \overline{f(s)}\varphi * \tilde{\varphi}(s)\, ds = \int |\Phi(\hat{s})|^2\, d\mu(\hat{s})$$

whenever $\varphi(s, t) = \psi(s)\lambda(t)$, $\psi \in L_c^2(R^a)$, $\lambda \in L_c^2(G_1)$. If $C \subset C_1 \times C_2$, where C_1 is compact in R^a and C_2 is compact in G_1, let λ be the function constructed in the proof of Lemma 1 so that $\lambda \in C_c(G_1)$, $\Lambda \in C_c(\hat{G}_1)$ and $\Lambda(\hat{t}) = 1$ for $\hat{t} \in C_2$. For some N, the set C_1 is contained in the a-fold product of the interval $[-N, N]$. Let ψ be the a-fold product of the function which is equal to $4N/\pi - 8N^2|x|/\pi^2$ for $|x| \leqq \pi/2N$ and is equal to 0 for $|x| \geqq \pi/2N$. Then $\Psi(\hat{s}) > 1$ for $\hat{s} \in C_1$ and thus $\Phi > 1$ on C. Since $\varphi * \tilde{\varphi} \in C_c(G)$, the assertions of the theorem follow from the inequality

$$0 \leqq \mu(\hat{s} + C) \leqq \int |\Phi(\hat{s} + \hat{t})|^2\, d\mu(\hat{t}) = \int \overline{f(s)}\varphi * \tilde{\varphi}(s)[\hat{s}, s]\, ds,$$

as in the proof of Theorem 3.2.

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