ON A PAPER BY M. IOSIFESCU AND S. MARCUS

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In this paper we will construct an example showing that the problem posed in [1] has a negative answer. Two more theorems on the subject treated in [1] will be included.

Let $I_o = [0, 1]$, R the reals, and let, for $A \subset R$, A^o be the interior of A. Let $\{x_n\}$ be a sequence in $[0, 1)$ such that $0 = x_1 < x_2 < \ldots$ and $\lim x_n = 1$. For each n, let I_n be a closed interval having x_n as its midpoint (except for $n = 1$ in which case x_1 is the left endpoint of I_1) such that $I_n \cap I_m = \phi$, $n \neq m$, and the metric density relative to I_o of $\bigcup_{n \geq 1} I_n$ at 1 is zero. Let J_n be a closed interval in I_n concentric with I_n (except for $n = 1$, where J_1 has x_1 as its left endpoint) whose length is half that of I_n.

We recall that a function $f: I_o \to R$ is approximately continuous at x_o, $x_o \in I_o$, if there exists a measurable set $E \subset I_o$ such that E has metric density 1 at x_o relative to I_o and $\lim f(x) = f(x_o)$, $x \in E$, $x \to x_o$ [2, p. 132].

THEOREM 1. There exist $f, g: I_o \to R$ bounded on I_o such that f is approximately continuous on I_o, g is a derivative on I_o, and $f^2 + g^2$ does not possess the Darboux property on I_o.

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Proof. We will first construct g. Let $x_n = 1 - \frac{1}{n}$, and let $g(1) = 0$, $g(x_n) = 0$, $n = 1, 2, \ldots$. On the intervals $[x_n, x_{n+1}]$ define g continuously such that $\{x: x \in [x_n, x_{n+1}]$ and $|g(x)| > \frac{1}{2}\} = [x_n, x_{n+1}] - (J_n \cup J_{n+1})$ and $\int_{x_n}^{x_{n+1}} |g| = \frac{1}{2^n}$.

On $[x_n, x_{n+1}]$, let g be non-negative if n is odd and non-positive if n is even. We can clearly make g bounded on I_o. Define $G: I_o \to R$ by

$$G(t) = \begin{cases} \int_0^t g, & 0 \le t < 1 \\ 1 - \frac{1}{2^2} + \frac{1}{3^2} - + \ldots, & t = 1. \end{cases}$$

It is readily verified that $G'(t) = g(t)$, $t \in I_o$ (see e.g. the proof of theorem 2).

Define a bounded function $h: I_o \to R$ satisfying the conditions: $\{x: h(x) = 0\} = I_o - \bigcup_{n \ge 1} I_n^o$, $\{x: h(x) > \frac{1}{4}\} = \bigcup J_n^o$, $h(x) \ge 0$ on I_o and continuous on $[0, 1)$. Then h is approximately continuous on I_o and hence $f = \sqrt{h}$ is approximately continuous on I_o. However, $f^2(x) + g^2(x) \ge \frac{1}{4}$ for $x \in [0, 1)$ and $f^2(1) + g^2(1) = 0$. Hence $f^2 + g^2$ does not possess the Darboux property.

It is natural to inquire whether or not $f + g$ has the Darboux property if f is approximately continuous and g is a derivative. We will first show that this is not always the case.

THEOREM 2. There exist $f, g: I_o \to R$ such that f is approximately continuous on I_o, g is a derivative on I_o, and $f + g$ does not possess the Darboux property.

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Proof. Using the notation introduced in the second paragraph, let $x_n = 1 - \frac{1}{\sqrt{n}}$, and divide the two complementary intervals of $I_n - J_n^o (n > 1)$ into two equal intervals τ'_n, τ''_n and σ''_n, σ'_n, where τ'_n, σ''_n are to the left of τ''_n, σ'_n, respectively, and τ''_n is to the left of σ''_n. For $n = 1$, we have only σ'_1, σ''_1. Let g be continuous on J_n, zero at the endpoints of J_n, negative on J_n^o such that $\int_{J_n} g = -\frac{1}{2(n-1)}$ $(n > 1)$, and let g be zero on J_1. On each interval K_n of the form $[x_n, x_{n+1}] - (J_{n+1}^o \cup J_n^o)$ define g to be continuous, zero at the endpoints of K_n, positive on K_n^o such that $\int_{K_n} g = \frac{1}{2n-1}$, and $\{x: x \in K_n \text{ and } g(x) \leq 1\} = \sigma''_n \cup \tau''_{n+1}$. Let $g(1) = 0$. Then g is continuous on $[0, 1)$. Let

$$G(t) = \begin{cases} \int_0^t g, & 0 \leq t < 1 \\ 1 - \frac{1}{2} + \frac{1}{3} - + \dots, & t = 1 . \end{cases}$$

It follows that $G'(t) = g(t)$, $0 \leq t < 1$. To prove that $G'(1) = g(1) = 0$, let $h_i \to 1$, $h_i \in [0, 1)$. For each i, let n_i be the integer for which $x_{n_i} < h_i \leq x_{n_i + 1}$. Since $G(1) - G(h_i)$ is the remainder of an alternating convergent series, we have the inequality

$$|G(1) - G(h_i)| \leq \frac{1}{2(n_i - 1)} .$$

Hence

$$\left| \frac{G(1) - G(h_i)}{1 - h_i} \right| \leq \frac{1}{2(n_i - 1)} \Big/ \frac{1}{\sqrt{n_i + 1}} ,$$

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and thus $G'(1) = 0$.

Define $f: I_o \to R$ so that the following conditions are satisfied: $\{x: f(x) = 0\} = I_o - \bigcup_{n>1} I_n^o$, $f \geq 0$ on I_o, f continuous on $[0, 1)$, $\{x: f(x) \geq 1\} = \bigcup_{n>1} (\tau_n'' \cup J_n \cup \sigma_n'') \cup J_1 \cup \sigma_1''$, and $f(x) = -g(x) + 1$, $x \in J_n$. Then f is approximately continuous on I_o. But $f(x) + g(x) \geq 1$, $x \in [0, 1)$, and $f(1) + g(1) = 0$.

Both functions in the above proof are unbounded. In view of the next theorem this is an essential feature.

THEOREM 3. Let $f: I_o \to R$ be approximately continuous on I_o and let $g: I_o \to R$ be a derivative on I_o. If either f or g is bounded on I_o, then $h = f + g$ possesses the Darboux property on I_o.

Proof. If f is bounded on I_o, then f is a derivative on I_o [2, p. 132], and the conclusion follows. We assume that $|g| < M$ on I_o. Let $0 \leq \alpha < \beta \leq 1$ such that $h(\alpha) \neq h(\beta)$, say $h(\alpha) < h(\beta)$, and let $h(\alpha) < y_o < h(\beta)$. We will exhibit x_o, $\alpha < x_o < \beta$, such that $h(x_o) = y_o$. Define $f_n: I_o \to R$ by

$$f_n(x) = \begin{cases} n, & f(x) \geq n \\ f(x), & -n < f(x) < n \\ -n, & f(x) \leq -n . \end{cases}$$

Then f_n is bounded and approximately continuous on I_o, and hence f_n is a derivative on I_o. Thus $f_n + g = h_n$ has the Darboux property. Let n_o be an integer such that $-n_o + M < y_o < n_o - M$ and $h_{n_o}(\alpha) < y_o < h_{n_o}(\beta)$. There

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exists $x_o \in (\alpha, \beta)$ such that $h_{n_o}(x_o) = y_o$. We only need to show that $-n_o < f(x_o) < n_o$. If $|f(x_o)| \geq n_o$, then $|f_{n_o}(x_o)| = n_o$, and $f_{n_o}(x_o) + g(x_o) \neq y_o$, contrary to $h_{n_o}(x_o) = y_o$.

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