# USEFUL THEOREMS ON COMMUTATIVE NON-ASSOCIATIVE ALGEBRAS

*by* NORA LOSEY†

(Received 20th January 1966)

## 1. Introduction

Recently J. M. Osborn has investigated the structure of a simple commutative non-associative algebra with unity element satisfying a polynomial identity, (4), (5) and (6). From his work it seems likely that if such an algebra is of degree three or more it is necessarily power-associative. In (4) he establishes a hierarchy of identities with the property that each identity is satisfied by an algebra satisfying no preceding identity. Following (5), (6), the next identity to consider is

$$\begin{aligned} ay(x^3x)+(-c-h)y(x^2x^2)+c(yx^2)x^2+(-c-h)(yx)x^3 \\ +(-4a+5c+5h)(yx^3)x+(c+h)(yx\,.\,x^2)x+(6a-9c-8h)(yx^2\,.\,x)x \\ +(-3a+4c+3h)((yx\,.\,x)x)x+h(yx\,.\,x)x^2 = 0 \qquad (1.1) \end{aligned}$$

where $a$, $c$ and $h$ are elements of the ground field.

In this paper we give two theorems useful in investigations of this kind (6), and in § 3 we give a sketch of the application of our theorems to algebras satisfying (1.1).

By an algebra we shall mean a commutative non-associative algebra with a unity element over a field of characteristic not equal to 2, 3 or 5. An algebra is said to be of degree $m$ if the supremum of the cardinal numbers of all sets of mutually orthogonal idempotents is $m$. If $e$ is an idempotent of an algebra $A$ and $\alpha$ an element of the ground field, define $A_e(\alpha) = \{x \in A: xe = \alpha x\}$. Often $xy$ is written $xR_y$ where $R_y$ denotes the linear transformation induced by right multiplication by $y$. If $a \in A$ then the component of $a$ in the subspace $A_e(\alpha)$ is written $a_\alpha(e)$ or $a_\alpha$.

## 2. Theorems 1 and 2

**Theorem 1.** *Let $A$ be a simple algebra over a field $F$ and $\lambda \in F$, $\lambda \neq 0, 1, \frac{1}{2}$, be such that for each idempotent $e$ in $A$ the following hold:*

(i) *If $xe = \alpha x$, $x \in A$, $\alpha \in F$, then $\alpha = 0, 1, \lambda$ or $1-\lambda$.*

(ii) *$A = A_e(0)+A_e(\lambda)+A_e(\bar{\lambda})+A_e(1)$ where $\bar{\lambda} = 1-\lambda$.*

† Some of the material given here has appeared in a dissertation presented to the University of Wisconsin in partial fulfilment of the requirements for the Ph.D. in Mathematics. The dissertation was written under the supervision of Prof. J. M. Osborn.

(iii) $A_e(0) \,.\, A_e(1) = 0$;
$A_e(1) \,.\, A_e(1) \subseteq A_e(1)$;
$A_e(\alpha) \,.\, A_e(1) \subseteq A_e(\alpha)$ *for* $\alpha = \lambda$ *or* $\bar{\lambda}$.

(iv) *For all* $y, w, z \in A$,

$$y_{\bar{\lambda}} \,.\, w_0 z_0 = \beta(y_{\bar{\lambda}} w_0 \,.\, z_0 + y_{\bar{\lambda}} z_0 \,.\, w_0)$$
$$y_{\lambda} \,.\, w_0 z_0 = \delta(y_{\lambda} w_0 \,.\, z_0 + y_{\lambda} z_0 \,.\, w_0)$$

*for fixed* $\beta, \delta \in F$.

*Then A is of degree at most* 2.

**Proof.** We first show that if $e$ is an idempotent of $A$ such that $e \neq 0$ or 1 then $A_e(0)$ is a Jordan subalgebra of $A$. If $A_e(\lambda)+A_e(\bar{\lambda}) = 0$ then $A_e(0)$ is an ideal of $A$. Hence we must have $A_e(0) = 0$ or $A_e(1) = 0$ which implies $e = 1$ or $e = 0$, contradicting the hypothesis. Thus $A_e(\lambda)+A_e(\bar{\lambda}) \neq 0$. Let

$$H = \{x \in A_e(0)\colon\ bx = 0 \text{ for all } b \in A_e(\lambda)+A_e(\bar{\lambda})\}.$$

Then $H$ is an ideal of $A$. Indeed let $x \in H$ and $a \in A$. Then

$$xa = xa_0 + xa_\lambda + xa_{\bar{\lambda}} + xa_1 = xa_0$$

by (iii) and the definition of $H$. Thus for all $b \in A_e(\lambda)+A_e(\bar{\lambda})$ we get $b \,.xa = b \,.xa_0$ by (iv); so $H$ is an ideal as desired. If $H = A$ then $A = A_e(0)$ and $e = 0$. So $H = 0$. Then for $p, q \in A_e(0)$ let $r = p^2 q \,.\, p - p^2 \,.\, qp$. It follows from repeated application of (iv) that $br = 0$ for all $b \in A_e(\lambda)+A_e(\bar{\lambda})$. Thus $r \in H$ so $r = 0$ and $A_e(0)$ is a Jordan algebra.

If $A$ has degree 3 or more then $1 = e+f+g$ where $e, f$ and $g$ are mutually orthogonal idempotents. Consider the subspace

$$B(\rho, \sigma, \tau) = A_e(\rho) \cap A_f(\sigma) \cap A_g(\tau)$$

where $\rho, \sigma, \tau$ range over $0, \lambda, \bar{\lambda}, 1$. If $b \in B(\rho, \sigma, \tau)$ and $b \neq 0$, then

$$b = b \,.\, 1 = b(e+f+g) = (\rho+\sigma+\tau)b.$$

Thus $\rho+\sigma+\tau = 1$. That is $B(\rho, \sigma, \tau) = 0$ unless at least one of $\rho, \sigma, \tau$ is 0. If $b \in B(0, \sigma, \tau)$ then $bf = \sigma f$ in the Jordan algebra $A_e(0)$. Thus $\sigma$ is one of $0, 1, \frac{1}{2}$ by (1). We have excluded the last by hypothesis. Thus $B(\rho, \sigma, \tau) \neq 0$ only if exactly one of $\rho, \sigma, \tau$ is not 0. Now

$$A = B(1, 0, 0)+B(0, 1, 0)+B(0, 0, 1).$$

Furthermore $e \in B(1, 0, 0) = A_e(1)$, a non-zero proper ideal of $A$, contradicting the simplicity of $A$. Hence $A$ is of degree at most 2 as desired.

**Theorem 2.** *Let A be a simple algebra over a field F with the following properties for each idempotent e*:

(i) $A = A_e(1)+A_e(\frac{1}{2})+A_e(0)$.

(ii) $A_e(1) \,.\, A_e(1) \subseteq A_e(1)$;
$A_e(0) \,.\, A_e(1) = 0$;
$A_e(0) \,.\, A_e(\frac{1}{2}) \subseteq A_e(\frac{1}{2})$;
$A_e(\frac{1}{2}) \,.\, A_e(\frac{1}{2}) \subseteq A_e(1)+A_e(0)$.

(iii) *If* $y_0, z_0 \in A_e(0)$ *and* $w_2 \in A_e(\frac{1}{2})$ *then* $y_0z_0 . w_2 = z_0 . y_0w_2 + y_0 . z_0w_2$.

(iv) *If* $y_2, z_2 \in A_e(\frac{1}{2})$ *and* $w_0 \in A_e(0)$ *then*

$$(y_2z_2 . w_0)_0 = (y_2 . z_2w_0)_0 + (z_2 . y_2w_0)_0$$

*and* $(y_2 . w_0z_2)_1 = (y_2w_0 . z_2)_1$.

*Then if A is an algebra of degree* 3 *or more A is Jordan.*

It is enough to show that $A$ is power-associative, by (**3**). This will follow from Lemmas 2.1-2.3. In each of these lemmas we assume that $A$ is an algebra satisfying the hypotheses of Theorem 2.

**Lemma 2.1.** *If* $e \neq 0$ *or* 1 *then* $A_e(0)$ *is a Jordan sub-algebra of* $A$.

**Proof.** Let $K = \{x \in A_e(0)\colon xb = 0 \text{ for all } b \in A_e(\frac{1}{2})\}$. Then if $a \in A$, $k \in K$ we have $ka = ka_0$ by (ii). If $b \in A_e(\frac{1}{2})$ then

$$ka . b = ka_0 . b = k . a_0b + a_0 . kb = 0$$

by (iii) and the definition of $K$. Hence $K$ is a proper ideal of $A$ and so must be 0. Also if $p, q \in A_e(0)$ and $r = p^2q . p - p^2 . qp$ then $br = 0$ for all $b \in A_e(\frac{1}{2})$ by (iii). Thus $r \in K$, $r = 0$ and $A_e(0)$ is Jordan.

If $A$ has degree 3 or more then $1 = e+f+g$ for mutually orthogonal idempotents $e, f$ and $g$. As in Theorem 1 we define

$$B(\rho, \sigma, \tau) = A_e(\rho) \cap A_f(\sigma) \cap A_g(\tau)$$

where $\rho, \sigma, \tau$ range over 0, $\frac{1}{2}$, 1. The same argument as before tells us that

$$A = B(1, 0, 0) + B(0, 1, 0) + B(0, 0, 1) + B(\tfrac{1}{2}, \tfrac{1}{2}, 0) + B(\tfrac{1}{2}, 0, \tfrac{1}{2}) + B(0, \tfrac{1}{2}, \tfrac{1}{2}).$$

A standard idempotent is defined to be any one of $e, f, g, e+f, e+g, f+g$ where $e+f+g = 1$.

**Lemma 2.2.** *Let h be a standard idempotent of A. Let* $y_0 \in A_h(0)$, $x_1 \in A_h(1)$, $z_2 \in A_e(\frac{1}{2})$. *Then*

$$y_0 . z_2x_1 = y_0z_2 . x_1.$$

**Proof.** Assume $h = e$, and $e+f+g = 1$. This results in no loss of generality for if $h = f+g$ then $1-h = e$ and by interchanging the roles of $x$ and $y$ we are back at the first case.

Thus we have $x_1 \in B(1, 0, 0)$. We may assume $z_2 \in B(\frac{1}{2}, \frac{1}{2}, 0)$. We must consider the three cases, $y_0 \in B(0, 1, 0)$, $y_0 \in B(0, 0, 1)$, and $y_0 \in B(0, \frac{1}{2}, \frac{1}{2})$. In the first case the lemma holds because $x_1$, $x_2$ and $y_0$ are all in $A_g(0)$, and the result holds in Jordan algebras (**1**). In the second case both sides of the equation are zero by (ii). In the final case the lemma follows from (iii) since $x_1, z_2 \in A_g(0)$. Here $x_1z_2 . y_0 = x_1 . z_2y_0 + x_1y_0 . z_2$ where the last term is zero by (ii).

For fields of characteristic not equal to 2, 3 or 5, the commutative power-associative identity is equivalent to $x^3x = x^2x^2$, by (**2**). By repeated linearisation this is seen to be equivalent to $P(x, y, v, z) = 0$ where

$$\begin{aligned} P(x, y, v, z) = {} & 4yz . xv + 4yv . xz + 4yx . vz - (yz . v)x - (yz . x)v \\ & -(xv . z)y - (xy . z)v - (vy . x)z - (vy . z)x - (xy . v)z \\ & -(vy . y)x - (xz . y)v - (xv . y)z - (vz . x)y - (xz . v)y. \end{aligned}$$

https://doi.org/10.1017/S001309150001172X Published online by Cambridge University Press

**Lemma 2.3.** *Let $h$ be a standard idempotent of $A$. If $x$, $y$, $v$, $z$ each belong to $A_h(1)$ or $A_h(0)$ then $P(x, y, v, z) = 0$.*

**Proof.** If all four elements are in $A_h(1)$ or $A_h(0)$ then they are all in a Jordan subalgebra. But a Jordan algebra is power-associative (1). Thus $P(x, y, v, z) = 0$. If some of $x$, $y$, $v$, $z$ are in $A_h(1)$ and some are in $A_h(0)$ then all terms in the expression are zero by (ii) and $P = 0$ trivially.

To check that $A$ is power-associative it is sufficient to show $P(x, y, v, z) = 0$ for $x$, $y$, $v$, $z$ each belonging to one of the six $B(\rho, \sigma, \tau)$ that are non-zero. To do this it is necessary to calculate $P(x, y, v, z)$ in four specific cases. These are given in the table below. Let $x \in A_h(\alpha)$, $y \in A_h(\beta)$, $v \in A_h(\gamma)$, $z \in A_h(\delta)$ where $h$ is a standard idempotent.

| Case | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $P(x, y, v, z) = 0$ by: |
|---|---|---|---|---|---|
| 1 | $\frac{1}{2}$ | 0 | 0 | 1 | (ii), (iii), Lemma 2.2 |
| 2 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | (ii), (iv), Lemma 2.2 |
| 3 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | (ii), (iv), Lemma 2.2 |
| 4 | 0 | $\frac{1}{2}$ | 0 | 0 | (ii), (iii). |

All possible cases have been covered by Lemma 2.3 and the above table by suitable choice of the standard idempotent $h$. The proof of Theorem 2 is now complete.

## 3. Application to algebras satisfying an identity of degree five.

In this section we show that except for a few values of the parameters $a$, $c$ and $h$, simple algebras satisfying (1.1) of degree three or more are of the form $A = A_e(0) + A_e(\frac{1}{2}) + A_e(1)$ and are Jordan.

We shall need the following well-known lemma:

**Lemma 3.1.** *Let $A$ be an algebra over a field $F$ such that every subfield of the centre of $A$ is in $F$, and such that $r \in F$ and $r = k^2$ for no $k \in F$. Then if $A$ is simple so is $A_K$ where $K = F(k)$ and $k$ is such that $k^2 = r$.*

Let $e$ be an idempotent in an algebra $A$ in which every pair of elements $x$, $y$ satisfy equation (1.1). Setting $x = e$ in (1.1) and factoring gives

$$yR_e(R_e - 1)[(-3a + 4c + 3h)(R_e^2 - R_e) + (-a + c + h)] = 0. \tag{3.1}$$

If $a = h$ and $c = 0$ equation (3.1) gives no information. If $c \neq 0$ but

$$-3a + 4c + 3h = 0,$$

equation (3.1) becomes $yR_e(R_e - 1) = 0$, and it can be shown that simple algebras satisfying (1.1) and this condition must be of degree one. Otherwise (3.1) becomes

$$yR_e(R_e - 1)(R_e - \lambda)(R_e - \bar{\lambda}) = 0 \tag{3.2}$$

where $\lambda$ and $\bar{\lambda}$ are roots of the equation

$$(-3a + 4c + 3h)X^2 + (3a - 4c - 3h)X + (-a + c + h) = 0,$$

that is,

$$\lambda, \bar{\lambda} = \tfrac{1}{2} \pm \tfrac{1}{2}\left(\frac{a-h}{-3a+4c+3h}\right)^{\frac{1}{2}}.$$

In view of Lemma 3.1 there is no loss in assuming that $\lambda$, $\bar{\lambda} \in F$. If $A$ satisfies (1.1) then so does $A_{F(\lambda)}$. If $A_{F(\lambda)}$ is of degree $m$ or is power-associative then so is $A$. Note that $\lambda+\bar{\lambda} = 1$. If $a = h$, and of course $c \neq 0$, we have $\lambda = \bar{\lambda} = \frac{1}{2}$. Finally we note that $\lambda$ or $\bar{\lambda}$ is 0 or 1 exactly if $a = c+h$, and $c \neq 0$.

If $A$ satisfies (1.1) and (3.2) with four distinct roots then

$$A = A_e(0)+A_e(\lambda)+A_e(\bar{\lambda})+A_e(1).$$

By using this decomposition, linearisation, and by finding suitable ideals, one can prove the following theorem.†

**Theorem 3.** *Simple algebras satisfying* (1.1) *are either of degree one or satisfy the conditions for Theorem* 1 *and so are of degree at most two, unless the coefficients are so that* $a = h$, $a = c+h$, $(-2a+3c+3h)+(3a-5c-5h)\mu = 0$ *for* $\mu = \lambda$ *or* $\bar{\lambda}$, *or so that* $(a, c, h)$ *is a multiple of one of the following* $(8, 7, -1)$, $(6, -3, 7)$, $(4, 1, 2)$, $(16, -9, 19)$, $(40, -19, 49)$, $(22, -9, 25)$, $(-26\pm 10\sqrt{5}, 121\pm 33\sqrt{5}, -158\mp 34\sqrt{5})$.

We next consider an algebra satisfying (1.1) with $a = h$ and $c \neq 0$. Our identity then is

$$y\{aR_{x^3x}+(-c-a)R_{x^2x^2}+c(R_{x^2})^2+(-c-a)R_xR_{x^3}+(a+5c)R_{x^3}R_x$$
$$+(a+c)R_xR_{x^2}R_x+(-2a-9c)R_{x^2}R_x^2+4cR_x^4+aR_x^2R_{x^2}\} = 0. \tag{3.3}$$

We recall that

$$yR_e(R_e-1)(R_e-\tfrac{1}{2})^2 = 0$$

for all idempotents $e$ of $A$. It follows that $A$ can be written as the supplementary sum

$$A = A_e(0)+B_e(\tfrac{1}{2})+A_e(1)$$

where

$$B_e(\tfrac{1}{2}) = \{x \in A\colon\ x(2R_e-1)^2 = 0\}.$$

The usual techniques of non-associative algebra show us that

$$\{yR_e(R_e-1)(R_e-\tfrac{1}{2})\}$$

is an ideal of $A$ for most values of the parameters.

**Lemma 3.2.** *Let $A$ be a simple algebra satisfying* (3.3) *with* $c \neq 0$ *and the pair* $(a, c)$ *not a multiple of any of the following*: $(14, -3)$, $(3, -1)$, $(2, 1)$, $(2, -1)$. *Then* $A = A_e(0)+A_e(\frac{1}{2})+A_e(1)$ *for any idempotent $e$ of $A$.*

We use Theorem 2 to give us the following:

**Theorem 4.** *Let $A$ be a simple algebra satisfying* (3.3) *with* $c \neq 0$ *and the pair* $(a, c)$ *not a multiple of any of the following*: $(14, -3)$, $(3, -1)$, $(2, 1)$, $(2, -1)$, $(4, -1)$. *Then if $A$ is of degree* $\geqq 3$, *$A$ is Jordan.*

† Most of the details of this and succeeding calculations have been omitted. A copy of the complete calculations is available on request from the author.

## REFERENCES

(1) A. A. Albert, A structure theory for Jordan algebras, *Ann. Math.* **48** (1947), 546-567.

(2) A. A. Albert, On the power-associativity of rings, *Summa Brasiliensis Mathematicae*, **2** (1948), 21-23.

(3) A. A. Albert, A theory of power-associative commutative algebras, *Trans. Amer. Math. Soc.* **69** (1950), 503-527.

(4) J. M. Osborn, Identities on non-associative algebras, *Canadian J. Math.* **17** (1965), 78-92.

(5) J. M. Osborn, An identity of degree four, *Proc. Amer. Math. Soc.* **16** (1965), 1114-1119.

(6) J. M. Osborn, A generalization of power-associativity, to appear in *Pacific J. Math.*

(7) J. M. Osborn, On commutative non-associative algebras, *J. Algebra*, **2** (1965), 48-79.

University of Manitoba