Forum of Mathematics, Sigma (2022), **Vol. 10**:e35 1–50
doi:10.1017/fms.2022.30

CrossMark

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Cluster categories of formal DG algebras and singularity categories

Norihiro Hanihara

Kavli Institute for the Physics and Mathematics of the Universe (WPI),The University of Tokyo Institutes for Advanced Study, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan; E-mail: norihiro.hanihara@ipmu.jp.

Received: 30 January 2022; **Accepted:** 24 March 2022

Keywords: Cluster category, DG algebra, Calabi-Yau algebra, cluster tilting subcategory, d-representation-infinite algebra, Iwanaga-Gorenstein algebra, derived orbit category, singularity category, DG orbit category

2020 Mathematics Subject Classification: *Primary* – 16E45, 18G35, 18G80; *Secondary* – 16E35, 16S38, 16G50

Abstract
Given a negatively graded Calabi-Yau algebra, we regard it as a DG algebra with vanishing differentials and study its cluster category. We show that this DG algebra is sign-twisted Calabi-Yau and realise its cluster category as a triangulated hull of an orbit category of a derived category and as the singularity category of a finite-dimensional Iwanaga-Gorenstein algebra. Along the way, we give two results that stand on their own. First, we show that the derived category of coherent sheaves over a Calabi-Yau algebra has a natural cluster tilting subcategory whose dimension is determined by the Calabi-Yau dimension and the a-invariant of the algebra. Second, we prove that two DG orbit categories obtained from a DG endofunctor and its homotopy inverse are quasi-equivalent. As an application, we show that the higher cluster category of a higher representation infinite algebra is triangle equivalent to the singularity category of an Iwanaga-Gorenstein algebra, which is explicitly described. Also, we demonstrate that our results generalise the context of Keller–Murfet–Van den Bergh on the derived orbit category involving a square root of the AR translation.

Contents

1. Introduction

Cluster tilting theory emerged at the beginning of this century in the contexts of categorification of cluster algebras [BMRRT] and higher-dimensional Auslander-Reiten theory [I1], which has led to fruitful connections between various areas of mathematics. A central role is played by the concept of cluster tilting objects in Calabi-Yau (CY) triangulated categories, which gives a categorification of Fomin–Zelevinsky's cluster algebra [FZ]; see, for example, [Ke5] for an introduction. A general construction of such triangulated categories is given as Amiot's generalised cluster categories [Am], which is based on the formalism of differential graded (DG) algebras [Ke3]. Throughout, we fix a field k, and every module will be a right module unless stated otherwise. Recall that a DG k-algebra Λ is *bimodule* $(n+1)$*-CY* [Gi] if it is homologically smooth and there is an isomorphism

$$\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)[n+1] \simeq \Lambda$$

in the derived category $\mathcal{D}(\Lambda^e)$ of the enveloping algebra $\Lambda^e := \Lambda^{\mathrm{op}} \otimes_k \Lambda$. Then the *cluster category* $\mathcal{C}(\Lambda)$ of Λ is the Verdier quotient of the perfect derived category $\operatorname{per} \Lambda$ by the thick subcategory $\mathcal{D}^b(\Lambda)$ consisting of DG modules of finite-dimensional total cohomology. The fundamental result due to Amiot and its generalization by Guo [Am, Gu] states that if Λ is a bimodule $(n+1)$-CY DG algebra whose cohomology is concentrated degree ≤ 0 and each cohomology is finite dimensional, then $\mathcal{C}(\Lambda)$ is an n-CY triangulated category and $\Lambda \in \mathcal{C}(\Lambda)$ is an n-cluster tilting object.

The aim of this paper is to give some descriptions of this cluster category for a certain class of DG algebras, namely *formal* DG algebras. Recall that a DG algebra is formal if it is isomorphic to its cohomology in the homotopy category of DG algebras. We therefore start our discussion with a graded (non-DG) algebra R and view it as a DG algebra R^{dg} with trivial differentials.

1.1. Cluster categories and orbit categories

Our first observation is that we can obtain a class of CY DG algebras from certain graded (non-DG) algebras. Recall the distinct notion of CY algebra for graded non-DG algebras; a graded algebra R over a field k is *homologically smooth* if it perfect as a graded bimodule and is *bimodule* $(d+1)$*-CY of* a*-invariant* a if it is homologically smooth and there is an isomorphism

$$\mathrm{RHom}_{R^e}(R, R^e)(a)[d+1] \simeq R$$

in the derived category $\mathcal{D}(\operatorname{Mod}^{\mathbb{Z}} R^e)$ of (all) graded bimodules $\operatorname{Mod}^{\mathbb{Z}} R^e$. (This should not be confused with the derived category $\mathcal{D}((R^{\mathrm{dg}})^e)$ of the DG algebra $(R^{\mathrm{dg}})^e$.) Here, (1) is the degree shift functor on the graded modules, while $[1]$ is the suspension in the derived category. Such algebras arise naturally and are studied extensively in representation theory and commutative or non-commutative algebraic geometry [AS, YZ, Boc1, KS, IR, BS, BSW, MM, AIR, VdB2, RR].

It is well-known that among CY algebras, those of a-invariant 1 are fundamental in the sense that they are higher preprojective algebras [IO] of their degree 0 part [Ke6, MM, HIO, AIR]. Although our results are already non-trivial for $a = 1$, we study CY algebras of arbitrary a-invariant, which exhibits some additional symmetries.

Let R be a CY algebra. We view it as a DG algebra with vanishing differentials, which we denote by R^{dg}, and study its properties. Note that the gradings on R and on R^{dg} are of different nature; the first one is 'algebraic' while the second one is 'cohomological' (see [Y, Section 3.1, 15.1]). Such homological properties of DG algebras have been investigated, for example, in [HM, MGYC]. The following observation shows a relationship between the CY properties of R and R^{dg}. In particular, we obtain from a graded CY algebra a DG algebra that is always very close to being CY and often, in fact, is CY. We refer to Theorem 5.2 for a precise statement. Here we do not need any additional assumptions on R such as (R0), and so on, below.

Proposition 1.1 (Theorem 5.2). *Let R be a graded bimodule $(d+1)$-CY algebra of a-invariant a. Then R^{dg} is sign twisted bimodule $(d+a+1)$-CY.*

For a DG algebra Λ satisfying $\operatorname{per}\Lambda \supset \mathcal{D}^b(\Lambda)$, we set

$$\mathcal{C}(\Lambda) := \operatorname{per}\Lambda/\mathcal{D}^b(\Lambda)$$

and call it, by abuse of language, the *cluster category* of Λ. If Λ is a CY DG algebra (e.g., the derived preprojective algebra [Ke6] of a finite-dimensional algebra, or the Ginzburg DG algebra [Gi] associated to a quiver with potential), then $\mathcal{C}(\Lambda)$ is the usual cluster category introduced in [Am]. Although our DG algebra R^{dg} is not CY in general, it is close enough to CY that we can define the cluster category $\mathcal{C}(R^{\mathrm{dg}})$, which gives rise to a cluster tilting object. To understand this category, we first study the categories arising from the graded algebra R and then compare with those arising from R^{dg}.

We now assume the following on the CY algebra R:

(R0) R is negatively graded.
(R1) Each R_i is finite dimensional.

We note that the condition (R0) can be replaced by positive grading up to Theorem 1.2 below, but *negative* grading will be essential in the later discussion.

Let $\operatorname{per}^{\mathbb{Z}}R$ be the perfect derived category of R: that is, the thick subcategory of $\mathcal{D}(\operatorname{Mod}^{\mathbb{Z}}R)$ generated by the finitely generated graded projective modules. Also let $\mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}}R)$ be the bounded derived category of the category $\operatorname{fl}^{\mathbb{Z}}R$ of graded R-modules of finite length. We set

$$\operatorname{qper}^{\mathbb{Z}}R := \operatorname{per}^{\mathbb{Z}}R/\mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}}R).$$

When R is Noetherian (or, more generally, graded coherent), we have $\operatorname{per}^{\mathbb{Z}}R = \mathcal{D}^b(\operatorname{mod}^{\mathbb{Z}}R)$ since R is homologically smooth, where $\mathcal{D}^b(\operatorname{mod}^{\mathbb{Z}}R)$ is the bounded derived category of finitely presented graded R-modules $\operatorname{mod}^{\mathbb{Z}}R$. Then the Verdier quotient $\operatorname{qper}^{\mathbb{Z}}R$ is nothing but the derived category of the Serre quotient $\operatorname{qgr}R = \operatorname{mod}^{\mathbb{Z}}R/\operatorname{fl}^{\mathbb{Z}}R$ (see [Miy, Theorem 3.2]), which is regarded as the category of coherent sheaves over the non-commutative projective scheme [AZ] and plays an essential role in non-commutative algebraic geometry. Our category $\operatorname{qper}^{\mathbb{Z}}R$ is thus a generalization of the derived category $\mathcal{D}^b(\operatorname{qgr}R)$.

Our first main result is the existence of a natural cluster tilting subcategory in $\operatorname{qper}^{\mathbb{Z}}R$, which is of independent interest. More importantly, we prove that the construction of $\operatorname{qper}^{\mathbb{Z}}R$ as the Verdier quotient $\operatorname{per}^{\mathbb{Z}}R/\mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}}R)$ lies on the context of Iyama–Yang's formulation [IYa1] of Amiot's cluster category (see Theorem 2.6 and Theorem 4.6), which consequently yields a cluster tilting subcategory.

Theorem 1.2 (Theorem 4.6(4)). *Let R be a graded bimodule $(d+1)$-CY algebra of a-invariant a satisfying (R0) and (R1). Then the subcategory*

$$\mathrm{add}\{R(-i)[i] \mid i \in \mathbb{Z}\} \subset \mathrm{qper}^{\mathbb{Z}} R$$

is a $(d+a)$-cluster tilting subcategory.

For example, by setting R to be the polynomial ring with standard positive grading, we deduce that the derived category of coherent sheaves over the projective space $\mathbb{P}^d$ has a $(2d+1)$-cluster tilting subcategory $\mathrm{add}\{\mathcal{O}(i)[i] \mid i \in \mathbb{Z}\}$.

Now we compare the derived categories of the graded algebra R and the DG algebra R^{dg}. An important step is to consider the *total module* (see Section 5), which gives a DG functor

$$\mathrm{Tot}\colon \mathcal{C}_{\mathrm{dg}}(\mathrm{Mod}^{\mathbb{Z}} R) \to \mathcal{C}_{\mathrm{dg}}(R^{\mathrm{dg}})$$

on the DG categories of complexes of graded R-modules and DG R^{dg}-modules, and in turn induces a functor on the derived categories. We deduce the following result as a consequence of Theorem 1.2 above.

Corollary 1.3 (Theorem 6.1). *The functor* Tot *induces a fully faithful functor*

$$\mathrm{qper}^{\mathbb{Z}} R/(-1)[1] \to \mathcal{C}(R^{\mathrm{dg}})$$

whose image generates $\mathcal{C}(R^{\mathrm{dg}})$ as a triangulated subcategory.

This is a cluster category analogue of the result in [KY, Theorem 1.3] for the perfect derived category. Note that this gives a reasonable description of the cluster category since on $\mathcal{C}(R^{\mathrm{dg}})$, the degree shift and the suspension are identified and more accessible in the sense that derived categories are sometimes explicitly described.

Now we apply Minamoto–Mori's equivalence [MM] (see Proposition 4.9 below); there exists a triangle equivalence $\mathrm{qper}^{\mathbb{Z}} R \simeq \mathcal{D}^b(\mathrm{mod}\, A)$ for the finite-dimensional algebra

$$A = A(R) = \begin{pmatrix} R_0 & 0 & \cdots & 0 \\ R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \end{pmatrix}, \tag{1.1}$$

where $\mathrm{mod}\, A$ is the category of finitely generated modules over A and $\mathcal{D}^b(\mathrm{mod}\, A)$ is its bounded derived category. This algebra is *d-representation infinite*, which is fundamental in higher-dimensional Auslander-Reiten theory [HIO] and non-commutative algebraic geometry [Min2, MM]. By the derived equivalence above, we deduce that the autoequivalence $\nu_d = - \otimes^{\mathrm{L}}_A DA[-d]$ of $\mathcal{D}^b(\mathrm{mod}\, A)$ has an ath root $\nu_d^{1/a}$ (see equation (4.1)). Then we can rewrite Corollary 1.3 as a fully faithful functor

$$\mathcal{D}^b(\mathrm{mod}\, A)/\nu_d^{-1/a}[1] \hookrightarrow \mathcal{C}(R^{\mathrm{dg}}) \tag{1.2}$$

whose image generates $\mathcal{C}(R^{\mathrm{dg}})$ as a thick subcategory. Note that we can formally write $\nu_d^{-1/a}[1]$ as $\nu_{d+a}^{-1/a}$; thus $\mathcal{C}(R^{\mathrm{dg}})$ can be regarded as a '$\mathbb{Z}/a\mathbb{Z}$-quotient' of the $(d+a)$-cluster category of A in the sense that it is obtained from an ath root of the automorphism ν_{d+a}.

The existence of a square root of the AR translation appears in [KMV] and was important in their structure theorem for certain CY categories [KMV, Theorem 1.4]. Our result in equation (1.2) is an interpretation and a generalization of a situation of their theorem. We discuss in examples (see Example 4.13 and 6.6) how our results specialise to their setting.

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

1.2. Cluster categories and singularity categories

We further describe the cluster category as a singularity category. Recall that the *singularity category* $\mathcal{D}_{\mathrm{sg}}(\Lambda)$ of a Noetherian ring Λ is the Verdier quotient $\mathcal{D}^b(\operatorname{mod}\Lambda)/\operatorname{per}\Lambda$, which is widely studied in representation theory and algebraic geometry. If Λ is *Iwanaga-Gorenstein* in the sense that the free module Λ has finite injective dimension on the left and right, then $\mathcal{D}_{\mathrm{sg}}(\Lambda)$ is canonically equivalent to the stable category $\underline{\mathrm{CM}}\,\Lambda$ of Cohen-Macaulay modules [Bu]. In the context of cluster tilting theory, Iwanaga-Gorenstein algebras that are stably CY and admit cluster tilting objects, together with the relationship between the cluster categories, have been of particular interest [GLS, I1, KR1, IYo, Am, BIRS, AIRT, ART, KMV, IO, AO, AIR, TV].

Let a finite-dimensional algebra $A = A(R)$ as in equation (1.1) and an (A, A)-bimodule U be

$$U = U(R) = \begin{pmatrix} R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \\ R_{-a} & R_{-(a-1)} & \cdots & R_{-1} \end{pmatrix}.$$

This is a ‘relative’ d-APR tilting of A. We have a trivial extension algebra

$$B = B(R) = A \oplus U,$$

which turns out to be d-Iwanaga-Gorenstein (Proposition 6.3). Our second main result is a description of the cluster category $\mathcal{C}(R^{\mathrm{dg}})$ of a DG algebra in terms of a finite-dimensional Iwanaga-Gorenstein algebra B.

Theorem 1.4 (Theorem 6.4). *There exists a triangle equivalence*

$$\mathcal{C}(R^{\mathrm{dg}}) \simeq \mathcal{D}_{\mathrm{sg}}(B).$$

In particular, $\mathcal{D}_{\mathrm{sg}}(B)$ is a twisted $(d + a)$-CY category with a $(d + a)$-cluster tilting object.

To build the equivalence above, we need a general result on DG orbit categories (Theorem 1.5 below). Let us explain the connection, for simplicity, in the case $a = 1$.

In this case, R is bimodule $(d + 1)$-CY of a-invariant 1; thus it is the $(d + 1)$-preprojective algebra of its degree 0 part [AIR, Theorem 3.3], which is the d-representation infinite algebra A in equation (1.1). Then R^{dg} is the *derived $(d + 2)$-preprojective algebra* (or the *$(d + 2)$-CY completion*)

$$\mathbf{\Pi}_{d+2}(A) = T_A^{\mathrm{L}} \operatorname{RHom}_A(DA, A)[d + 1]$$

in the sense of [Ke6]; thus its cluster category $\mathcal{C}(R^{\mathrm{dg}})$ is the $(d+1)$-cluster category $\mathcal{C}_{d+1}(A)$ of A. (Note that by convention, our n-cluster category is n-CY.) On the other hand, we have another description of this cluster category $\mathcal{C}_{d+1}(A)$ as a certain singularity category; setting

$$C = A \oplus DA[-d - 2],$$

there exists an equivalence

$$\mathcal{C}(R^{\mathrm{dg}}) = \operatorname{per} R^{\mathrm{dg}}/\mathcal{D}^b(R^{\mathrm{dg}}) \simeq \operatorname{thick}_{\mathcal{D}(C)} A/\operatorname{per} C$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

by the relative Koszul dual ([Am, Proof of Theorem 4.10]; see also Lemma 8.3 below). Therefore, the equivalence we need is one between the singularity categories

$$\mathrm{thick}_{\mathcal{D}(C)} A/\mathrm{per}\, C \simeq \mathrm{thick}_{\mathcal{D}(B)} A/\mathrm{per}\, B = \mathcal{D}_{\mathrm{sg}}(B).$$

Note that they are precisely Keller's description of triangulated hulls [Ke2], and their equivalence is a consequence of a general equivalence of triangulated hulls, which is our third main result.

Let $\mathcal{A}$ be a pretriangulated DG category, and let F, G be DG endofunctors on $\mathcal{A}$ inducing mutually inverse equivalences on $H^0\mathcal{A}$. We then have DG orbit categories $\mathcal{A}/F$ and $\mathcal{A}/G$, whose derived categories $\mathrm{tria}(\mathcal{A}/F)$ and $\mathrm{tria}(\mathcal{A}/G)$, the smallest triangulated subcategories of $\mathcal{D}(\mathcal{A}/F)$ and $\mathcal{D}(\mathcal{A}/G)$ containing the representable modules, give triangulated hulls of $H^0\mathcal{A}/H^0F \simeq H^0\mathcal{A}/H^0G$. Our result shows that these triangulated hulls are equivalent.

Theorem 1.5 (Theorem 7.1). *Suppose there exists a natural transformation $G \circ F \to 1_{\mathcal{A}}$ inducing a natural isomorphism on $H^0\mathcal{A}$. Then the DG orbit categories $\mathcal{A}/F$ and $\mathcal{A}/G$ are quasi-equivalent. In particular, the triangulated hulls* $\mathrm{tria}(\mathcal{A}/F)$ *and* $\mathrm{tria}(\mathcal{A}/G)$ *are equivalent.*

We obtain the singular equivalence of B and C by applying this general result to $\mathcal{A} = \mathcal{C}^b(\mathrm{proj}\, A)$, $F = - \otimes_A p(\mathrm{RHom}_A(U[1], A))$ and $G = - \otimes_A p(U[1])$ (Corollary 7.6), where $p(-)$ means a bimodule projective resolution.

As one of the applications and examples of our main results, we give a realization of certain higher cluster categories as singularity categories. We assume, for simplicity, that our base field k is perfect for the following result.

Theorem 1.6 (Theorem 9.1). *Any m-cluster category of a d-representation infinite algebra with $m > d$ is a singularity category of a d-Iwanaga-Gorenstein algebra.*

For example, any (higher) cluster category of a non-Dynkin quiver is the singularity category of a 1-Iwanaga-Gorenstein algebra. Moreover, we can explicitly describe the Iwanaga-Gorenstein algebra; see Theorem 9.1 and Proposition 9.4. This should be compared with the results in [HJ], where they describe higher cluster categories of 1-representation finite algebras (or of Dynkin types) in terms of singularity categories of self-injective algebras, using a combinatorial method.

We also give systematic examples for the case R is a polynomial ring (Section 10) and consider examples arising from dimer models (Section 11).

2. Preliminaries

2.1. Cluster-like categories

We recall some basic concepts on certain structures in triangulated categories. At the end of this subsection, we state Iyama–Yang's result (Theorem 2.6), which gives a general framework for the construction of 'cluster-like' categories.

Let us start with the following fundamental notion.

Definition 2.1. An object or a subcategory $\mathcal{M}$ in a triangulated category $\mathcal{T}$ is *silting* if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[>0]) = 0$ and $\mathrm{thick}\, \mathcal{M} = \mathcal{T}$.

We assume throughout this paper that our silting subcategory $\mathcal{M}$ satisfies $\mathcal{M} = \mathrm{add}\, \mathcal{M}$: that is, it is closed under direct sums and summands. The standard example of a silting object is $\Lambda \in \mathrm{per}\, \Lambda$ for a negative DG algebra Λ: that is, a DG algebra with $H^{>0}\Lambda = 0$. The same holds for negative DG categories.

Let $\mathcal{C}$ and $\mathcal{D}$ be subcategories of a triangulated category $\mathcal{T}$. We set

$$\mathcal{C} * \mathcal{D} = \{X \in \mathcal{T} \mid \text{there is a triangle } C \to X \to D \to C[1] \text{ for some } C \in \mathcal{C},\ D \in \mathcal{D}\}.$$

By the octahedral axiom, the operation $*$ is associative. One obtains a co-t-structure (or weight structure) [Bon, P] from a silting subcategory, which is given as follows.

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

Proposition-Definition 2.2 (See, e.g., [AI, Proposition 2.23][IYa1, Proposition 2.8]). *Let $\mathcal{T}$ be an idempotent complete triangulated category with a silting subcategory $\mathcal{M}$. Set*

$$\begin{aligned} t_{\geq 0} &= \bigcup_{l\geq 0} \mathcal{M}[-l] * \cdots * \mathcal{M}[-1] * \mathcal{M}, \\ t_{\leq 0} &= \bigcup_{l\geq 0} \mathcal{M} * \mathcal{M}[1] * \cdots * \mathcal{M}[l] \end{aligned}.$$

Then $(t_{\geq 0}, t_{\leq 0})$ is a co-t-structure. We call it the co-t-structure associated to $\mathcal{M}$.

In what follows, we will simply write $t_{\geq 0} = \cdots * \mathcal{M}$, $t_{\leq 0} = \mathcal{M} * \cdots$ and so on. It is important for us that some co-t-structures and t-structures are related.

Definition 2.3 ([Bon]). Let $\mathcal{M} \subset \mathcal{T}$ be a silting subcategory and $(t_{\geq 0}, t_{\leq 0})$ the associated co-t-structure. Let $t = (t^{\leq 0}, t^{\geq 0})$ be a t-structure in $\mathcal{T}$.

1. We say t is *right adjacent to $\mathcal{M}$* if $t_{\leq 0} = t^{\leq 0}$.
2. We say t is *left adjacent to $\mathcal{M}$* if $t_{\geq 0} = t^{\geq 0}$.

For example, if Λ is a negative DG algebra that is homologically smooth such that each cohomology is finite dimensional, then the standard t-structure on $\operatorname{per}\Lambda$ is right adjacent to a silting object $\Lambda \in \operatorname{per}\Lambda$. It follows that its image under the duality $\operatorname{RHom}_\Lambda(-,\Lambda)\colon \operatorname{per}\Lambda \leftrightarrow \operatorname{per}\Lambda^{\mathrm{op}}$ is left adjacent to a silting object $\Lambda \in \operatorname{per}\Lambda^{\mathrm{op}}$.

Now let us recall the notion of (relative) Serre functors.

Definition 2.4. Let $\mathcal{T}$ be a k-linear Hom-finite triangulated category and $\mathcal{T}^{\mathrm{fd}} \subset \mathcal{T}$ a thick subcategory.

1. An autoequivalence $S\colon \mathcal{T} \to \mathcal{T}$ is a *Serre functor* on $\mathcal{T}$ if there is a functorial isomorphism
$$\operatorname{Hom}_{\mathcal{T}}(X,Y) \simeq D\operatorname{Hom}_{\mathcal{T}}(Y,SX)$$
for all $X, Y \in \mathcal{T}$.
2. A triangle autoequivalence $S\colon \mathcal{T} \to \mathcal{T}$ is a *relative Serre functor* for $\mathcal{T}^{\mathrm{fd}} \subset \mathcal{T}$ if it restricts to an autoequivalence on $\mathcal{T}^{\mathrm{fd}}$ and the above functorial isomorphism holds for all $X \in \mathcal{T}^{\mathrm{fd}}$ and $Y \in \mathcal{T}$.
3. We say that $(\mathcal{T}, \mathcal{T}^{\mathrm{fd}}, S, \mathcal{M})$ is a *relative Serre quadruple* if S is a relative Serre functor for $\mathcal{T}^{\mathrm{fd}} \subset \mathcal{T}$ and $\mathcal{M}$ is a silting subcategory of $\mathcal{T}$.

Note that we require a relative Serre functor S to be an triangle autoequivalence on $\mathcal{T}$ (as was not the case in [IYa1]). We need this for a slightly generalised formulation of their results, which we present below in Theorem 2.6. Note also that the natural isomorphism $S \circ [1] \simeq [1] \circ S$ associated to the triangle functor S is not necessarily the identity even when $S = [d+1]$; see [Ke4, Section 2.5].

We have one more notion to recall.

Definition 2.5. A k-linear idempotent-complete category $\mathcal{C}$ is a *dualising variety* if $D = \operatorname{Hom}_k(-,k)$ induces a duality $\operatorname{mod}\mathcal{C} \leftrightarrow \operatorname{mod}\mathcal{C}^{\mathrm{op}}$ between the category of finitely presented $\mathcal{C}$-modules.

For example, the category $\operatorname{proj}^{\mathbb{Z}}\Lambda$ of finitely generated graded projective modules over a finite-dimensional graded algebra Λ is a dualising variety.

We are now ready to state the following generalised formulation of Amiot's cluster category.

Theorem 2.6 ([IYa1, IYa2]). *Let $(\mathcal{T}, \mathcal{T}^{\mathrm{fd}}, S, \mathcal{M})$ be a relative Serre quadruple such that $\mathcal{M}$ is a dualising variety, and $(t_{\geq 0}, t_{\leq 0})$ be the co-t-structure associated to $\mathcal{M}$.*

1. *The silting subcategory $\mathcal{M}$ has a right-adjacent t-structure with $t_{\leq 0}^{\perp} \subset \mathcal{T}^{\mathrm{fd}}$ if and only if it has a left-adjacent t-structure with ${}^{\perp}t_{\geq 0} \subset \mathcal{T}^{\mathrm{fd}}$. Suppose in what follows that the equivalent conditions above are satisfied.*
2. *The category $\mathcal{T}/\mathcal{T}^{\mathrm{fd}}$ has a Serre functor $S \circ [-1]$.*

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

3. *The quotient functor* $\pi\colon \mathcal{T} \to \mathcal{T}/\mathcal{T}^{\mathrm{fd}}$ *induces bijections* $\mathrm{Hom}_{\mathcal{T}}(X,Y) \to \mathrm{Hom}_{\mathcal{T}/\mathcal{T}^{\mathrm{fd}}}(X,Y)$ *for all* $X \in t_{\leq 0}$ *and* $Y \in St_{\geq 2}$. *Moreover, the composition* $t_{\leq 0} \cap St_{\geq 2} \subset \mathcal{T} \to \mathcal{T}/\mathcal{T}^{\mathrm{fd}}$ *is an additive equivalence.*
4. *Let* $n \geq 1$, *and assume that* $\mathcal{M}$ *is stable under* $S_{n+1} = S \circ [-n-1]$. *Then* $\pi(\mathcal{M}) \subset \mathcal{T}/\mathcal{T}^{\mathrm{fd}}$ *is an* n*-cluster tilting subcategory.*

Proof. (1) is [IYa1, Theorem 4.10]. For (2) and (3), adapt the proof of [IYa1, Theorem 5.8(a)(b)]. We include a proof of (4) since it requires a modification from [IYa1, Theorem 5.8]. Since $\mathcal{M}$ is stable under S_{n+1}, so is $t_{\geq i}$ for each $i \in \mathbb{Z}$; thus we have $St_{\geq 2} = S_{n+1}t_{\geq 1-n} = t_{\geq 1-n} = \cdots * \mathcal{M}[n-1]$. Therefore, we deduce that the fundamental domain $t_{\leq 0} \cap St_{\geq 2}$ is $\mathcal{M} * \cdots * \mathcal{M}[n-1]$, hence the result. $\square$

If Λ is a bimodule $(n+1)$-CY negative DG algebra with finite-dimensional $H^0\Lambda$, then $(\mathrm{per}\,\Lambda, \mathcal{D}^b(\Lambda), [n+1], \Lambda)$ is a relative Serre quadruple. One can apply the above theorem and recover the original results of Amiot and Guo.

2.2. Comparing the derived categories of graded rings

We collect some basic results on derived categories of graded rings based on the comparison of derived categories of an abelian category and its Serre subcategory. Although various conditions on nice behavior for these derived categories are well-known (e.g., [Ve, III, Section 2.2]), we could not find a precise reference for our setting, so we include this section for the convenience of the reader.

Recall that a subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is *wide* if it is closed under kernels, cokernels and extensions. This ensures that $\mathcal{D}^*_{\mathcal{B}}(\mathcal{A})$, the full subcategory of $\mathcal{D}^*(\mathcal{A})$ consisting of complexes with cohomologies in $\mathcal{B}$, is a thick subcategory of $\mathcal{D}^*(\mathcal{A})$ for each boundedness condition $* \in \{b, +, -, \{\}\}$. Similarly, we denote by $\mathcal{C}^*_{\mathcal{B}}(\mathcal{A})$ the full subcategory of $\mathcal{C}^*(\mathcal{A})$ consisting of complexes with cohomologies in $\mathcal{B}$.

Proposition 2.7. *Let* $\mathcal{B} \subset \mathcal{A}$ *be a wide subcategory. Suppose that for any morphism* $Y \to Y_1$ *in* $\mathcal{C}^*(\mathcal{A})$ *with* $Y \in \mathcal{C}^*(\mathcal{B})$ *and* $Y_1 \in \mathcal{C}^*_{\mathcal{B}}(\mathcal{A})$, *there exist quasi-isomorphisms* $Y_1 \leftarrow Y_2 \to Y_3$ *such that*

- $Y_2 \to Y_1$ *is injective (in* $\mathcal{C}(\mathcal{A})$*),*
- $Y \to Y_1$ *factors through* $Y_2 \to Y_1$,
- $Y_3 \in \mathcal{C}(\mathcal{B})$,

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, dashed] & Y_1 \\ & Y_2 \arrow[u, hook] \arrow[r] & Y_3. \end{tikzcd}$$

Then the natural functor $\mathcal{D}^*(\mathcal{B}) \to \mathcal{D}^*_{\mathcal{B}}(\mathcal{A})$ *is an equivalence.*

Proof. We first show that the functor is faithful. Let $X \to Y$ be a morphism in $\mathcal{K}^*(\mathcal{B})$ whose image in $\mathcal{D}^*(\mathcal{A})$ is 0. Then there is a quasi-isomorphism $Y \to Y_1$ in $\mathcal{K}^*(\mathcal{A})$ such that the composite $X \to Y \to Y_1$ is 0 in $\mathcal{K}^*(\mathcal{A})$. Adding to Y a null-homotopic complex $N \in \mathcal{C}^*(\mathcal{B})$, for example the mapping cone of the identity map of X, we have a zero map $X \to Y \oplus N \to Y_1$ in $\mathcal{C}^*(\mathcal{A})$. Now, applying the assumption to the quasi-isomorphism $Y \oplus N \to Y_1$, there exist quasi-isomorphisms $Y_1 \hookleftarrow Y_2 \to Y_3$ satisfying the above conditions, giving rise to the following commutative diagram in $\mathcal{C}(\mathcal{A})$:

$$\begin{tikzcd} X \arrow[r] & Y \oplus N \arrow[r, "\mathrm{qis}"] \arrow[dr, dashed] & Y_1 \\ & & Y_2 \arrow[u, hook, "\mathrm{qis}"'] \arrow[r, "\mathrm{qis}"] & Y_3 \end{tikzcd}$$

Clearly, the composite $Y \oplus N \to Y_2 \to Y_3$ is a quasi-isomorphism. Also, since $X \to Y \oplus N \to Y_1$ is 0 and $Y_2 \to Y_1$ is injective, it follows that $X \to Y \oplus N \to Y_3$ is 0. By $Y_3 \in \mathcal{C}(\mathcal{B})$, we conclude that $X \to Y$ is 0 in $\mathcal{D}(\mathcal{B})$.

We next show that the functor is full. Let $X, Y \in \mathcal{C}^*(\mathcal{B})$, and present a morphism $X \to Y$ in $\mathcal{D}^*(\mathcal{A})$ by the diagram $X \to Y_1 \leftarrow Y$ in $\mathcal{C}^*(\mathcal{A})$. Applying the assumption to $X \oplus Y \to Y_1$, we have quasi-isomorphisms $Y_1 \leftarrow Y_2 \to Y_3$, giving rise to the commutative diagram

$$\begin{tikzcd}
 & Y_1 & \\
X \arrow[ur] \arrow[r, dashed] & Y_2 \arrow[u, "\mathrm{qis}"] \arrow[d, "\mathrm{qis}"] & Y \arrow[ul, "\mathrm{qis}"'] \arrow[l, dashed] \\
 & Y_3 & .
\end{tikzcd}$$

This shows that the diagram $X \to Y_3 \leftarrow Y$ in $\mathcal{C}(\mathcal{B})$ defines the same morphism in $\mathcal{D}(\mathcal{A})$ as $X \to Y_1 \leftarrow Y$.

Finally, noting that the image of $Y_3 \in \mathcal{C}(\mathcal{B})$ in $\mathcal{D}(\mathcal{B})$ lies in $\mathcal{D}^*(\mathcal{B})$, we see that the functor is dense by setting $Y = 0$ in the assumption. □

Applying the above criterion, we can prove the following, which will be implicitly used throughout this paper.

Proposition 2.8. *Let $R = \bigoplus_{i \leq 0} R_i$ be a negatively graded algebra such that each R_i is finite dimensional. Then the natural functor $\mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R) \to \mathcal{D}^b_{\mathrm{fl}^{\mathbb{Z}} R}(\mathrm{Mod}^{\mathbb{Z}} R)$ is an equivalence.*

Proof. We check the condition in Proposition 2.7 with $\mathcal{B} = \mathrm{fl}^{\mathbb{Z}} R$ and $\mathcal{A} = \mathrm{Mod}^{\mathbb{Z}} R$. Let $Y \in \mathcal{C}^b(\mathrm{fl}^{\mathbb{Z}} R)$, $Y_1 \in \mathcal{C}^b_{\mathrm{fl}^{\mathbb{Z}} R}(\mathrm{Mod}^{\mathbb{Z}} R)$ and $Y \to Y_1$ be a morphism in $\mathcal{C}^b(\mathrm{Mod}^{\mathbb{Z}} R)$. Since the terms of Y are of finite length, the morphism $Y \to Y_1$ factors through $(Y_1)_{\leq n} \to Y_1$ for sufficiently large n. Also, since the cohomology of Y_1 is bounded and each is of finite length, the truncation $(Y_1)_{\leq n} \to Y_1$ with respect to the grading is a quasi-isomorphisms for sufficiently large n. Taking the commonly large n, we obtain an injective quasi-isomorphism $Y_2 := (Y_1)_{\leq n} \to Y_1$ through which $Y \to Y_1$ factors. Similarly, the truncation $Y_2 \to (Y_2)_{\geq -m}$ is a quasi-isomorphism for sufficiently large m. Put $Y_3' := (Y_2)_{\geq -m}$ for such m. Then each term of Y_3' is concentrated in degree $[-m, n]$; thus it is a module over the finite-dimensional factor algebra $R_{\geq -n-m}$ of R. Now we work over this finite-dimensional algebra. Then one can take quasi-isomorphism $Y_3' \to Y_3$ with $Y_3 \in \mathcal{C}^{+,b}(\mathrm{mod}^{\mathbb{Z}} R_{\geq -n-m})$ (e.g., with injective terms), and we have $Y_3 \in \mathcal{C}(\mathrm{fl}^{\mathbb{Z}} R)$. □

3. *t*-structure in $\mathrm{per}^{\mathbb{Z}} R$

We will be interested in a negatively graded CY algebra with an a-invariant. Before that, we will place ourselves in a slightly more general setting. Let $R = \bigoplus_{i \leq 0} R_i$ be a negatively graded algebra. We assume the following on R:

(R1) Each R_i is finite dimensional.
(R2) $\mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R)$ is contained in $\mathrm{per}^{\mathbb{Z}} R$.

The condition (R2) is clearly satisfied if R is homologically smooth. The aim of this section is to show that there is a t-structure in $\mathrm{per}^{\mathbb{Z}} R$ in this setting. For each $j \in \mathbb{Z}$, we denote by $\mathrm{Mod}^{\leq j} R$ (respectively, $\mathrm{Mod}^{\geq j} R$) the full subcategory of $\mathrm{Mod}^{\mathbb{Z}} R$ consisting of graded modules concentrated in degree $\leq j$ (respectively, $\geq j$).

Theorem 3.1. *Let R be a negatively graded algebra satisfying (R1) and (R2). Set*

$$t^{\leq 0} = \{X \in \mathrm{per}^{\mathbb{Z}} R \mid H^i(X) \in \mathrm{Mod}^{\leq -i} R \text{ for all } i \in \mathbb{Z}\},$$
$$t^{\geq 0} = \{X \in \mathrm{per}^{\mathbb{Z}} R \mid H^i(X) \in \mathrm{Mod}^{\geq -i} R \text{ for all } i \in \mathbb{Z}\}.$$

Then $(t^{\leq 0}, t^{\geq 0})$ is a t-structure in $\mathrm{per}^{\mathbb{Z}} R$.

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

Remark 3.2. Under the assumption that R is Noetherian (or, more generally, graded coherent), there is a version of this for $\mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$ without the 'smoothness' condition (R2), which is in practice far more general than Theorem 3.1 above. We give this general result in Appendix B.

Let us collect some basic notions for big triangulated categories. Let $\mathcal{T}$ be a triangulated category with arbitrary (set-indexed) coproducts. An object $C \in \mathcal{T}$ is *compact* if the natural map $\bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{T}}(C, X_i) \to \mathrm{Hom}_{\mathcal{T}}(C, \bigoplus_{i\in I} X_i)$ is an isomorphism for every set $\{X_i \mid i \in I\}$ of objects. A subcategory is said to *generate* $\mathcal{T}$ if the smallest triangulated subcategory containing it and closed under arbitrary coproducts is the whole $\mathcal{T}$. We call a subcategory of $\mathcal{T}$ a *compact set of generators* if it is skeletally small, consists of compact objects and generates $\mathcal{T}$.

Now recall from [AI, Definition 4.1] that a subcategory $\mathcal{S}$ of $\mathcal{T}$ is *silting* if it forms a compact set of generators such that $\mathrm{Hom}_{\mathcal{T}}(A, B[>0]) = 0$ for all $A, B \in \mathcal{S}$. Note that this is a modified version of Definition 2.1.

In the remainder of this section, let S be an arbitrary negatively graded algebra. Note that we do not need (R1) or (R2) until the proof of Theorem 3.1. We will simply write $\mathcal{D}$ for $\mathcal{D}(\mathrm{Mod}^{\mathbb{Z}} S)$. Let us start our discussion with the following observation.

Proposition 3.3. *The subcategory $\mathcal{M} = \mathrm{add}\{S(-i)[i] \mid i \in \mathbb{Z}\} \subset \mathcal{D}$ is silting.*

Proof. Clearly, $\mathcal{M}$ is a compact set of generators for $\mathcal{D}$. It remains to show that $\mathrm{Hom}_{\mathcal{D}}(S, S(-i)[i][j])$ vanishes for each $i \in \mathbb{Z}$ and $j > 0$. We only have to consider the case $i = -j < 0$, in which case $\mathrm{Hom}_{\mathcal{D}}(S, S(-i)[i][j]) = \mathrm{Hom}_{\mathrm{Mod}^{\mathbb{Z}} S}(S, S(j)) = 0$ since S is negatively graded. □

We deduce by [Ke1, Theorem 4.3] that $\mathcal{D}$ is triangle equivalent to the derived category $\mathcal{D}(\mathcal{A})$ of a negative DG category $\mathcal{A}$. Then we have the standard t-structure $(\mathcal{D}^{\leq 0}_{\mathcal{M}}, \mathcal{D}^{\geq 0}_{\mathcal{M}})$ associated to $\mathcal{M}$, which is given by

$$\mathcal{D}^{\leq 0}_{\mathcal{M}} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(M, X[i]) = 0 \text{ for all } M \in \mathcal{M} \text{ and } i > 0.\},$$
$$\mathcal{D}^{\geq 0}_{\mathcal{M}} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(M, X[i]) = 0 \text{ for all } M \in \mathcal{M} \text{ and } i < 0.\}.$$

As usual, we put $\mathcal{D}^{\leq n}_{\mathcal{M}} = \mathcal{D}^{\leq 0}_{\mathcal{M}}[-n]$ and $\mathcal{D}^{\geq n}_{\mathcal{M}} = \mathcal{D}^{\geq 0}_{\mathcal{M}}[-n]$ for each $n \in \mathbb{Z}$.

Now we use the following computation.

Lemma 3.4. *We have*

$$\mathcal{D}^{\leq 0}_{\mathcal{M}} = \{X \in \mathcal{D} \mid H^i(X) \in \mathrm{Mod}^{\leq -i} S \textit{ for all } i \in \mathbb{Z}\},$$
$$\mathcal{D}^{\geq 0}_{\mathcal{M}} = \{X \in \mathcal{D} \mid H^i(X) \in \mathrm{Mod}^{\geq -i} S \textit{ for all } i \in \mathbb{Z}\}.$$

Proof. Since $\mathcal{D}^{\leq 0}_{\mathcal{M}} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(M[i], X) = 0 \text{ for all } M \in \mathcal{M} \text{ and } i < 0\}$, we have

$$\begin{aligned}\mathcal{D}^{\leq 0}_{\mathcal{M}} &= \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(S(i)[-i][-j], X) = 0 \text{ for all } i \in \mathbb{Z} \text{ and } j > 0\}\\ &= \{X \in \mathcal{D} \mid H^{i+j}(X)_{-i} = 0 \text{ for all } i \in \mathbb{Z} \text{ and } j > 0\}\\ &= \{X \in \mathcal{D} \mid H^{i}(X)_{-i+j} = 0 \text{ for all } i \in \mathbb{Z} \text{ and } j > 0\},\end{aligned}$$

thus the first assertion. By $\mathcal{D}^{\geq 0}_{\mathcal{M}} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(M[i], X) = 0 \text{ for all } M \in \mathcal{M} \text{ and } i > 0\}$, we similarly have the second equation. □

We need one lemma to ensure that the above t-structure in $\mathcal{D}$ restricts to the small derived category.

Lemma 3.5. *Let $X' \to X \to X'' \to X'[1]$ be a triangle in $\mathcal{D}$ with $X' \in \mathcal{D}^{\leq 0}_{\mathcal{M}}$, $X'' \in \mathcal{D}^{\geq 1}_{\mathcal{M}}$, and let $i \in \mathbb{Z}$.*

1. *The triangle induces a short exact sequence*

$$0 \longrightarrow H^i X' \longrightarrow H^i X \longrightarrow H^i X'' \longrightarrow 0.$$

2. *The above exact sequence is isomorphic to the truncation*

$$0 \longrightarrow (H^i X)_{\leq -i} \longrightarrow H^i X \longrightarrow (H^i X)_{>-i} \longrightarrow 0$$

of $H^i X \in \mathrm{Mod}^{\mathbb{Z}} S$ *with respect to the grading.*

Proof. (1) It is enough to show that the connecting homomorphism $H^i X'' \to H^{i+1} X'$ is 0 for each $i \in \mathbb{Z}$. Since $X' \in \mathcal{D}^{\leq 0}_{\mathcal{M}}$ and $X'' \in \mathcal{D}^{\geq 1}_{\mathcal{M}}$, we have $H^i X'' \in \mathrm{Mod}^{\geq -i+1} S$ and $H^{i+1} X' \in \mathrm{Mod}^{\leq -i-1} S$, hence the assertion.

(2) Similarly, we have $H^i X' \in \mathrm{Mod}^{\leq -i} S$ and $H^i X'' \in \mathrm{Mod}^{\geq -i+1} S$; thus the exact sequence in (1) has to be the truncation of $H^i X$ with respect to the grading. □

We are now ready to prove the main result.

Proof of Theorem 3.1. We show that the above t-structure in $\mathcal{D} = \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}} R)$ restricts to $\mathrm{per}^{\mathbb{Z}} R$. Let $X \in \mathrm{per}^{\mathbb{Z}} R$. We have to show that its truncation X', X'' in Lemma 3.5 are perfect. We may replace X by a bounded complex of finitely generated graded projective R-modules. Then we have that $H^i X = 0$ for almost all i and that each $H^i X \in \mathrm{Mod}^{\mathbb{Z}} R$ is bounded above. Moreover, by assumption (R1), each vector space $(H^i X)_j$ is finite dimensional. Then by Lemma 3.5(1), the cohomology $H^i X''$ is 0 for almost all i, and by Lemma 3.5(2), each $H^i X'' \in \mathrm{Mod}^{\mathbb{Z}} R$ is bounded below. Therefore, $H^i X''$ lies in $\mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R)$, hence in $\mathrm{per}^{\mathbb{Z}} R$ by (R2). We conclude that the remaining term X' is also perfect. □

4. Cluster tilting in $\mathrm{qper}^{\mathbb{Z}} R$ and the ath root of the AR translation

4.1. Cluster tilting

Let us first recall the notion of *(twisted) Calabi-Yau algebras*, in the graded case, which is of our central interest. Let α be a graded automorphism of a graded ring Λ. We say that α is *inner* if there exists a homogeneous invertible element $a \in \Lambda$ such that $\alpha(m) = ama^{-1}$ for all $m \in \Lambda$. We denote by $(-)_\alpha$ the twist automorphism on $\mathrm{Mod}^{\mathbb{Z}} \Lambda$; thus for each $M \in \mathrm{Mod}^{\mathbb{Z}} \Lambda$, the graded module M_α has the same underlying graded vector space as M, with Λ action $m \cdot x = m\alpha(x)$ for $m \in M$ and $x \in \Lambda$. Similarly, given two graded automorphisms α, β of Λ and a graded bimodule M, we denote by ${}_\alpha M_\beta$ the twisted graded bimodule that has Λ-action $x \cdot m \cdot y = \alpha(x) m \beta(y)$ for $m \in M$ and $x, y \in \Lambda$. For example, the notation ${}_\alpha M_1$ shows that the action is twisted by α on the left, while it is non-twisted on the right.

Definition 4.1. A graded algebra R is *bimodule twisted n-Calabi-Yau of a-invariant a* if it satisfies the following conditions:

- R is homologically smooth: that is, $R \in \mathrm{per}^{\mathbb{Z}} R^e$.
- There exists a graded automorphism α of R such that $\mathrm{RHom}_{R^e}(R, R^e)(a)[n] \simeq {}_\alpha R_1$ in $\mathcal{D}(\mathrm{Mod}^{\mathbb{Z}} R^e)$.

We refer to α as the *Nakayama automorphism*, which is uniquely determined up to inner automorphism. We say that R is *Calabi-Yau* if α is inner.

Let R be a negatively graded bimodule twisted n-CY algebra of a-invariant a with Nakayama automorphism α. We moreover assume that each R_i is finite-dimensional over k.

Let us first note a basic fact on the derived categories of R.

Proposition 4.2. $\mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R) \subset \mathrm{per}^{\mathbb{Z}} R$ *has a relative Serre functor* $(-)_\alpha(a)[n]$.

Proof. This follows by adapting [Ke4, Lemma 4.1]. Note that all the isomorphisms appear in the proof, which therein preserves gradings when the relevant objects are graded, so the argument carries over. □

It follows using this relative Serre duality that R is *Artin-Schelter regular over R_0 of dimension n and a-invariant a* in the sense of [MM, Definition 3.1]: that is, we have $\mathrm{gl.dim}\, R = n$, $\mathrm{gl.dim}\, R_0 < \infty$ and

$$\mathrm{Ext}^i_R(R_0, R) = \begin{cases} (DR_0)(-a) & (i = n) \\ 0 & (i \neq n) \end{cases}$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

in $\mathrm{Mod}^{\mathbb{Z}}R$ and in $\mathrm{Mod}^{\mathbb{Z}}R^{\mathrm{op}}$. Note that we can get rid of the Nakayama automorphism when restricted to one-sided modules. Note also that since R is negatively graded, the condition $\mathrm{gl.dim}\,R_0 < \infty$ is automatic by the following observation.

Lemma 4.3. *Let $R = \bigoplus_{i\leq 0} R_i$ be a negatively graded ring. If R is homologically smooth, then so is R_0.*

Proof. We can pick a bimodule projective resolution of R whose terms are concentrated in degree ≤ 0. Taking its degree 0 part yields a bimodule projective resolution of R_0. □

Let us also note the following information on the a-invariant of homologically smooth algebras.

Lemma 4.4. *Let $R = \bigoplus_{i\leq 0} R_i$ be a negatively graded, bimodule twisted n-CY algebra of a-invariant a such that each R_i is finite dimensional. If $n > 0$, then we have $a > 0$.*

Proof. Let $0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to R_0 \to 0$ be the minimal projective resolution of R_0 in $\mathrm{Mod}^{\mathbb{Z}}R$. We have $Q_0 = R$, and since R is negatively graded, the remaining terms are generated in degree < 0: that is, $Q_l \in \mathrm{add}\{R(i) \mid i > 0\}$ for $l > 0$. On the other hand, we have $Q_n \in \mathrm{add}\,R(a)$ by [MM, Proposition 3.4(3)]. This forces $a > 0$ for $n > 0$. □

Now we return to our original discussion. Let R be a negatively graded bimodule twisted $(d+1)$-CY algebra of a-invariant a. We need the following reformulation of Proposition 3.3.

Proposition 4.5. *$\mathcal{M} = \mathrm{add}\{R(-i)[i] \mid i \in \mathbb{Z}\}$ is a silting subcategory of $\mathrm{per}^{\mathbb{Z}}R$.*

Proof. We have seen in Proposition 3.3 that $\mathcal{M}$ has no positive self-extensions. Also, we clearly have $\mathrm{thick}\,\mathcal{M} = \mathrm{per}^{\mathbb{Z}}R$. □

Therefore, we obtain a relative Serre quadruple $(\mathrm{per}^{\mathbb{Z}}R, \mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}}R), (-)_\alpha(a)[d+1], \mathcal{M})$. The first main result of this paper is that this lies on a context of Theorem 2.6. This also explains the well-known fact (2) on the Serre functor on the derived category of non-commutative projective spaces.

Theorem 4.6. *Let $d \geq 0$, and let R be a negatively graded bimodule twisted $(d+1)$-CY algebra of a-invariant a such that each R_i is finite dimensional.*

1. *$\mathcal{M}$ is a dualising variety with left- and right-adjacent t-structures.*
2. *$\mathrm{qper}^{\mathbb{Z}}R$ has a Serre functor $(-)_\alpha(a)[d]$.*
3. *The quotient functor $\pi\colon \mathrm{per}^{\mathbb{Z}}R \to \mathrm{qper}^{\mathbb{Z}}R$ induces bijections $\mathrm{Hom}_{\mathrm{per}^{\mathbb{Z}}R}(X,Y) \to \mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}}R}(X,Y)$ for each $X \in \mathcal{M}*\cdots$ and $Y \in \cdots*\mathcal{M}[d+a-1]$. Moreover, the composition $\mathcal{M}*\cdots*\mathcal{M}[d+a-1] \subset \mathrm{per}^{\mathbb{Z}}R \xrightarrow{\pi} \mathrm{qper}^{\mathbb{Z}}R$ is an equivalence.*
4. *$\pi(\mathcal{M}) = \mathrm{add}\{R(-i)[i] \mid i \in \mathbb{Z}\} \subset \mathrm{qper}^{\mathbb{Z}}R$ is a $(d+a)$-cluster tilting subcategory.*

In the proof below, we write $\mathcal{T} = \mathrm{per}^{\mathbb{Z}}R$ and $\mathcal{T}^{\mathrm{fd}} = \mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}}R)$.

Proof. (1) Since $\mathcal{M} = \mathrm{add}\{R(-i)[i] \mid i \in \mathbb{Z}\}$, we have $\mathcal{M} \simeq \mathrm{proj}^{\mathbb{Z}}\Lambda$ with $\Lambda = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(R, R(-i)[i]) = R_0$, hence $\mathcal{M}$ is a dualising variety.

We next show that the silting subcategory $\mathcal{M} \subset \mathcal{T}$ has left- and right-adjacent t-structures. By Theorem 2.6(1), it suffices to show the existence of the right-adjacent t-structure with $t_{\leq 0}^{\perp} \subset \mathcal{T}^{\mathrm{fd}}$. Set $t^{\leq 0} := t_{\leq 0}$ and $t^{\geq 0} := (t^{\leq -1})^{\perp}$, where $t^{\leq n} = t^{\leq 0}[-n]$ and so on. Note that $t^{\leq 0} = \mathcal{M}[<0]^{\perp}$ and $t^{\geq 0} = (t^{\leq -1})^{\perp} = (t_{\leq -1})^{\perp} = \mathcal{M}[>0]^{\perp}$. Then as in Lemma 3.4, we have

$$t^{\leq 0} = \{X \in \mathcal{T} \mid H^i(X) \in \mathrm{Mod}^{\leq -i}R \text{ for all } i \in \mathbb{Z}\},$$
$$t^{\geq 0} = \{X \in \mathcal{T} \mid H^i(X) \in \mathrm{Mod}^{\geq -i}R \text{ for all } i \in \mathbb{Z}\}.$$

Now the assertion that $(t^{\leq 0}, t^{\geq 0})$ is a t-structure is precisely what we showed in Theorem 3.1, and clearly $t^{\geq 0} \subset \mathcal{T}^{\mathrm{fd}}$.

(2) This follows from Theorem 2.6(2).

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

(3)(4) Let $S = (-)_\alpha(a)[d+1]$ be the relative Serre functor for $\mathcal{T}^{\mathrm{fd}} \subset \mathcal{T}$. Then $S_{d+a+1} = (-)_\alpha(a)[-a]$ preserves $\mathcal{M}$. Therefore, we have $St_{\geq 2} = S_{d+a+1}t_{\geq -d-a+1} = t_{\geq -d-a+1}$, hence (3) by Theorem 2.6(3), and (4) by Theorem 2.6(4). $\square$

We end this subsection with the following computation.

Lemma 4.7. *The natural map* $\mathrm{Hom}_{\mathrm{per}^{\mathbb{Z}}R}(R, R(i)) \to \mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}}R}(R, R(i))$ *is an isomorphism whenever* $d > 0$ *or* $i < a$.

Proof. Note that $R(a)[b] \in \mathcal{M}[a+b]$ for each $a, b \in \mathbb{Z}$. Then, using Theorem 4.6(3), we have the assertion for $i \leq d+a-1$, so it remains to consider $i \geq d+a$. Let, more generally, $i > 0$. Then the left-hand side is clearly 0. By Serre duality, we have $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}}R}(R, R(i)) = D\,\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}}R}(R(i), R(a)[d])$ with $R(i) \in \mathcal{M}[1] * \cdots$ and $R(a)[d] \in \mathcal{M}[d+a]$. Therefore, Theorem 4.6(3) shows that this is $D\,\mathrm{Hom}_{\mathrm{per}^{\mathbb{Z}}R}(R(i), R(a)[d])$, which vanishes when $d > 0$ or $i < a$. $\square$

4.2. *Tilting and the ath root of the AR-translation*

In this subsection, we note a result due to Minamoto–Mori [MM] and give a finite-dimensional algebra A, which will play a crucial role in the sequel. Before that, let us recall the following notion.

Definition 4.8 ([HIO]). A finite-dimensional algebra Λ is *d-representation infinite* if $\mathrm{gl.dim}\,\Lambda \leq d$ and

$$\nu_d^{-i}\Lambda \in \mathrm{mod}\,\Lambda$$

holds for all $i \geq 0$, where ν_d is the autoequivalence $-\otimes^{\mathrm{L}}_{\Lambda} D\Lambda[-d]$ on $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$.

Note that we allow $d = 0$ and understand '0-representation infinite' algebras as being semisimple.

Proposition 4.9 (compare [MM, Theorem 4.12]). *Let R be a negatively graded bimodule twisted $(d+1)$-CY algebra of a-invariant a such that each R_i is finite dimensional.*

1. $T = \bigoplus_{l=0}^{a-1} R(-l)$ *is a tilting object in* $\mathrm{qper}^{\mathbb{Z}}R$.
2. $A = \mathrm{End}_{\mathrm{qper}^{\mathbb{Z}}R}(T)$ *is d-representation infinite.*

Therefore, there exists a triangle equivalence $\mathrm{qper}^{\mathbb{Z}}R \simeq \mathcal{D}^b(\mathrm{mod}\,A)$.

This is a compact version of [MM, Theorem 4.12] as well as a generalization of [MM, Theorem 4.14] to non-coherent algebras. We will give a proof using Theorem 4.6 in Appendix C.

We are now in the position to state the following important consequence. Suppose in what follows that $(-)_\alpha \simeq 1$ on $\mathrm{qper}^{\mathbb{Z}}R$: for example, that R is CY. Let A be the d-representation-infinite algebra given in Proposition 4.9, and let F be the autoequivalence on $\mathcal{D}^b(\mathrm{mod}\,A)$, making the diagram below commutative:

$$\begin{tikzcd} \mathrm{qper}^{\mathbb{Z}}R \arrow[r, "\simeq"] \arrow[d, "(1)"'] & \mathcal{D}^b(\mathrm{mod}\,A) \arrow[d, "F"] \\ \mathrm{qper}^{\mathbb{Z}}R \arrow[r, "\simeq"] & \mathcal{D}^b(\mathrm{mod}\,A). \end{tikzcd} \tag{4.1}$$

Corollary 4.10. *We have $F^a = \nu_d$ as autoequivalences of $\mathcal{D}^b(\mathrm{mod}\,A)$.*

Proof. Comparing the Serre functors on $\mathrm{qper}^{\mathbb{Z}}R \simeq \mathcal{D}^b(\mathrm{mod}\,A)$, the autoequivalences (a) on $\mathrm{qper}^{\mathbb{Z}}R$ and ν_d on $\mathcal{D}^b(\mathrm{mod}\,A)$ are compatible, hence we obtain the desired result. $\square$

We can therefore regard F as an ath root of the d-AR translation ν_d and denote $F =: \nu_d^{1/a}$ and also $F^{-1} =: \nu_d^{-1/a}$.

Let us give some easy examples of ath roots of the AR translation; see Lemma 10.1 for the CY property of polynomial rings.

Example 4.11. Let $R = k[x, y]$ with $\deg x = \deg y = -1$, so R is 2-CY of a-invariant 2. Applying Proposition 4.9, we have a well-known equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{mod} A)$ with A the Kronecker algebra. The AR-quiver of this category has a component that looks like

$$\begin{tikzcd}
 & R(1) \arrow[dr, shift left] \arrow[dr, shift right] & & R(-1) \arrow[dr, shift left] \arrow[dr, shift right] & & R(-3) \arrow[dr, shift left] \arrow[dr, shift right] & \\
\cdots \arrow[ur, shift left] \arrow[ur, shift right] & & R \arrow[ur, shift left] \arrow[ur, shift right] & & R(-2) \arrow[ur, shift left] \arrow[ur, shift right] & & \cdots .
\end{tikzcd}$$

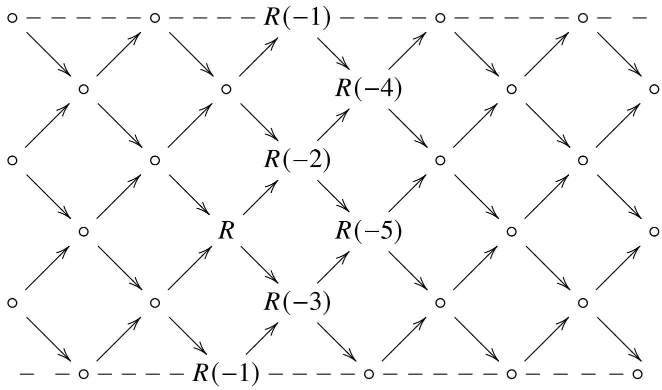
By diagram (4.1) above, this shows that $\nu_1^{-1/2}$ on $\mathcal{D}^b(\operatorname{mod} A)$ acts by 'moving one place to the right'.

We next look at a higher root of the AR translation.

Example 4.12. Let $R = k[x, y]$ with $\deg x = -2$ and $\deg y = -3$, so R is 2-CY of a-invariant 5. By Proposition 4.9, there is a triangle equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{mod} A)$, where A is the path algebra over k of the following quiver of type $\widetilde{A_4}$:

$$\begin{tikzcd}
 & 0 \arrow[ddl] \arrow[ddr] & \\
4 & & 1 \arrow[ll] \arrow[dll] \\
3 & & 2 \arrow[ull]
\end{tikzcd},$$

with the vertex i corresponding to the summand $R(-i)$. By the triangle equivalence, we see that the AR-quiver of the triangulated category $\mathcal{D}^b(\operatorname{qgr} R)$ has the following connected component:



,

where the horizontal ends are identified. We see that $\nu_1^{-1/5} = (-1)$ acts on this component by 'moving one place down'.

We refer to Section 10 for more general examples for polynomial rings.

The existence of a square root of the AR translation appears in [KMV] for generalised Kronecker quivers. We show in the following example how to recover their context.

Example 4.13. Let $m \geq 2$, and set

$$R = k\langle x_1, \ldots, x_m\rangle/(x_1^2 + \cdots + x_m^2), \quad \deg x_i = -1.$$

This is a non-Noetherian Artin-Schelter regular algebra of dimension 2 (see [Z]) and is graded coherent (see [MM, Theorem 4.16]). We need the following bimodule resolution of R.

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

Lemma 4.14. *The complex*

$$0 \longrightarrow R \otimes R(2) \xrightarrow{d_2} \bigoplus_{i=1}^{m} R \otimes R(1) \xrightarrow{d_1} R \otimes R \xrightarrow{\mu} R \longrightarrow 0$$

with maps

$$\begin{aligned} \mu(1 \otimes 1) &= 1 \\ d_1((1 \otimes 1)_i) &= x_i \otimes 1 - 1 \otimes x_i \\ d_2(1 \otimes 1) &= (x_i \otimes 1 + 1 \otimes x_i)_i \end{aligned}$$

gives a bimodule projective resolution of R.

Proof. The multiplication map μ is clearly surjective, and since R is generated by $x_1, \ldots, x_m$, it is exact at $R \otimes R$. We next prove the exactness at $R \otimes R(2)$: that is, d_2 is injective. For this, we look at each graded component and show that $(R \otimes R)_{-n} \to \bigoplus_{i=1}^{m}(R \otimes R)_{-n-1}$ is injective for all $n \geq 0$. Let $\sum_{p+q=n} a_{pq} \in (R \otimes R)_{-n}$ with $a_{pq} \in R_{-p} \otimes R_{-q}$ be a non-zero element and let p be the smallest index such that $a_{pq} \neq 0$. Note that $R_{-p} \otimes R_{-q}$ is mapped into the direct sum of $(R_{-p} \otimes R_{-q-1}) \oplus (R_{-p-1} \otimes R_{-q})$; thus a_{pq} is the component with only possible p, which has non-zero image in $\bigoplus_{i=1}^{m} R_{-p} \otimes R_{-q-1}$. Now, this map $R_{-p} \otimes R_{-q} \to \bigoplus_{i=1}^{m} R_{-p} \otimes R_{-q-1}$ is just $f \otimes g \mapsto (f \otimes x_i g)_i$, which is injective by [Min1, Lemma 2.2]. It follows that the $\bigoplus_{i=1}^{m}(R_{-p} \otimes R_{-q-1})$-component of $d_2(a_{pq})$ is non-zero; thus d_2 is injective. It remains to prove the acyclicity at $\bigoplus_{i=1}^{m} R \otimes R(1)$. It suffices to show that the terms have 'correct dimensions': that is, putting $d_n = \dim R_{-n}$, we have $\sum_{p+q=n} d_p d_q - m \sum_{p+q=n+1} d_p d_q + \sum_{p+q=n+2} d_p d_q - d_{n+2} = 0$. This is easily seen by using $d_{n+2} = m d_{n+1} - d_n$ (see [Min1, Lemma 2.2]). □

Applying $\operatorname{Hom}_{R^e}(-, R^e)$ to the above complex, we deduce that R is graded bimodule twisted 2-CY of a-invariant 2 with Nakayama automorphism $\sigma\colon x_i \mapsto -x_i$.

By Proposition 4.9, we have a derived equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{mod} A)$ with $A = \begin{pmatrix} R_0 & 0 \\ R_{-1} & R_0 \end{pmatrix}$, which is the path algebra of the m-Kronecker quiver $Q_m = (\circ \xrightarrow{m} \circ)$. Now the twist automorphism $(-)_\sigma$ is isomorphic to the identity functor on $\operatorname{Mod}^{\mathbb{Z}} R$, so we have an autoequivalence $\nu_1^{-1/2}$ on $\mathcal{D}^b(\operatorname{mod} kQ_m)$. The AR quiver of the derived category has a connected component

$$\begin{tikzcd}[column sep=small] & R(1) \arrow[dr, "m"] & & R(-1) \arrow[dr, "m"] & & R(-3) \arrow[dr, "m"] & \\ \cdots \arrow[ur, "m"] & & R \arrow[ur, "m"] & & R(-2) \arrow[ur, "m"] & & \cdots . \end{tikzcd}$$

We see that $\nu_1^{-1/2}$ acts by 'moving one place to the right'; compare Example 4.11.

5. CY algebras as DG algebras

We will consider a graded algebra R as a DG algebra with the same underlying graded ring and the vanishing differential. We write R^{dg} when considering R as a DG algebra.

We first collect some sign conventions that are heavily used in this section. Throughout this section, we denote by $|x|$ the degree of a homogeneous element x in a graded vector space.

Convention 5.1. Let Λ and Γ be DG algebras.

1. Let X be a DG *right* Λ-module. Then its shift $X[1]$ has the same right Λ-action as X.
2. Let X be a DG *left* Λ-module. Then its shift $X[1]$ has a left Λ-action $a \cdot x = (-1)^{|a|} ax$ for $a \in \Lambda$ and $x \in X[1]$.
3. There is an isomorphism $\mathcal{H}om_\Lambda(\Lambda[-l], \Lambda) \simeq \Lambda[l]$ of DG left Λ-modules by $f \mapsto (-1)^{l|f|} f(1)$.

4. We let $\Lambda^e = \Lambda^{\mathrm{op}} \otimes \Lambda$ the enveloping algebra and identify $(\Lambda^e)^{\mathrm{op}} \simeq \Lambda^e$ via $x \otimes y \leftrightarrow (-1)^{|x||y|} y \otimes x$.
5. We identify a DG (Λ, Γ)-bimodule X and a DG $\Lambda^{\mathrm{op}} \otimes_k \Gamma$-module via $\lambda \cdot x \cdot \gamma = (-1)^{|\lambda||x|} x \cdot (\lambda \otimes \gamma)$.

We say that a DG algebra Λ is *twisted bimodule n-CY* if it is homologically smooth and there exists a DG automorphism σ of Λ such that we have an isomorphism $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)[n] \simeq {}_1\Lambda_\sigma$ in $\mathcal{D}(\Lambda^e)$. The aim of this section is to note the following observation. Note that the term 'CY algebra' means precisely as in Definition 4.1, and *no* additional conditions are imposed.

Theorem 5.2. *Let R be a graded twisted bimodule n-CY algebra of a-invariant a with Nakayama automorphism α. Then R^{dg} is twisted bimodule $(n+a)$-CY. Precisely, R^{dg} is homologically smooth, and we have the following:*

1. *If a is odd, then* $\mathrm{RHom}_{(R^{\mathrm{dg}})^e}(R, (R^{\mathrm{dg}})^e)[n+a] \simeq {}_\alpha R_1$.
2. *If a is even, then* $\mathrm{RHom}_{(R^{\mathrm{dg}})^e}(R, (R^{\mathrm{dg}})^e)[n+a] \simeq {}_\alpha R_\sigma$ *for the automorphism $\sigma \colon x \mapsto (-1)^{|x|} x$ of R.*

Let us introduce some notations. For a DG algebra Λ, we denote by $\mathrm{C}(\Lambda) = Z^0\mathcal{C}_{\mathrm{dg}}(\Lambda)$ the (abelian) category of DG Λ-modules. In what follows, we denote by S an arbitrary graded algebra with the corresponding formal DG algebra S^{dg}. Let X be a graded (S, S)-bimodule. We can view it as a DG $(S^{\mathrm{dg}}, S^{\mathrm{dg}})$-bimodule X^{dg} with trivial differentials, hence as an $(S^{\mathrm{dg}})^e$-module, which gives a fully faithful functor

$$(-)^{\mathrm{dg}} \colon \operatorname{Mod}^{\mathbb{Z}} S^e \longrightarrow \mathrm{C}((S^{\mathrm{dg}})^e),$$

where $S^e = S^{\mathrm{op}} \otimes S$ is the enveloping algebra (as a graded algebra). By our convention, the action of $a \otimes b \in S^e$ on an element $x \otimes y$ in the free S^e-module S^e is given by $(x \otimes y) \cdot (a \otimes b) = ax \otimes yb$; thus free modules have outer bimodule structures.

We note the following sign-conventional lemmas. The proofs are left to the reader.

Lemma 5.3. *Let $F = S^e(l)$ be a free S^e-module. Then F^{dg} is isomorphic to the free DG $(S^{\mathrm{dg}})^e$-module $(S^{\mathrm{dg}})^e[l]$. The isomorphism is given by*

$$F^{\mathrm{dg}} \longrightarrow (S^{\mathrm{dg}})^e[l], \quad x \otimes y \mapsto (-1)^{l|x|} x \otimes y.$$

Lemma 5.4. *Let $X \in \operatorname{Mod}^{\mathbb{Z}} S^e$ and $l \in \mathbb{Z}$. Then we have an isomorphism $({}_{\sigma^l}X(l))^{\mathrm{dg}} \simeq X^{\mathrm{dg}}[l]$ in $\mathrm{C}((S^{\mathrm{dg}})^e)$, given by the identity map on the underlying graded vector space.*

We immediately obtain the following relationship for the homological smoothness of graded algebras and the corresponding formal DG algebras.

Proposition 5.5. *Let S be a graded algebra that is homologically smooth. Then S^{dg} is homologically smooth.*

Proof. There exists a projective resolution $0 \to P_n \to \cdots \to P_0 \to S \to 0$ over S^e with each P_i a finitely generated projective S^e-module. Applying $(-)^{\mathrm{dg}}$, we have an exact sequence $0 \to P_n^{\mathrm{dg}} \to \cdots \to P_0^{\mathrm{dg}} \to S^{\mathrm{dg}} \to 0$ in $\mathrm{C}((S^{\mathrm{dg}})^e)$ with each $P_i^{\mathrm{dg}} \in \operatorname{add}\{(S^{\mathrm{dg}})^e[l] \mid l \in \mathbb{Z}\}$ by Lemma 5.3. This shows $S^{\mathrm{dg}} \in \operatorname{per}(S^{\mathrm{dg}})^e$: that is, S^{dg} is homologically smooth. □

Let us now recall the notion of a total module of a complex of DG modules. We have two variations: *total sum* Tot and *total product* $\widehat{\mathrm{Tot}}$. Let $X = (\cdots \to X^{i-1} \xrightarrow{\delta_X^{i-1}} X^i \xrightarrow{\delta_X^i} X^{i+1} \to \cdots)$ be a complex of DG Λ-modules for a DG algebra Λ; thus each X^i is a DG Λ-module, each δ_X^i is a morphism of DG Λ-modules and $\delta_X^{i+1} \circ \delta_X^i = 0$. Then define

$$\operatorname{Tot} X = \bigoplus_{i \in \mathbb{Z}} X^i[-i], \qquad \widehat{\operatorname{Tot}} X = \prod_{i \in \mathbb{Z}} X^i[-i]$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

as graded Λ-modules and with differentials $\delta_X + \sum_i d_{X^i[-i]}$. Here, δ_X is the differential of the complex X and $d_{X^i[-i]}$ is the differential of the DG module $X^i[-i]$. Then $\operatorname{Tot} X$ and $\widehat{\operatorname{Tot}}\, X$ are DG Λ-modules. When a complex X of DG Λ-modules is bounded, we have $\operatorname{Tot} X = \widehat{\operatorname{Tot}}\, X$, which is acyclic whenever X is acyclic.

For a complex $X = (\cdots \to X^{i-1} \to X^i \to X^{i+1} \to \cdots)$ of DG Λ-modules and a DG Λ-module Y, we denote by $\mathcal{H}om_\Lambda(X, Y)$ the complex

$$\cdots \longrightarrow \mathcal{H}om_\Lambda(X^{i+1}, Y) \xrightarrow{\cdot\delta_X^i} \mathcal{H}om_\Lambda(X^i, Y) \xrightarrow{\cdot\delta_X^{i-1}} \mathcal{H}om_\Lambda(X^{i-1}, Y) \longrightarrow \cdots$$

of DG k-modules with $\mathcal{H}om_\Lambda(X^i, Y)$ at degree $-i$, where $\mathcal{H}om_\Lambda(-,-)$ is the Hom-complex between DG Λ-modules defined as follows: for DG Λ-modules M and N, the degree n part of the complex $\mathcal{H}om_\Lambda(M, N)$ is the space $\operatorname{Hom}_{\operatorname{Mod}^{\mathbb{Z}}\Lambda}(M, N[n])$ of homogeneous map of graded Λ-modules of degree n, with differential $df = d_N f - (-1)^n f d_M$ for $f \in \operatorname{Hom}_{\operatorname{Mod}^{\mathbb{Z}}\Lambda}(M, N[n])$.

The following is quite useful for computations.

Lemma 5.6. *Let Λ and Γ be DG algebras and X a complex of DG Λ-modules. Then for any DG (Γ, Λ)-bimodule Y, we have an isomorphism $\mathcal{H}om_\Lambda(\operatorname{Tot} X, Y) \simeq \widehat{\operatorname{Tot}}\, \mathcal{H}om_\Lambda(X, Y)$ of left DG Γ-modules.*

Proof. It is easily verified that the degree n part of each side is $\prod_{i\in\mathbb{Z}} \mathcal{H}om_\Lambda(X^i, Y)^{i+n}$; thus two sides coincide as graded vector spaces. Now we check that this identification is compatible with the differentials and the Γ-actions.

On the one hand, the differential on the left-hand side maps $f \in \mathcal{H}om_\Lambda(X^i, Y)^{i+n} \subset \mathcal{H}om_\Lambda(\operatorname{Tot} X, Y)$ to $d_Y f - (-1)^n f d_{\operatorname{Tot} X} = d_Y f - (-1)^n f \delta_X^{i-1} - (-1)^{n+i} f d_{X^i}$.

On the other hand, the differential of the right-hand side maps $f \in \mathcal{H}om_\Lambda(X^i, Y)^{i+n} \subset \widehat{\operatorname{Tot}}\, \mathcal{H}om_\Lambda(X, Y)$ to $f\delta_X^{i-1} + (-1)^{-i} d_{\mathcal{H}om_\Lambda(X^i,Y)} f = f\delta_X^{i-1} + (-1)^i (d_Y f - (-1)^{i+n} f d_{X^i}) = f\delta_X^{i-1} + (-1)^i d_Y f - (-1)^n f d_{X^i}$.

Then one can check that the map $\mathcal{H}om_\Lambda(\operatorname{Tot} X, Y) \to \widehat{\operatorname{Tot}}\, \mathcal{H}om_\Lambda(X, Y)$ given by $f \mapsto (-1)^{i(i+1)/2+in} f$ for $f \in \mathcal{H}om_\Lambda(X^i, Y)^{i+n}$ is an isomorphism of left DG Γ-modules. □

An important step for the proof of Theorem 5.2 is the following observation. We denote by $\operatorname{mod}^{\mathbb{Z}} S^e$ the category of finitely presented graded S^e-modules. Note that we are not assuming that S^e is graded coherent, and $\operatorname{mod}^{\mathbb{Z}} S^e$ is just an additive category.

Lemma 5.7. *Let σ be the automorphism $x \mapsto (-1)^{|x|} x$ on S. The following two functors are naturally isomorphic:*

(a) $F\colon \operatorname{mod}^{\mathbb{Z}} S^e \xrightarrow{\operatorname{Hom}_{S^e}(-, S^e)} \operatorname{Mod}^{\mathbb{Z}} S^e \xrightarrow{\sigma(-)} \operatorname{Mod}^{\mathbb{Z}} S^e$.

(b) $G\colon \operatorname{mod}^{\mathbb{Z}} S^e \xrightarrow{(-)^{\mathrm{dg}}} \mathsf{C}((S^{\mathrm{dg}})^e) \xrightarrow{\mathcal{H}om_{(S^{\mathrm{dg}})^e}(-, (S^{\mathrm{dg}})^e)} \mathsf{C}((S^{\mathrm{dg}})^e) \xrightarrow{\text{forget}} \operatorname{Mod}^{\mathbb{Z}} S^e$.

In particular, these functors are naturally isomorphic on $\operatorname{proj}^{\mathbb{Z}} S^e$.

Proof. We first prove that two functors are naturally isomorphic on the category of finitely generated free modules. We will then extend the natural isomorphism to $\operatorname{mod}^{\mathbb{Z}} S^e$.

Let us start by defining an isomorphism $\varphi_P\colon F(P) \to G(P)$ at the free module $P = S^e(l)$. Clearly, we have $F(P) = S \otimes S(-l)$, with the action of S given by $b \cdot (x \otimes y) \cdot a = (-1)^{|b|} xa \otimes by$. On the other hand, using Lemma 5.3, we see $G(P) = (S \otimes S)[-l]$, with the S-action $b \cdot (x \otimes y) \cdot a = (-1)^{|a|(|b|+|y|)+|b|(l+|x|)} xa \otimes by$. Now we define an isomorphism $F(P) = S \otimes S(-l) \to (S \otimes S)[-l] = G(P)$ by the formula

$$\varphi_P\colon x \otimes y \mapsto (-1)^{(|x|+l+1)|y|} x \otimes y.$$

We can readily check that this is (S, S)-bilinear.

We next show that this isomorphism is natural. Let $P \to Q$ be a morphism of finitely generated free modules. We may assume that this is of the form $S^e(l) \to S^e(m)$ with $1 \otimes 1 \mapsto \sum_i u_i \otimes v_i$. Under the functor $(-)^{\mathrm{dg}}$ and the isomorphism in Lemma 5.3, it becomes an $(S^{\mathrm{dg}})^e$-linear map $(S^{\mathrm{dg}})^e[l] \to (S^{\mathrm{dg}})^e[m]$ sending $1 \otimes 1$ to $\sum_i (-1)^{m|u_i|} u_i \otimes v_i$. Note that we have $|u_i| + |v_i| - m = -l$. Our task is to show that in the diagram below, the middle square is commutative in $\mathrm{Mod}^{\mathbb{Z}} S^e$:

$$\begin{tikzcd}
{}_\sigma \mathrm{Hom}_{S^e}(S^e(m), S^e) \arrow[r, equal] \arrow[d] & {}_\sigma S^e(-m) \arrow[r, "\varphi_Q"] \arrow[d, "f"'] & (S^{\mathrm{dg}})^e[-m] \arrow[r, equal] \arrow[d, "g"] & \mathcal{H}om_{(S^{\mathrm{dg}})^e}((S^{\mathrm{dg}})^e[m], (S^{\mathrm{dg}})^e) \arrow[d] \\
{}_\sigma \mathrm{Hom}_{S^e}(S^e(l), S^e) \arrow[r, equal] & {}_\sigma S^e(-l) \arrow[r, "\varphi_P"] & (S^{\mathrm{dg}})^e[-l] \arrow[r, equal] & \mathcal{H}om_{(S^{\mathrm{dg}})^e}((S^{\mathrm{dg}})^e[l], (S^{\mathrm{dg}})^e).
\end{tikzcd}$$

By our sign conventions, the map f is just a left S^e-linear map with $1 \otimes 1 \mapsto \sum_i u_i \otimes v_i$, while g is given by $1 \otimes 1 \mapsto (-1)^{m(l+1)} \sum_i (-1)^{m|u_i|} u_i \otimes v_i$. Using these, we can verify the desired commutativity.

It is now routine to extend this natural isomorphism to $\mathrm{mod}^{\mathbb{Z}} S^e$. To define an isomorphism $\varphi_M \colon F(M) \to G(M)$ at $M \in \mathrm{mod}^{\mathbb{Z}} S^e$, take a presentation $P_1 \to P_0 \to M \to 0$ by finitely generated free modules. Since F and G are restrictions of left exact functors on $\mathrm{Mod}^{\mathbb{Z}} S^e$, we can define φ_M to be the unique map, making the following diagram of exact sequences (in $\mathrm{Mod}^{\mathbb{Z}} S^e$) commutative. Note that the right square is commutative by our previous claim:

$$\begin{tikzcd}
0 \arrow[r] & F(M) \arrow[r] \arrow[d, dashed, "\varphi_M"'] & F(P_0) \arrow[r] \arrow[d, "\varphi_{P_0}"] & F(P_1) \arrow[d, "\varphi_{P_1}"] \\
0 \arrow[r] & G(M) \arrow[r] & G(P_0) \arrow[r] & G(P_1).
\end{tikzcd}$$

It is easy to verify that φ_M is independent of the presentation, and clearly φ_M is an isomorphism.

Now we show that this isomorphism is natural on $\mathrm{mod}^{\mathbb{Z}} S$. Let $f \colon M \to N$ be a morphism in $\mathrm{mod}^{\mathbb{Z}} S$. Taking presentations $P_1 \to P_0 \to M \to 0$ and $Q_1 \to Q_0 \to N \to 0$ by finitely generated free modules, we can lift f to a morphism of presentations. Applying F and G to these, we obtain a commutative diagram

$$\begin{tikzcd}
& 0 \arrow[r] & F(M) \arrow[r] \arrow[dd, "\varphi_M" near start] & F(P_0) \arrow[r] \arrow[dd, "\varphi_{P_0}" near start] & F(P_1) \arrow[dd, "\varphi_{P_1}"] \\
0 \arrow[r] & F(N) \arrow[ur] \arrow[r] \arrow[dd, "\varphi_N" near start] & F(Q_0) \arrow[ur] \arrow[r] \arrow[dd, "\varphi_{Q_0}" near start] & F(Q_1) \arrow[ur] \arrow[dd, "\varphi_{Q_1}" near start] & \\
& 0 \arrow[r] & G(M) \arrow[r] & G(P_0) \arrow[r] & G(P_1) \\
0 \arrow[r] & G(N) \arrow[ur] \arrow[r] & G(Q_0) \arrow[ur] \arrow[r] & G(Q_1) \arrow[ur] & ,
\end{tikzcd}$$

from which we deduce the desired naturality on f. □

We are now ready to prove the main theorem of this section.

Proof of Theorem 5.2. We know R^{dg} is homologically smooth by Proposition 5.5. We compute the inverse dualising complex of R^{dg}. Let $P = (P_n \to \cdots \to P_1 \to P_0)$ be a projective resolution of R in $\mathcal{C}^b(\mathrm{proj}^{\mathbb{Z}} R^e)$. Then the cohomology of $P^\vee = \mathrm{Hom}_{R^e}(P, R^e)$ is concentrated in degree n, where it is ${}_\alpha R(-a)$. Considering P as a complex P^{dg} of DG bimodules as above, the total sum of P^{dg} gives an $(R^{\mathrm{dg}})^e$-cofibrant resolution of the DG $(R^{\mathrm{dg}})^e$-module R^{dg}. Indeed, $\mathrm{Tot}\, P^{\mathrm{dg}} \to R^{\mathrm{dg}}$ is a quasi-isomorphism since Tot preserves acyclicity, and since each P_i^{dg} is cofibrant by Lemma 5.3 and $\mathrm{Tot}\, P^{\mathrm{dg}}$ is a successive extension of P_i^{dg}'s that is split as graded $(R^{\mathrm{dg}})^e$-modules, it follows that $\mathrm{Tot}\, P^{\mathrm{dg}}$ is cofibrant (see [Ke1, Section 3.1]). Then $\mathrm{RHom}_{(R^{\mathrm{dg}})^e}(R^{\mathrm{dg}}, (R^{\mathrm{dg}})^e) = \mathcal{H}om_{(R^{\mathrm{dg}})^e}(\mathrm{Tot}\, P^{\mathrm{dg}}, (R^{\mathrm{dg}})^e)$, which is isomorphic by

Lemma 5.6 to the total product (which equals the total sum by boundedness) of the complex

$$Q = (\mathcal{H}om_{(R^{\mathrm{dg}})^e}(P_0^{\mathrm{dg}}, (R^{\mathrm{dg}})^e) \to \mathcal{H}om_{(R^{\mathrm{dg}})^e}(P_1^{\mathrm{dg}}, (R^{\mathrm{dg}})^e) \to \cdots \to \mathcal{H}om_{(R^{\mathrm{dg}})^e}(P_n^{\mathrm{dg}}, (R^{\mathrm{dg}})^e)).$$

Now, by Lemma 5.7, this complex is isomorphic to ${}_{\sigma}(P^\vee)$ as complexes of graded (R, R)-bimodules; thus it has cohomology only at degree n, where it is isomorphic to ${}_{\sigma\alpha}R(-a)$. Then we have a quasi-isomorphism $Q \to ({}_{\sigma\alpha}R(-a))^{\mathrm{dg}}[-n]$ of complexes of DG $(R^{\mathrm{dg}})^e$-modules, so $\operatorname{Tot} Q$ is quasi-isomorphic to $({}_{\sigma\alpha}R(-a))^{\mathrm{dg}}[-n]$. We conclude by Lemma 5.4 that it is ${}_{\alpha}R[-n-a]$ if a is odd, ${}_{\sigma\alpha}R[-n-a]$ if a is even. □

Example 5.8. Let $R = k[x_1, \cdots, x_n]$ be the polynomial ring, which is a CY algebra (see Lemma 10.1).

1. Set $\deg x_i = -1$ for all $1 \le i \le n$. Then R is bimodule n-CY of a-invariant n. By Theorem 5.2, we see that
   - R^{dg} is $2n$-CY if n is odd;
   - R^{dg} is twisted $2n$-CY if n is even.

   See Example 5.9 below for an illustration in $n = 2$ how R^{dg} fails to be CY.
2. Set $\deg x_i = 1$ for all $1 \le i \le n$. Then R is bimodule n-CY of a-invariant $-n$. By Theorem 5.2, we see that
   - R^{dg} is 0-CY if n is odd;
   - R^{dg} is twisted 0-CY if n is even.

   This partially recovers [MGYC, Theorem 6.4]; see also [HM, Example 6.1].
3. Set $\deg x_i = 2$ for all $1 \le i \le n$. Then R is bimodule n-CY of a-invariant $-2n$. By Theorem 5.2, we have $\mathrm{RHom}_{(R^{\mathrm{dg}})^e}(R, (R^{\mathrm{dg}})^e)[-n] \simeq {}_1R_\sigma$. Note that the automorphism σ is the identity since R is concentrated in even degrees. Therefore, we conclude that R^{dg} is $(-n)$-CY. This partially recovers [MGYC, Theorem 6.2].

We explicitly demonstrate how R^{dg} fails to be CY.

Example 5.9. Let $R = k[x, y]$ with $\deg x = \deg y = -1$. Then the graded ring R is bimodule 2-CY of a-invariant 2 and has the Koszul resolution, which we depict in the following way:

$$\begin{tikzcd} & R\otimes R(1) \arrow[dr, "x\otimes 1-1\otimes x"] & \\ R\otimes R(2) \arrow[ur, "-y\otimes 1+1\otimes y"] \arrow[dr, "x\otimes 1-1\otimes x"'] & \oplus & R\otimes R \\ & R\otimes R(1) \arrow[ur, "y\otimes 1-1\otimes y"'] & \end{tikzcd},$$

where the values on the arrows show the image of $1 \otimes 1$. Now consider this resolution as a complex of DG modules over $S := (R^{\mathrm{dg}})^e$. Under the isomorphism in Lemma 5.3, it becomes

$$\begin{tikzcd} & S[1] \arrow[dr, "x\otimes 1-1\otimes x"] & \\ S[2] \arrow[ur, "y\otimes 1+1\otimes y"] \arrow[dr, "-x\otimes 1-1\otimes x"'] & \oplus & S \\ & S[1] \arrow[ur, "y\otimes 1-1\otimes y"'] & \end{tikzcd}.$$

Applying $\mathcal{H}om_S(-, S)$, we get the complex of left DG S-modules

$$\begin{tikzcd} & S[-1] \arrow[dr, "-y\otimes 1-1\otimes y"] & \\ S \arrow[ur, "x\otimes 1-1\otimes x"] \arrow[dr, "y\otimes 1-1\otimes y"'] & \oplus & S[-2] \\ & S[-1] \arrow[ur, "x\otimes 1+1\otimes x"'] & \end{tikzcd},$$

whose total module is $\mathrm{RHom}_S(R, S)$ by Lemma 5.6. We therefore see that $\mathrm{RHom}_S(R, S)[4] \simeq {}_1R_\sigma$.

The appearance of the twist automorphism σ suggests that we should twist the multiplication of the CY algebra R in order for the DG algebra R^{dg} to be CY. Let us give an instance where R is twisted CY and R^{dg} is CY.

Example 5.10. Let $m \geq 2$ and

$$R = k\langle x_1, \dots, x_m\rangle/(x_1^2 + \cdots + x_m^2). \quad \deg x_i = -1,$$

If $m = 2$, this is a skew polynomial ring with two variables; compare Example 5.9 above. Recall from Example 4.13 that this is twisted bimodule 2-CY algebra of a-invariant 2 with Nakayama automorphism $\sigma\colon x_i \mapsto -x_i$. Therefore, by Theorem 5.2(2), we deduce that R^{dg} is (non-twisted) 4-CY. Let us explicitly demonstrate this.

Recall that the bimodule projective resolution of R is given by the complex

$$0 \longrightarrow R \otimes R(2) \xrightarrow{d_2} \bigoplus_{i=1}^{m} R \otimes R(1) \xrightarrow{d_1} R \otimes R \longrightarrow 0$$

with maps

$$\begin{aligned} d_1((1 \otimes 1)_i) &= x_i \otimes 1 - 1 \otimes x_i \\ d_2(1 \otimes 1) &= \sum_{i=1}^{m} (x_i \otimes 1 + 1 \otimes x_i). \end{aligned}$$

We follow the computation in Example 5.9 above. Set $S = (R^{\mathrm{dg}})^e$. Applying the functor $(-)^{\mathrm{dg}}$ and the isomorphism in Lemma 5.3, the above complex becomes

$$0 \longrightarrow S[2] \xrightarrow{d_2} \bigoplus_{i=1}^{m} S[1] \xrightarrow{d_1} S \longrightarrow 0,$$

where the maps are right S-linear morphisms such that

$$\begin{aligned} d_1((1 \otimes 1)_i) &= x_i \otimes 1 - 1 \otimes x_i \\ d_2(1 \otimes 1) &= \sum_{i=1}^{m} (-x_i \otimes 1 + 1 \otimes x_i). \end{aligned}$$

Applying $\mathcal{H}om_S(-, S)$, we get an isomorphic complex up to shifts; thus we see that $\mathrm{RHom}_S(R, S)[4] \simeq R$ in $\mathcal{D}(S)$.

6. Cluster categories, derived orbit categories and singularity categories

Let R be a CY algebra. We state the main result of this paper, which describes the cluster category of R^{dg} as an orbit category and a singularity category.

6.1. Cluster categories and orbit categories

Let R be a negatively graded twisted bimodule $(d + 1)$-CY algebra of a-invariant a such that each R_i is finite dimensional. In this subsection, we compare the derived category of R considered as a graded ring and that of R considered as a DG algebra R^{dg} with vanishing differentials. By Theorem 5.2, we know that R^{dg} is twisted bimodule $(d + a + 1)$-CY.

Recall the notion of total module from the previous section. Consider the DG functor

$$\mathrm{Tot}\colon \mathcal{C}^b_{\mathrm{dg}}(\mathrm{Mod}^{\mathbb{Z}} R) \to \mathcal{C}_{\mathrm{dg}}(R^{\mathrm{dg}})$$

from the DG category of complexes over $\mathrm{Mod}^{\mathbb{Z}} R$ to the DG category of DG R^{dg}-modules. This is given on morphisms by taking a map $f\colon X^i \to Y^j$ of graded R-modules considered as an element of $\mathcal{H}om_{\mathrm{Mod}^{\mathbb{Z}} R}(X, Y)$ to itself (without any signs) viewed as an element of $\mathcal{H}om_{R^{\mathrm{dg}}}(\mathrm{Tot}\, X, \mathrm{Tot}\, Y)$.

Taking the 0th cohomology, it induces a triangle functor $\mathcal{K}^b(\mathrm{Mod}^{\mathbb{Z}} R) \to \mathcal{K}(R^{\mathrm{dg}})$, which clearly takes acyclic complexes to acyclic DG modules. We therefore obtain a triangle functor

$$\mathrm{Tot}\colon\ \mathrm{per}^{\mathbb{Z}} R \to \mathrm{per}\, R^{\mathrm{dg}}.$$

Note that this restricts to $\mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R) \to \mathcal{D}^b(R^{\mathrm{dg}})$; thus it again induces a triangle functor,

$$\mathrm{Tot}\colon \mathrm{qper}^{\mathbb{Z}} R \to \mathcal{C}(R^{\mathrm{dg}}).$$

Now we have a natural isomorphism $\mathrm{Tot} \circ (-1)[1] \simeq \mathrm{Tot}$, hence a functor

$$\mathrm{Tot}\colon \mathrm{qper}^{\mathbb{Z}} R/(-1)[1] \to \mathcal{C}(R^{\mathrm{dg}}).$$

The following result gives a natural and more concrete description of the cluster category of R^{dg}. Similar types of results for derived or singularity categories were recently obtained in [KY, Bri].

Theorem 6.1. *Let R be a negatively graded twisted bimodule $(d+1)$-CY algebra of a-invariant a with $d \geq 0$. The functor* $\mathrm{Tot}\colon \mathrm{qper}^{\mathbb{Z}} R \to \mathcal{C}(R^{\mathrm{dg}})$ *induces a fully faithful functor*

$$\mathrm{qper}^{\mathbb{Z}} R/(-1)[1] \longrightarrow \mathcal{C}(R^{\mathrm{dg}})$$

whose image generates $\mathcal{C}(R^{\mathrm{dg}})$ as a triangulated subcategory.

Proof. Note that the cluster tilting subcategory $\mathcal{M} = \mathrm{add}\{R(-i)[i] \mid i \in \mathbb{Z}\} \subset \mathrm{qper}^{\mathbb{Z}} R$ given in Theorem 4.6 is mapped to a cluster tilting object $R \in \mathcal{C}(R^{\mathrm{dg}})$, and the functor Tot induces an equivalence $\mathcal{M}/(-1)[1] \xrightarrow{\simeq} \mathrm{add}\, R$. Indeed, $\mathcal{M}/(-1)[1]$ has an additive generator R, and we have an isomorphism $\mathrm{End}_{\mathcal{M}/(-1)[1]}(R) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{M}}(R, R(-i)[i])) \xrightarrow{\simeq} \mathrm{End}_{\mathcal{C}(R^{\mathrm{dg}})}(R)$ of graded rings since the left-hand side vanishes for $i \neq 0$, and thus both sides are R_0. Our assertion is now a consequence of the covering version of the 'cluster-Beilinson criterion' (compare [KR2, Lemma 4.5]), which we sketch below.

To save space, we will denote $\mathcal{D} := \mathrm{qper}^{\mathbb{Z}} R$ and $\mathcal{C} := \mathcal{C}(R^{\mathrm{dg}})$ and sometimes ${}_{\mathcal{A}}(-,-)$ for the morphism spaces in a category $\mathcal{A}$. We have to show that Tot induces isomorphisms

$$\mathrm{Hom}_{\mathcal{D}/(-1)[1]}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathrm{Tot}\, X, \mathrm{Tot}\, Y) \tag{6.1}$$

for all $X, Y \in \mathcal{D}$. We first prove that the above map is an isomorphism for all $X \in \mathcal{M}$ and $Y \in \mathcal{M} * \cdots * \mathcal{M}[j]$ with $0 \leq j \leq d + a - 1$ by induction on j. We have already seen this above for $j = 0$. Let $0 \leq j < d + a - 1$ and $Y \in \mathcal{M} * \cdots * \mathcal{M}[j] * \mathcal{M}[j+1]$. Pick a triangle $Y' \to M \to Y \to Y'[1]$ in $\mathcal{D}$ with $M \in \mathcal{M}$ and $Y' \in \mathcal{M} * \cdots * \mathcal{M}[j]$. This gives a commutative diagram of exact sequence

$$\begin{tikzcd}
{}_{\mathcal{D}/(-1)[1]}(X, Y') \arrow[r] \arrow[d, "\wr"] & {}_{\mathcal{D}/(-1)[1]}(X, M) \arrow[r] \arrow[d, "\wr"] & {}_{\mathcal{D}/(-1)[1]}(X, Y) \arrow[r] \arrow[d] & {}_{\mathcal{D}/(-1)[1]}(X, Y'[1]) = 0 \arrow[d] \\
{}_{\mathcal{C}}(\mathrm{Tot}\, X, \mathrm{Tot}\, Y') \arrow[r] & {}_{\mathcal{C}}(\mathrm{Tot}\, X, \mathrm{Tot}\, M) \arrow[r] & {}_{\mathcal{C}}(\mathrm{Tot}\, X, \mathrm{Tot}\, Y) \arrow[r] & {}_{\mathcal{C}}(\mathrm{Tot}\, X, \mathrm{Tot}\, Y'[1]) = 0,
\end{tikzcd}$$

in which the left two vertical maps are isomorphisms by induction hypothesis and the right-end terms are 0 since $\mathcal{M} \subset \mathcal{D}$ and $R \in \mathcal{C}$ are $(d + a)$-cluster tilting. Therefore, the remaining vertical map is also an isomorphism. Since $\mathcal{D} = \mathcal{M} * \cdots \mathcal{M}[d+a-1]$, this induction shows that equation (6.1) is an isomorphism for all $X \in \mathcal{M}$ and $Y \in \mathcal{D}$.

We next prove that equation (6.1) is an isomorphism for all $X, Y \in \mathcal{D}$. We can perform a similar induction on j to show that this is indeed the case for $X \in \mathcal{M} * \cdots * \mathcal{M}[j]$, which completes the proof. $\square$

Let A be the d-representation infinite algebra given in Proposition 4.9 as the endomorphism ring of a tilting object in $\mathrm{qper}^{\mathbb{Z}} R$. Explicitly, we have

$$A = \begin{pmatrix} R_0 & 0 & \cdots & 0 \\ R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \end{pmatrix}.$$

Assuming that the Nakayama automorphism α of R in Theorem 6.1 satisfies $(-)_\alpha \simeq 1$ on $\mathrm{qper}^{\mathbb{Z}} R$, we deduce the following by means of the diagram in equation (4.1).

Corollary 6.2. *Let R be as in Theorem 6.1, and suppose that $(-)_\alpha \simeq 1$ on $\mathrm{qper}^{\mathbb{Z}} R$. Then there exists a fully faithful functor*

$$\mathcal{D}^b(\mathrm{mod}\, A)/\nu_d^{-1/a}[1] \longrightarrow \mathcal{C}(R^{\mathrm{dg}})$$

whose image generates $\mathcal{C}(R^{\mathrm{dg}})$ as a triangulated subcategory.

6.2. Cluster categories and singularity categories

We present another description of the cluster category $\mathcal{C}(R^{\mathrm{dg}})$. Set

$$U = \begin{pmatrix} R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \\ R_{-a} & R_{-(a-1)} & \cdots & R_{-1} \end{pmatrix},$$

which is an (A, A)-bimodule, and let $B = A \oplus U$ be the trivial extension algebra.

Recall that a *cotilting bimodule* over a ring Λ is a (Λ, Λ)-bimodule Y satisfying the following:

- ∘ $\mathrm{inj.dim}_\Lambda Y < \infty$ and $\mathrm{inj.dim}_{\Lambda^{\mathrm{op}}} Y < \infty$.
- ∘ $\mathrm{Ext}^i_\Lambda(Y, Y) = 0$ and $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(Y, Y) = 0$ for $i > 0$.
- ∘ The natural maps $\Lambda \to \mathrm{End}_\Lambda(Y)$ and $\Lambda^{\mathrm{op}} \to \mathrm{End}_{\Lambda^{\mathrm{op}}}(Y)$ are isomorphisms.

We have the following basic information on U and B.

Proposition 6.3.

1. *U is a cotilting bimodule over A.*
2. *B is a d-Iwanaga-Gorenstein algebra.*

Proof. (1) Since A has finite global dimension, we clearly have $\mathrm{inj.dim}_A U < \infty$. Note also that we have $A = \mathrm{End}_{\mathrm{qper}^{\mathbb{Z}} R}(T)$ and $U = \mathrm{RHom}_{\mathrm{qper}^{\mathbb{Z}} R}(T, T(-1))$ for a tilting object T given in Proposition 4.9. This gives $\mathrm{Ext}^i_A(U, U) = 0$ for $i > 0$ and an isomorphism $A \to \mathrm{End}_A(U)$. It remains to verify the conditions on the opposite side. Let $(-)^* = \mathrm{RHom}_R(-, R)$ be the duality $\mathrm{per}^{\mathbb{Z}} R \leftrightarrow \mathrm{per}^{\mathbb{Z}} R^{\mathrm{op}}$. By the Artin-Schelter regularity of R (see the remark after Proposition 4.2), it restricts to $\mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R) \leftrightarrow \mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R^{\mathrm{op}})$ and hence induces a duality $\mathrm{qper}^{\mathbb{Z}} R \leftrightarrow \mathrm{qper}^{\mathbb{Z}} R^{\mathrm{op}}$, which we still denote by $(-)^*$. Then $T^* \in \mathrm{qper}^{\mathbb{Z}} R^{\mathrm{op}}$ is a tilting

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

object with $\mathrm{End}_{\mathrm{qper}^{\mathbb{Z}} R^{\mathrm{op}}}(T^*) = A^{\mathrm{op}}$, and we have $U \simeq \mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R^{\mathrm{op}}}(T^*, T^*(-1))$ as (A, A)-bimodules. This gives the desired assertions for the opposite side.

(2) This follows from (1) and [MY, Theorem 5.3]. □

Now we state the second main result of this paper.

Theorem 6.4. *Let R be as in Corollary 6.2. There exists a triangle equivalence*

$$\mathcal{C}(R^{\mathrm{dg}}) \simeq \mathcal{D}_{\mathrm{sg}}(B).$$

In particular, $\mathcal{D}_{\mathrm{sg}}(B)$ is a twisted $(d+a)$-CY category with a $(d+a)$-cluster tilting object.

We postpone the proof of this result to Section 8 since it requires a general result on DG orbit categories, which we give in the following section.

6.3. Examples

Before going on, let us give some demonstrations of our results. See Sections 9, 10 and 11 for systematic examples.

Example 6.5. Let us start with an almost trivial example. Let

$$R = k[x], \quad \deg x = -1.$$

This is bimodule 1-CY of a-invariant 1; thus R^{dg} is bimodule 2-CY by Theorem 5.2. We have $A = k$ (which is understood to be '0-representation infinite') and $U = k$; thus $B = A \oplus U = k[t]/(t^2)$. By Corollary 6.2 and Theorem 6.4, we have equivalences of triangulated categories

$$\mathcal{D}^b(\mathrm{mod}\, k)/[1] \simeq \mathcal{C}(R^{\mathrm{dg}}) \simeq \mathcal{D}_{\mathrm{sg}}(k[t]/(t^2)),$$

which are triangulated categories with 1 point. See Example A.5 for a generalization, where the case $\deg x = -n$ for arbitrary $n \geq 1$ is discussed.

Example 6.6. This is a continuation of Examples 4.13 and 5.10. Let $m \geq 2$, and set

$$R = k\langle x_1, \ldots, x_m\rangle/(x_1^2 + \cdots + x_m^2), \quad \deg x_i = -1,$$

which is twisted 2-CY of a-invariant 2 (Example 4.13), and R^{dg} is 4-CY (Example 5.10). The 1-representation infinite algebra A is the path algebra of the m-Kronecker quiver $Q_m\colon \circ \xrightarrow{m} \circ$, and the autoequivalence ν_1 of $\mathcal{D}^b(\mathrm{mod}\, kQ_m)$ has a square root (Example 4.13). Also, it is not difficult to see that the 1-Iwanaga-Gorenstein algebra B is presented by the following quiver with relations:

$$\begin{tikzcd} \circ \arrow[r, shift left=2, "x_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "x_m"'] & \circ \arrow[l, bend right=60, "u"'] \end{tikzcd}, \qquad \textstyle\sum_{i=1}^{m} x_i u x_i = 0,\ u x_i u = 0.$$

By Corollary 6.2 and Theorem 6.4, there exist triangle equivalences

$$\mathcal{D}^b(\mathrm{mod}\, kQ_m)/\nu_1^{-1/2}[1] \simeq \mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}(R^{\mathrm{dg}}).$$

Remark 6.7. In [KMV, Theorem 1.4], Keller–Murfet–Van den Bergh proved that any algebraic 3-CY triangulated category $\mathcal{T}$ with a 3-cluster tilting object T such that $\mathrm{End}_{\mathcal{T}}(T) = k$ and $\mathrm{Hom}_{\mathcal{T}}(T, T[-1]) = k^m$ is triangle equivalent to $\mathcal{D}^b(\mathrm{mod}\, kQ_m)/\tau^{-1/2}[1]$, where $\tau^{-1/2}$ is the square root of the AR translation defined in [KMV] using reflection functors.

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

Now, since $\mathcal{C} = \mathcal{C}(R^{\mathrm{dg}})$ for R in Example 6.6 above is a 3-CY category with a 3-cluster tilting object R satisfying $\mathrm{End}_{\mathcal{C}}(R) = k$ and $\mathrm{Hom}_{\mathcal{C}}(R, R[-1]) = k^m$, the above equivalent triangulated categories are precisely the 3-CY category in [KMV, Theorem 1.4]. Our result shows that the 3-CY category in [KMV] is also the singularity category of B and can be realised as the cluster category of the DG algebra R^{dg}. We refer to Example A.6 for a generalization, which covers the situation in [KMV, Remark 3.4.5].

7. Quasi-equivalence of DG orbit categories

Let $\mathcal{T}$ be a triangulated category and $F\colon \mathcal{T} \to \mathcal{T}$ an autoequivalence. In order to discuss when the orbit category $\mathcal{T}/F$ is triangulated, a triangulated hull of this orbit category was constructed by Keller [Ke2]. The idea was to take an orbit at the level of enhancement of $\mathcal{T}$.

Let $\mathcal{A}$ be a pretriangulated DG category and F a DG endofunctor on $\mathcal{A}$ inducing an equivalence on $H^0\mathcal{A}$. Then the *DG orbit category* of $\mathcal{A}$ with respect to F, which we denote by $\mathcal{B} = \mathcal{A}/F$, is the DG category with the same objects as $\mathcal{A}$ and with the morphism complex

$$\mathcal{B}(L, M) = \operatorname{colim}\left(\bigoplus_{n\geq 0}\mathcal{A}(F^nL, M) \xrightarrow{F} \bigoplus_{n\geq 0}\mathcal{A}(F^nL, FM) \xrightarrow{F} \bigoplus_{n\geq 0}\mathcal{A}(F^nL, F^2M) \xrightarrow{F} \cdots\right)$$

for each $L, M \in \mathcal{B}$. It follows that we have $H^0\mathcal{A}/H^0F = H^0\mathcal{B}$; hence $\operatorname{tria}\mathcal{B}$, the full triangulated subcategory of $\mathcal{D}(\mathcal{B})$ generated by the representable $\mathcal{B}$-modules, is a triangulated hull of $H^0\mathcal{A}/H^0F$ in the sense that there is a fully faithful functor $H^0\mathcal{A}/H^0F \hookrightarrow \operatorname{tria}\mathcal{B}$ whose image generates $\operatorname{tria}\mathcal{B}$ as a triangulated subcategory.

Observe that the above embedding is not dense in general, and thus it is not clear a naive expectation for orbit categories carries over to triangulated hulls. We give an answer to one such problem.

Let us briefly recall some relevant notions; see [Ke1, Section 7] for details. Let $\mathcal{B}$ and $\mathcal{C}$ be DG categories. A *quasi-functor* $\mathcal{C} \to \mathcal{B}$ is a $(\mathcal{C}, \mathcal{B})$-bimodule X whose value we denote by $X(B, C)$ for $B \in \mathcal{B}$ and $C \in \mathcal{C}$, such that the DG $\mathcal{B}$-module $X(-, C)$ is isomorphic in $\mathcal{D}(\mathcal{B})$ to a representable DG $\mathcal{B}$-module for each $C \in \mathcal{C}$. A quasi-functor $X\colon \mathcal{C} \to \mathcal{B}$ is a *quasi-equivalence* if $-\otimes^{\mathrm{L}}_{\mathcal{C}} X\colon \mathcal{D}(\mathcal{C}) \to \mathcal{D}(\mathcal{B})$ is an equivalence and restricts to an equivalence $H^0\mathcal{C} \xrightarrow{\simeq} H^0\mathcal{B}$. In this case, we deduce that there is a triangle equivalence $-\otimes^{\mathrm{L}}_{\mathcal{C}} X\colon \operatorname{tria}\mathcal{C} \xrightarrow{\simeq} \operatorname{tria}\mathcal{B}$. We say two DG categories $\mathcal{B}$ and $\mathcal{C}$ are *quasi-equivalent* if there exists a quasi-equivalence $\mathcal{C} \to \mathcal{B}$.

Theorem 7.1. *Let $\mathcal{A}$ be a pretriangulated DG category, and let F, G be DG endofunctors on $\mathcal{A}$ such that H^0F and H^0G are mutually inverse equivalences on $H^0\mathcal{A}$. Suppose that there is a morphism $G \circ F \to 1_{\mathcal{A}}$ of DG functors inducing a natural isomorphism on $H^0\mathcal{A}$. Then the DG orbit categories $\mathcal{B} = \mathcal{A}/F$ and $\mathcal{C} = \mathcal{A}/G$ are quasi-equivalent. In particular, the triangulated hulls* $\operatorname{tria}\mathcal{B}$ *and* $\operatorname{tria}\mathcal{C}$ *are triangle equivalent.*

Remark 7.2.

1. We do not need a natural transformation $F \circ G \to 1_{\mathcal{A}}$.
2. The assumption on the existence of a natural transformation $G \circ F \to 1_{\mathcal{A}}$ is satisfied in the following typical case: Let Λ be a finite-dimensional algebra of finite global dimension and $\mathcal{A} = \mathcal{C}^{-,b}(\operatorname{proj}\Lambda)$. Let X a complex of projective (Λ, Λ)-bimodules such that $F = -\otimes_\Lambda X\colon \mathcal{A} \to \mathcal{A}$ gives an equivalence on $H^0\mathcal{A} = \mathcal{D}^b(\operatorname{mod}\Lambda)$. Letting Y be the bimodule projective resolution of $\mathrm{RHom}_\Lambda(X, \Lambda)$, $G = -\otimes_\Lambda Y\colon \mathcal{A} \to \mathcal{A}$ gives an inverse of F on $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Moreover, a quasi-isomorphism $X \otimes_\Lambda Y \to \Lambda$ of (Λ, Λ)-bimodule complexes gives a natural transformation $G \circ F \to 1_{\mathcal{A}}$.

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

In view of relating the categories $\mathcal{B}$ and $\mathcal{C}$, consider the following direct system of complexes indexed by $\mathbb{N}\times\mathbb{N}$:

$$\begin{tikzcd}
\bigoplus_{n\geq 0}\mathcal{A}(F^nL,M) \arrow[r] \arrow[d] & \bigoplus_{n\geq 0}\mathcal{A}(GF^nL,M) \arrow[r] \arrow[d] & \bigoplus_{n\geq 0}\mathcal{A}(G^2F^nL,M) \arrow[r] \arrow[d] & \cdots \\
\bigoplus_{n\geq 0}\mathcal{A}(GF^nL,GM) \arrow[r] \arrow[d] & \bigoplus_{n\geq 0}\mathcal{A}(G^2F^nL,GM) \arrow[r] \arrow[d] & \bigoplus_{n\geq 0}\mathcal{A}(G^3F^nL,GM) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}, \tag{7.1}$$

where the vertical transition maps are induced by G and the horizontal ones by $G^{p+1}F^{1+n}L\to G^pF^nL$.

We first fix $q\geq 0$ and consider the colimit

$$U_q(L,M):=\operatorname*{colim}_{p\geq 0}\bigoplus_{n\geq 0}\mathcal{A}(G^{p+q}F^nL,G^qM)$$

of the $(q+1)$-st row. We can regard $U_q(-,M)$ as a DG $\mathcal{B}$-module for each $M\in\mathcal{A}$ as follows: Let $b\in\mathcal{B}(K,L)$ be a morphism represented by $F^mK\to F^rL$ and $u\in U_q(L,M)$ an element represented by $G^{p+q}F^nL\to G^qM$. Enlarging n if necessary, we may assume $r\leq n$. Then define $u\cdot b$ by the composition $G^{p+q}F^{m+n-r}K\xrightarrow{G^{p+q}F^{n-r}b}G^{p+q}F^nL\xrightarrow{u}G^qM$. Since there is a commutative diagram

$$\begin{tikzcd}
G^{p+q}F^{m+n-r}K \arrow[r, "G^{p+q}F^{n-r}b"] & G^{p+q}F^nL \arrow[r, "u"] & G^qM \\
G^{p+q+1}F^{1+m+n-r}K \arrow[r, "G^{p+q+1}F^{1+n-r}b"] \arrow[u] & G^{p+q+1}F^{1+n}L \arrow[r] \arrow[u] & G^qM \arrow[u, equal]
\end{tikzcd}$$

for each $r\leq n$, we see that this action is well-defined.

Let us note the following property of U_0.

Lemma 7.3. *The maps* $\mathcal{A}(F^n(-),F^pM)\xrightarrow{G^p}\mathcal{A}(G^pF^n(-),G^pF^pM)\to\mathcal{A}(G^pF^n(-),M)$ *induce a quasi-isomorphism*

$$\mathcal{B}(-,M)=\operatorname{colim}_{p\geq 0}\bigoplus_{n\geq 0}\mathcal{A}(F^n(-),F^pM)\longrightarrow\operatorname{colim}_{p\geq 0}\bigoplus_{n\geq 0}\mathcal{A}(G^pF^n(-),M)=U_0(-,M)$$

of DG $\mathcal{B}$*-modules.*

Proof. The commutative diagram

$$\begin{tikzcd}
\mathcal{A}(F^n(-),F^pM) \arrow[r, "G^p"] \arrow[d, "F"'] & \mathcal{A}(G^pF^n(-),G^pF^pM) \arrow[r] & \mathcal{A}(G^pF^n(-),M) \arrow[d] \\
\mathcal{A}(F^{n+1}(-),F^{p+1}M) \arrow[r, "G^{p+1}"] & \mathcal{A}(G^{p+1}F^{n+1}(-),G^{p+1}F^{p+1}M) \arrow[r] & \mathcal{A}(G^{p+1}F^{n+1}(-),M)
\end{tikzcd}$$

shows the existence of the morphism on the colimits.

It is clear that the induced morphism is a quasi-isomorphism since F and G are mutually inverse equivalences on $H^0\mathcal{A}$. □

Note that $U_q(-,-)$ constructed above does not have a $\mathcal{C}$-action. The next step toward relating $\mathcal{B}$ and $\mathcal{C}$ is constructing a DG $(\mathcal{C},\mathcal{B})$-bimodule that is quasi-isomorphic over $\mathcal{B}$ to U_0.

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

Lemma 7.4. *The vertical maps in equation (7.1) induce a sequence of quasi-isomorphisms*

$$U_0(-,M) \longrightarrow U_1(-,M) \longrightarrow U_2(-,M) \longrightarrow \cdots$$

of DG $\mathcal{B}$-modules for each $M \in \mathcal{A}$.

Proof. Since the vertical maps in equation (7.1) are quasi-isomorphisms, the induced map $U_q(L,M) \to U_{q+1}(L,M)$ is a quasi-isomorphism for each $q \geq 0$, $L \in \mathcal{B}$. It is easily verified that $U_q(-,M) \to U_{q+1}(-,M)$ is $\mathcal{B}$-linear. □

Now define the DG $\mathcal{B}$-module $U(-,M)$ by the colimit

$$U(-,M) := \operatorname*{colim}_{q\geq 0} U_q(-,M).$$

For each $L \in \mathcal{B}$, we have $U(L,M) = \operatorname{colim}_{p,q\geq 0} \bigoplus_{n\geq 0} \mathcal{A}(G^{p+q}F^nL, G^qM)$.

We observe that $U(L,M)$ has a left $\mathcal{C}$-action as follows: Let $c \in \mathcal{C}(M,N)$ be a morphism presented by $G^mM \to G^rN$ and $u \in U(L,M)$ an element presented by $G^{p+q}F^nL \to G^qM$. Enlarging m and p if necessary we may assume $q \leq m$ and $r \leq p$. Then define $c \cdot u$ by the composition $G^{p+m}F^nL \xrightarrow{G^{m-q}u} G^mM \xrightarrow{c} G^rN$. It is clear that this is well-defined and compatible with the right $\mathcal{B}$-action. We have therefore obtained a $(\mathcal{C},\mathcal{B})$-bimodule U.

As a final preparation, we describe an equivalence between the orbit categories $H^0\mathcal{B}$ and $H^0\mathcal{C}$.

Lemma 7.5. *The maps $H^0\mathcal{A}(F^nL, F^pM) \xrightarrow{G^{n+p}} H^0\mathcal{A}(G^{n+p}F^nL, G^{n+p}F^pM) \xrightarrow{\simeq} H^0\mathcal{A}(G^{n+p}F^nL, G^nM) \xleftarrow{\simeq} H^0\mathcal{A}(G^pL, G^nM) \to H^0\mathcal{C}(L,M)$ induce an equivalence $H^0\mathcal{B} \simeq H^0\mathcal{C}$.*

Proof. We have to show that the following diagram of isomorphisms is commutative, where we write ${}_h(G^aF^b, G^cF^d)$ for $H^0\mathcal{A}(G^aF^bL, G^cF^dM)$, and the unlabeled maps are induced by $G\circ F \to 1_{\mathcal{A}}$:

$$\begin{tikzcd}
{}_h(F^n,F^p) \arrow[r, "G^{n+p}"] \arrow[d, "F"'] & {}_h(G^{n+p}F^n, G^{n+p}F^p) \arrow[r] & {}_h(G^{n+p}F^n, G^n) & {}_h(G^p, G^n) \arrow[l] \arrow[d, "G"] \\
{}_h(F^{n+1},F^{p+1}) \arrow[r, "G^{n+p+2}"] & {}_h(G^{n+p+2}F^{n+1}, G^{n+p+2}F^{p+1}) \arrow[r] & {}_h(G^{n+p+2}F^{n+1}, G^{n+1}) & {}_h(G^{p+1}, G^{n+1}). \arrow[l]
\end{tikzcd}$$

We see this by filling and lifting the above diagram to a diagram of quasi-isomorphism in $\mathcal{A}$ as below. Here again we omit the objects L and M:

$$\begin{tikzcd}
\mathcal{A}(F^n,F^p) \arrow[r, "G^{n+p}"] \arrow[dd, "F"'] & \mathcal{A}(G^{n+p}F^n, G^{n+p}F^p) \arrow[r] \arrow[d, "G"] & \mathcal{A}(G^{n+p}F^n, G^n) \arrow[d, "G"] & \mathcal{A}(G^p, G^n) \arrow[l] \arrow[dd, "G"] \\
 & \mathcal{A}(G^{n+p+1}F^n, G^{n+p+1}F^p) \arrow[r] & \mathcal{A}(G^{n+p+1}F^n, G^{n+1}) \arrow[d] & \\
\mathcal{A}(F^{n+1},F^{p+1}) \arrow[r, "G^{n+p+2}"] & \mathcal{A}(G^{n+p+2}F^{n+1}, G^{n+p+2}F^{p+1}) \arrow[r] & \mathcal{A}(G^{n+p+2}F^{n+1}, G^{n+1}) & \mathcal{A}(G^{p+1}, G^{n+1}). \arrow[l]
\end{tikzcd}$$

It is now easy to verify that this diagram commutative. □

Now we are ready to prove the main theorem of this section.

Proof of Theorem 7.1. We show that the $(\mathcal{C},\mathcal{B})$-bimodule

$$U(L,M) = \operatorname*{colim}_{p,q\geq 0} \bigoplus_{n\geq 0} \mathcal{A}(G^{p+q}F^nL, G^qM)$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

constructed above gives a quasi-equivalence. Since the DG categories $\mathcal{B}$ and $\mathcal{C}$ have shifts (that is, the Yoneda embedding $\mathcal{B} \hookrightarrow \mathcal{C}_{\mathrm{dg}}(\mathcal{B})$ is closed under $[\pm 1]$ and the same for $\mathcal{C}$), we only have to show that $-\otimes^{\mathrm{L}}_{\mathcal{C}} U$ induces an equivalence $H^0\mathcal{C} \xrightarrow{\simeq} H^0\mathcal{B}$.

For each $M \in \mathcal{C}$, we have quasi-isomorphisms

$$u_M : \mathcal{B}(-, M) \to U_0(-, M) \to U(-, M)$$

of DG $\mathcal{B}$-modules by Lemma 7.3 and Lemma 7.4; thus U is a quasi-functor.

It remains to show that the induced map

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(\mathcal{C}(-, L), \mathcal{C}(-, M)) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(U(-, L), U(-, M))$$

is an isomorphism for each $L, M \in \mathcal{A}$. It suffices to show that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(\mathcal{C}(-, L), \mathcal{C}(-, M)) \arrow[r, "-\otimes^{\mathrm{L}}_{\mathcal{C}} U"] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(U(-, L), U(-, M)) \arrow[r, "\simeq", "u_L"'] & \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(\mathcal{B}(-, L), U(-, M)) \\
H^0\mathcal{C}(L, M) & H^0\mathcal{B}(L, M) \arrow[l, "\simeq"'] \arrow[r, equal] & \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(\mathcal{B}(-, L), \mathcal{B}(-, M)). \arrow[u, "u_M", "\wr"']
\end{tikzcd}$$

The equivalence $H^0\mathcal{B} \to H^0\mathcal{C}$ given in Lemma 7.5 shows that if $f \in H^0\mathcal{B}(L, M)$ is presented by a morphism $F^nL \to F^pM$ in $Z^0\mathcal{A}$, then the corresponding morphism $g \in H^0\mathcal{C}(L, M)$ is presented by $G^pL \to G^nM$ in $Z^0\mathcal{A}$ so that the diagram

$$\begin{tikzcd}
G^{n+p}F^nL \arrow[r, "G^{n+p}f"] \arrow[d] & G^{n+p}F^pM \arrow[d] \\
G^pL \arrow[r, "g"] & G^nM
\end{tikzcd} \tag{7.2}$$

is commutative in $H^0\mathcal{A}$.

Let f and g be of the form above. Then the commutativity we want amounts to saying that $u_M \cdot f = g \cdot u_L$ in $H^0U(L, M)$, where we may regard $\cdot$ as the right action of $\mathcal{B}$ (respectively, left action of $\mathcal{C}$) on U. Since $u_L \in U(L, L)$ is presented by the identity morphism in $\mathcal{A}$ the element $g \cdot u_L \in U(L, M)$ is presented by $G^{p+n}F^nL \to G^pL \xrightarrow{1} G^pL \xrightarrow{g} G^nM$. Similarly, since $u_M \in U(M, M)$ is presented by the identity morphism in $\mathcal{A}$, the element $u_M \cdot f \in U(L, M)$ is presented by $G^pF^nL \xrightarrow{G^pf} G^pF^pM \to M$, hence by $G^{n+p}F^nL \xrightarrow{G^{n+p}f} G^{n+p}F^pM \to G^nM$ obtained by applying G^n. Then the commutativity of the diagram (7.2) in $H^0\mathcal{A}$ implies that we have $u_M \cdot f = g \cdot u_L$ in $H^0U(L, M)$. □

Let us apply our general result to the setting of finite-dimensional algebras. Let Λ be a finite-dimensional algebra of finite global dimension, $\mathcal{A} = \mathcal{C}^{-,b}(\mathrm{proj}\,\Lambda)$, and X a complex of projective (Λ, Λ)-bimodules such that $F = -\otimes_\Lambda X : \mathcal{A} \to \mathcal{A}$ induces an autoequivalence on $H^0\mathcal{A} = \mathcal{D}^b(\mathrm{mod}\,\Lambda)$. Suppose that for each $L, M \in \mathcal{D}^b(\mathrm{mod}\,\Lambda)$, we have $\mathrm{Hom}_{\mathcal{D}(\Lambda)}(L, F^iM) = 0$ for almost all $i \in \mathbb{Z}$. Let $\mathcal{B} = \mathcal{A}/F$ be the DG orbit category and $\Gamma = \Lambda \oplus X[-1]$ the trivial extension DG algebra. Note that our assumptions imply that the orbit category $H^0\mathcal{B}$ is idempotent complete since we have $J_{H^0\mathcal{B}}(L, L) = J_{H^0\mathcal{A}}(L, L) \oplus \bigoplus_{i\neq 0} H^0\mathcal{A}(L, F^iL)$ for each L, that is indecomposable in $H^0\mathcal{A}$, and where J is the Jacobson radical of each category. It follows that the triangulated hull $\mathrm{tria}\,\mathcal{B}$ equals

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

the perfect derived category $\operatorname{per}\mathcal{B}$. Then Keller's theorem [Ke2, Theorem 2, also correction] gives an equivalence

$$\operatorname{per}\mathcal{B} \xrightarrow{\simeq} \operatorname{thick}_{\mathcal{D}(\Gamma)} \Lambda/\operatorname{per}\Gamma,$$

which is compatible with the natural functors from $\mathcal{D}^b(\operatorname{mod}\Lambda)$.

We have the same equivalence arising from the inverse of F. Let Y be a bimodule projective resolution of $\operatorname{RHom}_\Lambda(X,\Lambda)$ and $G = -\otimes_\Lambda Y \colon \mathcal{A} \to \mathcal{A}$, which induces a quasi-inverse to $F = -\otimes_\Lambda X$ on $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Let $\mathcal{C} = \mathcal{A}/G$ be the DG orbit category and $\Delta = \Lambda \oplus Y[-1]$ be the trivial extension DG algebra so that we have an equivalence

$$\operatorname{per}\mathcal{C} \xrightarrow{\simeq} \operatorname{thick}_{\mathcal{D}(\Delta)} \Lambda/\operatorname{per}\Delta\ .$$

By the above equivalences and Theorem 7.1, we deduce the following consequence.

Corollary 7.6. *There exists a triangle equivalence*

$$\operatorname{thick}_{\mathcal{D}(\Gamma)} \Lambda/\operatorname{per}\Gamma \xrightarrow{\simeq} \operatorname{thick}_{\mathcal{D}(\Delta)} \Lambda/\operatorname{per}\Delta\ ,$$

which is compatible with the projection functors from $\mathcal{D}^b(\operatorname{mod}\Lambda)$.

8. Proof of Theorem 6.4

We now give a proof of a main result, Theorem 6.4, of this paper using the result from the previous section. In fact, the essential part of the proof does not depend on our specific setup, so we first state the intermediate result in Proposition 8.1 below, which can also be viewed as a DG version of Theorem 6.4.

Let Λ be a finite-dimensional algebra that is homologically smooth and X a complex of (Λ,Λ)-bimodules such that $F = -\otimes^{\mathrm{L}}_\Lambda X$ gives an autoequivalence on $\mathcal{D}^b(\operatorname{mod}\Lambda)$. We assume the following on the tilting complex X:

(X1) For each $L, M \in \mathcal{D}^b(\operatorname{mod}\Lambda)$, we have $\operatorname{Hom}_{\mathcal{D}(\Lambda)}(L, F^i M) = 0$ for almost all $i \in \mathbb{Z}$.
(X2) X is concentrated in degree ≤ 0.

Let

$$\Gamma = \Lambda \oplus X[-1], \quad \Sigma = T^{\mathrm{L}}_\Lambda X$$

be the trivial extension DG algebra and, respectively, the derived tensor algebra: that is, the tensor algebra of a bimodule projective resolution of X. Then the condition (X2) shows that Σ is a negative DG algebra, and (X1) implies that its cohomology $H^0\Sigma = \bigoplus_{i\geq 0}\operatorname{Hom}_{\mathcal{D}(\Lambda)}(\Lambda, F^i\Lambda)$ is finite dimensional.

Recall from the introduction that we have denoted by

$$\mathcal{C}(\Pi) = \operatorname{per}\Pi/\mathcal{D}^b(\Pi)$$

and called it the cluster category for any DG algebra Π satisfying $\operatorname{per}\Pi \supset \mathcal{D}^b(\Pi)$. Our general intermediate result is an equivalence between the cluster category of the tensor algebra and the singularity category of the trivial extension algebra.

Proposition 8.1. *The DG algebra Σ satisfies $\operatorname{per}\Sigma \supset \mathcal{D}^b(\Sigma)$, and there exists a triangle equivalence*

$$\mathcal{C}(\Sigma) \simeq \operatorname{thick}_{\mathcal{D}(\Gamma)} \Lambda/\operatorname{per}\Gamma.$$

The first step is to apply Corollary 7.6. Let Y be a bimodule projective resolution of $\operatorname{RHom}_\Lambda(X,\Lambda)$, and set

$$\Delta = \Lambda \oplus Y[-1].$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

Then we have a triangle equivalence

$$\mathrm{thick}_{\mathcal{D}(\Gamma)} \Lambda/\mathrm{per}\,\Gamma \simeq \mathrm{thick}_{\mathcal{D}(\Delta)} \Lambda/\mathrm{per}\,\Delta. \tag{8.1}$$

We next use the following computation of a DG endomorphism algebra.

Lemma 8.2 (See [Am, Lemma 4.13]). *There exists an isomorphism* $\mathrm{RHom}_\Delta(\Lambda,\Lambda) \simeq T^{\mathrm{L}}_\Lambda X = \Sigma$ *in the homotopy category of DG algebras: that is, these two DG algebras are related by a zig-zag of quasi-isomorphisms of DG algebras.*

We also need the equivalence by relative Koszul dual.

Lemma 8.3. *The functor* $\mathrm{RHom}_\Delta(\Lambda,-)\colon \mathcal{D}(\Delta) \to \mathcal{D}(\Sigma)$ *restricts to equivalences* $\mathrm{thick}_{\mathcal{D}(\Delta)} \Lambda \simeq \mathrm{per}\,\Sigma$ *and* $\mathrm{per}\,\Delta \simeq \mathcal{D}^b(\Sigma)$. *Therefore, we have* $\mathrm{per}\,\Sigma \supset \mathcal{D}^b(\Sigma)$ *and an equivalence* $\mathrm{thick}_{\mathcal{D}(\Delta)} \Lambda/\mathrm{per}\,\Delta \xrightarrow{\simeq} \mathcal{C}(\Sigma)$.

Proof. The first assertion is clear. We prove the second one. Since $\Delta = \mathrm{RHom}_\Lambda(\Delta, Y[-1])$ as (Λ,Δ)-bimodules, we have $\mathrm{RHom}_\Delta(\Lambda,\Delta) = \mathrm{RHom}_\Delta(\Lambda, \mathrm{RHom}_\Lambda(\Delta, Y[-1])) = \mathrm{RHom}_\Lambda(\Lambda, Y[-1]) = Y[-1]$, which has finite-dimensional total cohomology. Therefore, the functor maps $\mathrm{per}\,\Delta$ into $\mathcal{D}^b(\Sigma)$. It remains to show the essential surjectivity. Since Σ is a negative DG algebra, the finite-dimensional derived category $\mathcal{D}^b(\Sigma)$ has a bounded t-structure whose heart $\mathcal{H}$ is equivalent to the category of finite-dimensional modules over $H^0\Sigma$. Indeed, we may assume that the terms of Σ are concentrated in degree ≤ 0, and we see that the functors $H^0\colon \mathcal{H} \to \mathrm{mod}\,H^0\Sigma$ and $\mathrm{mod}\,H^0\Sigma \to \mathcal{H}$ given by the restriction along $\Sigma \twoheadrightarrow H^0\Sigma$ yield mutually inverse equivalences by adapting [Am, Proposition 2.3(i)]. So it suffices to show that the heart is contained in the image of the functor. Note that $H^0\Sigma = T_\Lambda(H^0X)$ is a finite-dimensional graded algebra whose degree 0 part Λ has finite global dimension. Therefore, it is sufficient to prove that $D\Lambda$ is in the image. Clearly, $D\Lambda = \mathrm{RHom}_\Delta(\Lambda, D\Delta)$, so it remains to show $D\Delta \in \mathrm{per}\,\Delta$. But we have $D\Delta = \mathrm{RHom}_\Lambda(\Delta, D\Lambda)$ and $D\Lambda \in \mathrm{thick}_{\mathcal{D}(\Lambda)} Y[-1]$, hence the assertion. □

Proof of Proposition 8.1. It is a consequence of equation (8.1) and Lemma 8.3. □

We now return to our setup from Section 6. Recall that R is a negatively graded bimodule $(d+1)$-CY algebra of a-invariant a such that each R_i is finite dimensional and that A is a d-representation infinite algebra in Proposition 4.9 given by

$$A = \begin{pmatrix} R_0 & 0 & \cdots & 0 \\ R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \end{pmatrix},$$

whose d-AR-translation ν_d has an ath root defined by diagram (4.1). We have a cotilting bimodule $U = \nu_d^{-1/a}A$ and a d-Iwanaga-Gorenstein algebra $B = A \oplus U$ (see Proposition 6.3).

Let us first apply Proposition 8.1. Set $\Lambda = A$ and $X = U[1]$. Since U is a preprojective module over a d-representation infinite algebra, it clearly satisfies (X1) and (X2). Then the DG algebra $\Gamma = \Lambda \oplus X[-1]$ concentrates in degree 0 and is nothing but our Iwanaga-Gorenstein algebra $B = A \oplus U$. Since A has finite global dimension, the right-hand side of Proposition 8.1 is precisely the singularity category $\mathcal{D}_{\mathrm{sg}}(B)$ of B. Also, we see that the DG algebra Σ is the tensor algebra

$$S := T_A(U[1])$$

with trivial differentials. Proposition 8.1 for this case then gives the following equivalence.

Corollary 8.4. *There exists a triangle equivalence*

$$\mathcal{C}(S) \simeq \mathcal{D}_{\mathrm{sg}}(B).$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

To compare the cluster categories of S and R^{dg}, we need another intermediate DG algebra, which is

$$\widetilde{S} := \mathcal{E}\mathrm{nd}_{R^{\mathrm{dg}}}(\widetilde{T}), \text{ with } \widetilde{T} = R \oplus R[-1] \oplus \cdots \oplus R[-(a-1)],$$

where $\mathcal{E}\mathrm{nd}_{R^{\mathrm{dg}}}(-) = \mathcal{H}\mathrm{om}_{R^{\mathrm{dg}}}(-,-)$ is the endomorphism DG algebra.

Let us first state an easy relationship between R^{dg} and $\widetilde{S}$. We say that DG algebras Π_1 and Π_2 are *DG Morita equivalent* if there is a (Π_1, Π_2)-bimodule X such that $-\otimes^{\mathrm{L}}_{\Pi_1} X$ induces an equivalence $\mathcal{D}(\Pi_1) \xrightarrow{\simeq} \mathcal{D}(\Pi_2)$, or equivalently there exists a compact generator $M \in \mathcal{D}(\Pi_2)$ whose DG endomorphism algebra $\mathrm{RHom}_{\Pi_2}(M, M)$ is quasi-equivalent to Π_1 [Ke1, Theorem 8.2]; see also [Ke3, Section 3.8].

We immediately have the following lemma.

Lemma 8.5.

1. *The DG algebras R^{dg} and $\widetilde{S}$ are DG Morita equivalent.*
2. *We have* $\mathrm{per}\,\widetilde{S} \supset \mathcal{D}^b(\widetilde{S})$.
3. *The cluster categories of R^{dg} and $\widetilde{S}$ are equivalent.*

Proof. Since $\widetilde{S}$ is the DG endomorphism ring of a compact generator $\widetilde{T} \in \mathcal{D}(R^{\mathrm{dg}})$, we have (1). Then the assertions (2) and (3) follow. □

We next discuss the relationship between S and $\widetilde{S}$.

Lemma 8.6.

1. *S is a finite codimensional DG subalgebra of $\widetilde{S}$.*
2. *The cluster categories of S and $\widetilde{S}$ are equivalent.*

Proof. (1) Since both S and $\widetilde{S}$ have trivial differentials, we only have to regard them as graded algebras. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{qper}^{\mathbb{Z}} R \arrow[loop left, "(-1)"] \arrow[r, "\simeq"] & \mathcal{D}^b(\mathrm{mod}\, A) \arrow[loop right, "-\otimes^{\mathrm{L}}_A U"] \end{tikzcd}$$

taking the tilting object $T = \bigoplus_{l=0}^{a-1} R(-l)$ in Proposition 4.9 to A. We have $S = \bigoplus_{i\geq 0} \mathrm{Hom}_{\mathcal{D}(A)}(A, U^i)$, where U^i is the i-fold (derived) tensor product of U; thus it is isomorphic to $\bigoplus_{i\leq 0} \mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(T, T(i))$. On the other hand, since $\widetilde{T}$ is nothing but T regarded as a DG module, we have $\widetilde{S} = \mathrm{End}_R(T)$ as graded algebras. This is equal to $\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Mod}^{\mathbb{Z}} R}(T, T(i)) = \bigoplus_{i\leq a-1} \mathrm{Hom}_{\mathrm{Mod}^{\mathbb{Z}} R}(T, T(i))$, and by Lemma 4.7 to $\bigoplus_{i\leq a-1} \mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(T, T(i))$, which gives our assertion.

(2) Consider the pair of adjoint functors $F = -\otimes^{\mathrm{L}}_S \widetilde{S} \colon \mathcal{D}(S) \to \mathcal{D}(\widetilde{S})$ and $G = \mathrm{res} \colon \mathcal{D}(\widetilde{S}) \to \mathcal{D}(S)$.

Step 1: Restrictions of the adjoint pair. We observe that these functors restrict to the perfect and finite-dimensional derived categories. Clearly, $F = -\otimes^{\mathrm{L}}_S \widetilde{S}$ restricts to $\mathrm{per}\, S \to \mathrm{per}\, \widetilde{S}$ and $G = \mathrm{res}$ to $\mathcal{D}^b(\widetilde{S}) \to \mathcal{D}^b(S)$. Also, since $\widetilde{S}$ is perfect over S by (1) and Lemma 8.3, the remaining assertions follow. Therefore, the above functors induce an adjoint pair between the Verdier quotients.

Step 2: The unit and counit maps. We show that the unit and counit maps are isomorphisms. Let $X \in \mathcal{C}(S)$, and consider the unit map $u_X : X \to X\otimes^{\mathrm{L}}_S \widetilde{S}$. Since this is obtained by applying $X\otimes^{\mathrm{L}}_S -$ to an isomorphism $S \to \widetilde{S}$ in $\mathcal{C}(S)$, it is an isomorphism. Next let $Y \in \mathcal{C}(\widetilde{S})$ and $v_Y : FGY \to Y$ the counit. Note that G detects isomorphisms. Indeed, v_Y is an isomorphism in $\mathcal{C}(\widetilde{S})$ if and only if, as a map in $\mathrm{per}\, \widetilde{S}$, the cone of v_Y is in $\mathcal{D}^b(\widetilde{S})$. But this property is detected by the restriction functor G. Now the claim $G(v_Y)$ is an isomorphism follows from the fact that the composition $GY \xrightarrow{u_{GY}} GFGY \xrightarrow{G(v_Y)} GY$ equals the identity, which is a general property of an adjoint pair and the isomorphism of the unit. □

Now, Theorem 6.4 is a consequence of the following sequence of equivalences in the upper row:

$$\begin{tikzcd} \mathcal{D}_{\mathrm{sg}}(B) \arrow[rr, equal, "\text{Cor 8.4}"] \arrow[dr, equal, "\text{Cor 7.6}"'] & & \mathcal{C}(S) \arrow[r, equal, "\text{Lem 8.6}"] & \mathcal{C}(\widetilde{S}) \arrow[r, equal, "\text{Lem 8.5}"] & \mathcal{C}(R^{\mathrm{dg}}). \\ & \mathrm{thick}_{\mathcal{D}(C)} A/\mathrm{per}\, C \arrow[ur, equal, "\text{Lem 8.3}"'] & & & \end{tikzcd}$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

Here we have included $C = A \oplus \mathrm{RHom}_A(U[1], A)[-1]$, which is the DG algebra Δ for our specific setup.

We record the following formula for the d-representation infinite algebra A, the cotilting bimodule U and the d-Iwanaga-Gorenstein algebra B, which are determined by R.

$$A = \begin{pmatrix} R_0 & 0 & \cdots & 0 \\ R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \end{pmatrix}, \quad U = \begin{pmatrix} R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \\ R_{-a} & R_{-(a-1)} & \cdots & R_{-1} \end{pmatrix}, \quad B = A \oplus U. \tag{8.2}$$

Let us also record the equivalences we have shown:

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_d^{-1/a}[1] \hookrightarrow \mathcal{D}_{\mathrm{sg}}(B) \xrightarrow{\simeq} \mathcal{C}(R^{\mathrm{dg}}). \tag{8.3}$$

We end this section with the obvious lemma, which is useful for later computation.

Lemma 8.7. *Let J_0 be the Jacobson radical of R_0.*

1. *The Jacobson radical J_A of A is* $\begin{pmatrix} J_0 & 0 & \cdots & 0 \\ R_{-1} & J_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & J_0 \end{pmatrix}$.
2. *The Jacobson radical J_B of B is $J_A \oplus U$.*
3. *We have $J_B/J_B^2 = J_A/J_A^2 \oplus U/(J_AU + UJ_A)$, with*

$$U/(J_AU + UJ_A) = \begin{pmatrix} 0 & R_0/J_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_0/J_0 \\ \frac{R_{-a}}{J_0R_{-a} + R_{-a}J_0 + \sum_{i+j=a, i,j>0} R_{-i}R_{-j}} & 0 & \cdots & 0 \end{pmatrix}.$$

9. Higher cluster categories of higher representation infinite algebras

We give an application of Theorem 6.4, which is given by taking the CY algebra R as a (higher) preprojective algebra. We prove that any m-cluster category of a d-representation infinite algebra with $m > d$ is the singularity category of a d-Iwanaga-Gorenstein algebra. (Recall by convention that our m-cluster category is m-CY.) Moreover, we explicitly describe the quiver and relations of the Iwanaga-Gorenstein algebra for the case $d = 1$: that is, when A is hereditary.

Theorem 9.1. *Let A be a d-representation infinite algebra such that A/J_A is separable over k, and fix $n \geq 1$. Let $U = \mathrm{Ext}_A^d(DA, A)$ and*

$$B = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix} \oplus \begin{pmatrix} 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \\ U & 0 & \cdots & 0 \end{pmatrix}$$

the trivial extension algebra, where the matrix is $n \times n$. Then B is d-Iwanaga-Gorenstein, and there exists a triangle equivalence

$$\mathcal{C}_{d+n}(A) \simeq \mathcal{D}_{\mathrm{sg}}(B).$$

Proof. Let $\Pi = T_A U$ be the $(d+1)$-preprojective algebra of A. Give a grading on Π by setting $\deg U = -n$ so that Π is a bimodule $(d+1)$-CY algebra of a-invariant n. Note that Π^{dg} is quasi-isomorphic to the derived $(d+n+1)$-preprojective algebra (or the $(d+n+1)$-Calabi-Yau completion) $T_A^{\mathrm{L}}(\mathrm{RHom}_A(DA, A)[d+n])$, in the sense of [Ke6]. Therefore, the cluster category $\mathcal{C}(\Pi^{\mathrm{dg}})$ is nothing but the $(d+n)$-cluster category $\mathcal{C}_{d+n}(A)$ of A. By Theorem 6.4, we have a triangle equivalence $\mathcal{C}(\Pi^{\mathrm{dg}}) \simeq \mathcal{D}_{\mathrm{sg}}(B(\Pi))$ for the d-Iwanaga-Gorenstein algebra $B(\Pi)$ in equation (8.2) for the Calabi-Yau algebra Π, which is precisely B in the above statement. □

Remark 9.2. We give a general discussion in Appendix A of the effect of 'multiplying gradings' as we did in the above proof. See Corollary A.2 for a description as a derived orbit category, which predicts the above equivalence.

9.1. The case $d = 1$.

Let us record the special case $d = 1$: that is, when A is hereditary.

Corollary 9.3. *Let Q be a finite connected acyclic non-Dynkin quiver, $A = kQ$ its path algebra, and fix $n \geq 1$. Then we have a triangle equivalence*

$$\mathcal{C}_{n+1}(kQ) \simeq \mathcal{D}_{\mathrm{sg}}(B)$$

for the 1-Iwanaga-Gorenstein algebra B in Theorem 9.1.

In this case, we can explicitly describe the quiver and relations for B.

Proposition 9.4. *The 1-Iwanaga-Gorenstein algebra B in Theorem 9.1 is presented by the quiver $\widehat{Q}$ with*

(a) *vertices $Q_0 \times \{1, \dots, n\}$,*
(b) *three kinds of arrows:*
   (i) $a = a^l \colon (i, l) \to (j, l)$ *for each $a \colon i \to j$ in Q_1 and $1 \leq l \leq n$.*
   (ii) $v = v_i^l \colon (i, l+1) \to (i, l)$ *for each $i \in Q_0$ and $1 \leq l < n$.*
   (iii) $a^* \colon (j, 1) \to (i, n)$ *for each $a \colon i \to j$ in Q_1.*
(c) *three kinds of relations:*
   (i) $a^l v_i^l = v_j^l a^{l+1}$ *for each $a \colon i \to j$ in Q_1 and $1 \leq l < n$.*
   (ii) $\sum_{s(a)=i} a^* a^1 = \sum_{t(a)=i} a^n a^*$ *for all $i \in Q_0$.*
   (iii) $v_i^{l-1} v_i^l = 0$, $a^* v_j^1 = 0$, $v_i^{n-1} a^* = 0$ *for $a \colon i \to j$ in Q if $n \geq 2$, and $a^* b c^* = 0$ for any composable $a, b, c \in Q_1$ if $n = 1$.*

We use a reformulation of a well-known fact on preprojective algebras.

Lemma 9.5. *Let Q be a finite acyclic quiver, $A = kQ$ and $U = \mathrm{Ext}_A^1(DA, A)$. We denote by e_i the idempotent of A at the vertex i. Then there exists a subset $\mathcal{B} = \{u(a^*) \mid a \in Q_1\}$ of U satisfying the following conditions:*

- *If $a \colon i \to j$ is an arrow in Q, then $e_i u(a^*) = u(a^*) = u(a^*) e_j$ and $e_l u(a^*) = u(a^*) e_m = 0$ for $l \neq i$ and $m \neq j$.*
- *The image of $\mathcal{B}$ in $U/(J_A U + U J_A)$ is a k-basis.*
- $\sum_{a \in Q_1} (a u(a^*) - u(a^*) a) = 0$ *in U.*

Proof. Consider the two presentations $T_A U = k\overline{Q}/(\sum_{a \in Q_1}(a a^* - a^* a))$ of the preprojective algebra Π of Q, where $\overline{Q}$ is the double quiver of Q obtained by adding the opposite arrows $Q^* := \{a^* \colon j \to i \mid$

$a\colon i \to j$ in $Q\}$. Take the elements of U corresponding to $\{a^* \mid a \in Q_1\} \subset \overline{Q}_1$. This is a desired set since $U/(J_AU + UJ_A)$ is isomorphic as (kQ_0, kQ_0)-bimodules to the one spanned by Q^*. □

Proof of Proposition 9.4. By Lemma 8.7, we have

$$J_B/J_B^2 = \begin{pmatrix} J_A/J_A^2 & 0 & \cdots & 0 \\ 0 & J_A/J_A^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_A/J_A^2 \end{pmatrix} \oplus \begin{pmatrix} 0 & A/J_A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A/J_A \\ U/(J_AU+UJ_A) & 0 & \cdots & 0 \end{pmatrix}.$$

Therefore, we see that the quiver of B consists of the following:

- n copies $Q^1, \ldots, Q^n$ of Q,
- The arrows from Q^{l+1} to Q^l corresponding to the idempotents in A,
- The arrows from Q^1 to Q^n corresponding to the basis of $U/(J_AU + UJ_A)$ as (kQ_0, kQ_0)-bimodules.

In view of Lemma 9.5, these vertices and arrows are precisely the ones described in (a) and (b), hence the quiver of B is $\widehat{Q}$.

Now we determine the relations. To simplify the discussion below, we give a grading on B in Theorem 9.1 by setting the first factor to have degree 0 and the second one to have degree 1. Similarly, give a grading on $\widehat{Q}$ by setting the arrows in (i) to have degree 0 and in (ii), (iii) to have degree 1. Take a subset $\{u(a^*) \mid a \in Q_1\}$ of U given in Lemma 9.5, and consider the map $\widehat{Q} \to B$ defined by

- the natural embedding $kQ^l \to A \times \cdots \times A = B_0$ into the lth factor,
- $v_i^l \mapsto e_i$, where e_i is the corresponding idempotent of A in the $(l, l+1)$-component of B_1,
- $a^* \mapsto u(a^*)$, where $u(a^*) \in U$ is in the $(n, 1)$-component of B_1.

These maps induce a homogeneous homomorphism $\varphi\colon k\widehat{Q} \to B$, which clearly preserves the relations. Denoting by I the ideal generated by the relations, we obtain a homomorphism $k\widehat{Q}/I \to B$. Since it is an isomorphism in degree 0, it is enough to consider the degree 1 part. Let $e_{(i,l)}$ be the idempotent of $k\widehat{Q}/I$ at the vertex (i, l), and set $e_l = \sum_{i\in Q_0} e_{(i,l)}$. We denote their images under φ by the same symbols. It is sufficient to show that φ induces an isomorphism $e_l(k\widehat{Q}/I)e_m \to e_lBe_m$ for each $1 \le l, m \le n$. By the relation (iii), each term is 0 in degree 1 unless $m - l = 1$ or $(l, m) = (n, 1)$, so we only have to consider these two cases.

Case 1: The map $e_l(k\widehat{Q}/I)e_{l+1} \to e_lBe_{l+1}$ is an isomorphism for each $1 \le l < n$. By the relation (i), any element in $e_l(k\widehat{Q}/I)e_{l+1}$ can be written as $a \cdot (\sum_{i\in Q_0} v_i^l)$ for some $a \in kQ^l = A$. This observation immediately shows the map is an isomorphism.

Case 2: The map $e_n(k\widehat{Q}/I)e_1 \to e_nBe_1$ is an isomorphism. By the relation (ii), the space $e_n(k\widehat{Q}/I)e_1$ is isomorphic to the degree 1 part of the preprojective algebra of Q; thus to U. On the other hand, the space e_nBe_1 is also clearly U. □

We look at the most special case $d = 1$ and $n = 1$.

Example 9.6. Let Q be a finite connected acyclic non-Dynkin quiver and kQ its path algebra, which is 1-representation-infinite.

The 1-Iwanaga-Gorenstein algebra $B = kQ \oplus U$ with $U = \tau^{-1}kQ$ is a truncation of the preprojective algebra Π of Q, which is presented by the same quiver as Π and the additional relations 'the elements of U square to zero', as stated in Proposition 9.4.

The equivalence $\mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}_2(kQ)$ in Corollary 9.3 is given in [BIRS, ART] as the 2-CY category associated to the square of the Coxeter element in the Coxeter group of Q. Our proof is different from theirs since our equivalence comes from quasi-equivalence of DG orbit categories.

The next example is the case $d = 1$ and $n = 2$.

Example 9.7. Let Q be the following quiver

$$\begin{tikzcd} \circ \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \circ \arrow[r, "c"] & \circ \end{tikzcd},$$

thus $A = kQ$ is 1-representation infinite. Let $n = 2$, so by Proposition 9.4, the 1-Iwanaga-Gorenstein algebra B is presented by the following quiver with relations:

$$\begin{tikzcd} \circ \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \circ \arrow[r, "c"] \arrow[dl, shift right, "a^*"'] \arrow[dl, shift left, "b^*"] & \circ \arrow[dl, "c^*"'] \\ \circ \arrow[u, "v"] \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \circ \arrow[u, "v"] \arrow[r, "c"'] & \circ \arrow[u, "v"'] \end{tikzcd} \qquad \begin{gathered} av = va,\ bv = vb,\ cv = vc \\ a^*a + b^*b = 0,\ aa^* + bb^* = c^*c,\ cc^* = 0 \\ va^* = 0,\ vb^* = 0,\ vc^* = 0, \quad a^*v = 0,\ b^*v = 0,\ c^*v = 0. \end{gathered}$$

By Corollary 9.3, we have a triangle equivalence $\mathcal{C}_3(kQ) \simeq \mathcal{D}_{\mathrm{sg}}(B)$.

9.2. The case $n = 1$.

We now turn to another special case of $n = 1$. In this case, the algebra B is a truncation of the $(d + 1)$-preprojective algebra of A.

Corollary 9.8. *Let A be a d-representation infinite algebra, $U = \mathrm{Ext}^d_A(DA, A)$ and $B = A \oplus U$. Then B is d-Iwanaga-Gorenstein, and there is a triangle equivalence $\mathcal{C}_{d+1}(A) \simeq \mathcal{D}_{\mathrm{sg}}(B)$.*

This is a higher-dimensional analogue of Example 9.6 above. One can view [I2, Theorem 1.1(1)] as a prediction that $\mathcal{D}_{\mathrm{sg}}(B)$ has a $(d + 1)$-cluster tilting object. We deduce this from our equivalence with the $(d + 1)$-cluster category.

Let us now give an example. See also Example 11.2(1) for an example in $d = 2$.

Example 9.9. (1) Let A be the tensor product of two path algebras of Kronecker quivers; thus it is presented by the following quiver with relations:

$$\begin{tikzcd} 1 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] & 2 \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] \\ 3 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & 4 \end{tikzcd} \qquad xu = ux,\ xv = vx,\ yu = uy,\ yv = vy.$$

This is a 2-representation infinite algebra [HIO, Theorem 2.10]. This is also the endomorphism algebra of a tilting bundle $T = \mathcal{O} \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1) \oplus \mathcal{O}(1, 1)$ over $\mathbb{P}^1 \times \mathbb{P}^1$ (see, e.g., [Ki, Section 6, Section 8]), inducing a derived equivalence with compatible shifted Serre functors (see [Hu, Definition 3.11, Theorem 3.12])

$$\begin{tikzcd} \mathcal{D}^b(\mathrm{mod}\, A) \arrow[r, dash, "\simeq"] \arrow[loop below, "\nu_2^{-1}"] & \mathcal{D}^b(\mathrm{coh}\, \mathbb{P}^1 \times \mathbb{P}^1) \arrow[loop below, "(2,2)"] \end{tikzcd}.$$

Either by a direct calculation using $\Pi = \bigoplus_{i \geq 0} \mathrm{Hom}_{\mathrm{coh}\, \mathbb{P}^1 \times \mathbb{P}^1}(T, T \otimes \mathcal{O}(2i, 2i))$, or by [Ke6, Theorem 6.10], we see that the preprojective algebra Π of A is presented by the following quiver with suitable commutativity relations (where the four diagonal arrows represents xu, yu, xv, yv),

$$\begin{tikzcd} 1 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] & 2 \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] \\ 3 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & 4 \arrow[ul, shift left=3] \arrow[ul, shift left=1] \arrow[ul, shift right=1] \arrow[ul, shift right=3] \end{tikzcd}$$

thus so is its truncation B with some additional relations.

By Corollary 9.8, we obtain a triangle equivalence $\mathcal{C}_3(A) \simeq \mathcal{D}_{\mathrm{sg}}(B)$.

(2) Let A' be the algebra presented by the following quiver with relations:

$$\begin{tikzcd} 1 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & 2 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & 3 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & 4 \end{tikzcd}, \qquad xuy = yux,\ xvy = yvx.$$

This is obtained from A by a mutation: Take the left mutation in the sense of [AI, Definition 2.30] of T at the summand $\mathcal{O}(0,1)$. By the exact sequence $0 \to \mathcal{O}(0,1) \to \mathcal{O}(1,1)^{\oplus 2} \to \mathcal{O}(2,1) \to 0$, we obtain another tilting bundle $T' = \mathcal{O} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(2,1)$. Indeed, it is a silting object by [AI, Theorem 2.31], and there are certainly no negative self-extensions since it is in the heart $\operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1$. Then A' is the endomorphism ring of T'. Now we have $\operatorname{Hom}_{\mathcal{D}(A')}(A', \nu_2^{-i} A'[j]) = \operatorname{Hom}_{\mathcal{D}^b(\operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1)}(T', T' \otimes \mathcal{O}(2i, 2i)[j])$, which vanishes for $i \geq 0$ and $j > 0$. (We leave the verification to the reader.) We conclude that A' is also 2-representation infinite. Again either by a direct calculation of $\bigoplus_{i \geq 0} \operatorname{Hom}_{\operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1}(T', T' \otimes \mathcal{O}(2i, 2i))$ or by [Ke6, Theorem 6.10], its preprojective algebra Π' is given by the following quiver with commutativity relations,

$$\begin{tikzcd} 1 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & 2 \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] \\ 4 \arrow[u, shift left, "u"] \arrow[u, shift right, "v"'] & 3 \arrow[l, shift right, "x"'] \arrow[l, shift left, "y"] \end{tikzcd} \qquad \begin{aligned} &xuy = yux,\ xvy = yvx, \\ &uxv = vxu,\ uyv = vyu, \end{aligned}$$

thus its truncation B' by the same quiver with suitable additional relations. By Corollary 9.8, we have a triangle equivalence $\mathcal{C}_3(A') \simeq \mathcal{D}_{\mathrm{sg}}(B')$.

(3) The 2-representation infinite algebras A and A' above are derived equivalent, hence their cluster categories are equivalent; $\mathcal{C}_3(A) \simeq \mathcal{C}_3(A')$. Therefore, we deduce that all the relevant 3-CY categories are equivalent; $\mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}_3(A) \simeq \mathcal{C}_3(A') \simeq \mathcal{D}_{\mathrm{sg}}(B')$.

10. Examples: Polynomial rings

Let us start by recording the following well-known fact on the CY property of polynomial rings.

Lemma 10.1. *Let $R = k[x_0, \ldots, x_d]$ be a polynomial ring with $\deg x_i = -a_i$. Then it is bimodule $(d+1)$-CY algebra with a-invariant $a = \sum_{i=0}^{d} a_i$.*

Proof. This is seen by the Koszul bimodule resolution. Let V be the graded vector space with basis $\{x_0, x_1, \ldots, x_d\}$. Consider the complex

$$K:\ 0 \longrightarrow K_{d+1} \longrightarrow K_d \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_0 \longrightarrow 0$$

with $K_l = R \otimes \bigwedge^l V \otimes R$ and differential $K_l \to K_{l-1}$ given by the homogeneous R^e-linear map

$$1 \otimes (x_{i_1} \wedge \cdots \wedge x_{i_l}) \otimes 1 \mapsto \sum_{j=1}^{l} (-1)^{j-1} (x_{i_j} \otimes (x_{i_1} \wedge \cdots \wedge \widehat{x_{i_j}} \wedge \cdots \wedge x_{i_l}) \otimes 1 - 1$$
$$\otimes (x_{i_1} \wedge \cdots \wedge \widehat{x_{i_j}} \wedge \cdots \wedge x_{i_l}) \otimes x_{i_j}),$$

where $\widehat{x_{i_j}}$ indicates that x_{i_j} is skipped. Then the complex K together with the multiplication map $K_0 = R \otimes R \to R$ is a bimodule projective resolution of R (see [VdB1, Proposition 3.1]). It is easy to see that applying $\mathrm{Hom}_{R^e}(-, R^e)$ to K yields the isomorphic complex up to shifts, which shows our assertion. $\square$

The aim of this section is to apply our main results for polynomial rings and give a concrete description of the d-representation-infinite algebra A and the d-Iwanaga-Gorenstein algebra B (see equation (8.2) for the definitions). For a finite subgroup $G \subset \mathrm{GL}_{d+1}(k)$, which naturally acts on R, the *skew group algebra* $R * G$ is a vector space $R \otimes_k kG$ with multiplication $(a \otimes g)(b \otimes h) = ag(b) \otimes gh$. We have the following result on the algebras A and B.

Proposition 10.2. *Let $R = k[x_0, \dots, x_d]$ be a polynomial ring in $d+1 \geq 2$ variables, with $\deg x_i = -a_i < 0$ and $a = \sum_{i=0}^{d} a_i$. Suppose that a is invertible in k, that there exists a primitive ath root of unity $\zeta \in k$ and that $a_0, \dots, a_d$ are relatively prime.*

1. *The algebra A is presented by the quiver Q with the vertices $\{0, 1, \dots, a-1\}$, the arrows $x_i = x_i^l \colon l \to l + a_i$ for each $0 \leq i \leq d$ and $0 \leq l \leq a-1$ such that $l + a_i \leq a-1$ and with the commutativity relations $x_j^{l+a_i} x_i^l = x_i^{l+a_j} x_j^l$.*
2. *Let $g = \mathrm{diag}(\zeta^{a_0}, \dots, \zeta^{a_d}) \in \mathrm{SL}_{d+1}(k)$ and G the cyclic subgroup generated by g. Then the $(d+1)$-preprojective algebra of A is isomorphic to $R * G$.*
3. *A is d-representation-infinite of type $\widetilde{A}$, in the sense of [HIO].*
4. *B is presented by the quiver $\widetilde{Q}$ obtained by adding to Q the arrows $u = u^l \colon l \to l-1$ for each $1 \leq l \leq a-1$ and additional relations:*
   (i) $x_i^{l-1} u^l = u^{l+a_i} x_i^l$ *whenever* $1 \leq l$ *and* $l + a_i \leq a-1$.
   (ii) $x_j^{a_i-1} u^{a_i} x_i^0 = x_i^{a_j-1} u^{a_j} x_j^0$ *for every* (i, j) *such that* $a_i + a_j = a$.
   (iii) $u^{l-1} u^l = 0$ *for all* $1 \leq l \leq a-1$.
   (iv) $u^{a-1} x_i^0 u^1 = 0$ *for every* i *such that* $a_i = a-1$.

Proof. (1) Note that the category $\mathrm{proj}^{\mathbb{Z}} R$ is presented by the quiver with vertices set $\mathbb{Z}$, arrows $x_i^l \colon l \to l + a_i$ and with the commutativity relations. Since $A \simeq \mathrm{End}_{\mathrm{Mod}^{\mathbb{Z}} R}(R \oplus R(-1) \oplus \cdots R(-(a-1)))$, it is presented by its full subquiver with vertices $\{0, 1, \dots, a-1\}$ and with the induced relations.

(2)(3) We follow the construction of d-representation-infinite algebras of type $\widetilde{A}$ [HIO, Section 5].

Let L be the free $\mathbb{Z}$-module with basis $\alpha_1, \dots, \alpha_d$, and set $\alpha_0 := -\alpha_1 - \cdots - \alpha_d$. Let Q be the quiver with vertices $Q_0 = L$ and the set of arrows $Q_1 = \{x_i = x_i^l \colon l \to l + \alpha_i \mid l \in L, 0 \leq i \leq d\}$. Moreover, for each $l \in L$ and $0 \leq i < j \leq d$, define the relation $r_{ij} = r_{ij}^l = x_i x_j - x_j x_i$. Then we have a category $\mathcal{L}$ presented by this quiver and relations. We assign for each point $l = \sum_{i=1}^{d} l_i \alpha_i \in L$ the integer $m(l) = \sum_{i=1}^{d} l_i a_i$.

Now let $B \subset L$ the subgroup consisting of points l in L such that $m(l)$ is a multiple of a. The subgroup B has finite index a and acts on $\mathcal{L}$ by translation. We then have the orbit category $\mathcal{L}/B$, which can naturally be identified with an algebra Π presented by the (finite) quiver Q/B and induced relations (i.e., identify the vertices and arrows along the action of B and the same for the relations). By [HIO, Lemma 5.3], Π is isomorphic to the skew group algebra $R * G$.

Going back to the original quiver Q, we set

$$C = \{x \colon l \to l' \text{ in } Q_1 \mid m(l) < na \leq m(l') \text{ for some } n \in \mathbb{Z}\},$$

which is a periodic and bounding cut in the sense of [HIO, Definitions 5.4, 5.5] and is stable under B. Then C induces a grading on Q/B, hence on Π by

$$\deg x = \begin{cases} 1 & (x \in C) \\ 0 & (x \notin C) \end{cases}$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

for each $x \in Q_1$. Now, by [HIO, Theorem 5.6], Π is the preprojective algebra of its degree 0 part Π_0, and we see that $\Pi_0 \simeq A$, since they are presented by the quiver $(Q/B) \setminus (C/B)$ and the commutativity relations.

(4) We first compute the quiver of B using Lemma 8.7(3). Since R is generated in degree $> -a$, the vector space in the lower-left corner of $U/(J_A U + U J_A)$ is 0. Therefore, the arrows we have to add are just the ones corresponding to $1 \in R_0/J_0 = k$, and the quiver of B is $\widehat{Q}$. Then there exists a natural homomorphism $k\widehat{Q} \to B$, which sends the relations (i)-(iv) to 0 and thus induces a surjective homomorphism $\varphi\colon k\widehat{Q}/I \to B$, where I is the ideal generated by the relations. We show that φ is an isomorphism.

First we give a grading on $\widehat{Q}$ by $\deg x_i = 0$ and $\deg u = 1$ and similarly on $B = A \oplus U$ by $\deg A = 0$ and $\deg U = 1$. We have seen in (1) that φ is an isomorphism in degree 0. Also, the relations (iii)(iv) shows $k\widehat{Q}/I$ is concentrated in degree ≤ 1. To see this, we have to verify $ux_{i_l} \cdots x_{i_1} u = 0$ in $k\widehat{Q}/I$. If $l > 1$, the source of x_{i_l} is $\geq a_{i_1} + \cdots + a_{i_{l-1}} \geq 1$, so we can use the relation (i) and $ux_{i_l} \cdots x_{i_1} u = x_{i_l} u x_{i_{l-1}} \cdots x_{i_1} u$. By induction, we are reduced to proving $ux_i u = 0$, precisely $u^{j-1+a_i} x_i^{j-1} u^j = 0$ whenever $j \geq 1$ and $j - 1 + a_i \leq a - 1$. When $j > 1$, we can use (i) and $(u^{j-1+a_i} x_i^{j-1}) u^j = (x_i^{j-2} u^{j-1}) u^j$, which is 0 by (iii). Similarly, when $j - 1 + a_i < a - 1$, we have $u^{j-1+a_i}(x_i^{j-1} u^j) = u^{j-1+a_i}(u^{j+a_i} x_i^j) = 0$. If $j = 1$ and $j - 1 + a_i = a - 1$, this forces $a_i = a - 1$, in which case it is nothing but our 'exceptional' relation (iv).

Therefore, it remains to consider the degree 1 part. We may truncate by the idempotents; denote by e_i the idempotent of $k\widehat{Q}/I$ at vertex i, and we show that the induced map $e_j M e_i \to e_j U e_i$, where M is the degree 1 part of $k\widehat{Q}/I$, is an isomorphism for each $0 \leq i, j \leq a - 1$.

Now we give another grading on $\widehat{Q}$ defined from that on R: that is, we set $\deg x_i = -a_i$ and $\deg u = 0$. Clearly, each space $e_j M e_i$ is spanned by the monomials in $x_0^l, \ldots, x_d^l, u^l$ of degree $-(j - i + 1)$, each of which contains exactly one of the u^ls. We regard each monomial as a word in $\{x_0, \ldots, x_d, u\}$ by forgetting the superscripts. We say two monomials are *equivalent* if the associated words coincide up to a permutation. We claim that, under the relations, a complete set of representatives of equivalence classes of monomials actually span $e_j M e_i$. It is then clear by comparing the dimensions that φ is an isomorphism from $e_j M e_i$ to $e_j U e_i = R_{-(j-i+1)}$.

Case 1: $(i, j) \neq (0, a - 1)$. If $i \neq 0$, then any path in $e_j M e_i$ can be written as pu^i for some $p \in e_j(kQ)e_{i-1}$ under the relation (i), and p is equivalent to a monomial in x_is of degree $-(j - i + 1)$ by the commutativity relations in (1), which shows our claim. Similar argument works for the case $j \neq a - 1$.

Case 2: $(i, j) = (0, a - 1)$. Let m be a path from 0 to $a - 1$ in $\widehat{Q}$ containing exactly one u. Since $\deg x_i > -a$, the path m has to contain at least 2 of the x_i's (which is possibly the same). By the relation (i), we may assume that u is the second arrow in the path m. We have to show that a permutation of the x_is appearing in m does not affect m as an element in $k\widehat{Q}/I$. The assertion for permutations of the x_is appearing after u is clear by the commutativity relations in (1), so it remains to prove $x_j^{a_i - 1} u^{a_i} x_i^0 = x_i^{a_j - 1} u^{a_j} x_j^0$. If the target $a_i - 1 + a_j$ of this length 3 path is not $a - 1$, the assertion follows from Case 1 above. If $a_i - 1 + a_j = a - 1$, then this is nothing but the relation (ii). $\square$

Let us first look at the easiest case.

Example 10.3. This is a continuation of Example 4.11. Let $R = k[x, y]$ with $\deg x = \deg y = -1$, so R is 2-CY of a-invariant 2. By Theorem 5.2, R^{dg} is twisted 4-CY. As we have seen in Example 4.11, we have an equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{mod} A)$ with A the Kronecker algebra, and its AR translation ν_1 has a square root. On the other hand, the Iwanaga-Gorenstein algebra B is presented by the following quiver with relations:

$$\begin{tikzcd} 0 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & 1 \arrow[l, bend right=60, "u"'] \end{tikzcd}, \quad xuy = yux,\ uxu = uyu = 0.$$

By equation (8.3), there are equivalences

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_1^{-1/2}[1] \simeq \mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}(R^{\mathrm{dg}})$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

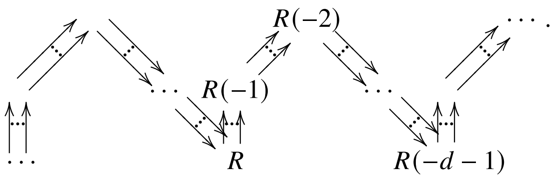
of twisted 3-CY categories; note that the embedding from $\mathcal{D}^b(\operatorname{mod} A)/\nu_1^{-1/2}[1]$ is dense by [Ke2, Theorem 1]. Using the description as an orbit category, we can classify the objects in $\mathcal{C}(R^{\mathrm{dg}})$ or $\mathcal{D}_{\mathrm{sg}}(B)$, which we leave to the reader.

We look at a higher-dimensional case.

Example 10.4. Assume $d > 1$, and let $R = k[x_0, x_1, \dots, x_d]$ with $\deg x_0 = \cdots = \deg x_d = -1$. Then R is $(d+1)$-CY of a-invariant $d+1$; thus, by Theorem 5.2, R^{dg} is sign-twisted $(2d+2)$-CY. It is well-known that $\operatorname{qgr} R$ is equivalent to the category $\operatorname{coh} \mathbb{P}^d$ of coherent sheaves over the projective space $\mathbb{P}^d$. The tilting object in $\mathcal{D}^b(\operatorname{qgr} R)$ given in Proposition 4.9 is the tilting bundle $T = \bigoplus_{l=0}^d \mathcal{O}_{\mathbb{P}^d}(l)$ on $\mathbb{P}^d$, whose endomorphism ring A is the d-Beilinson algebra. It is presented by the following quiver (with $d+1$ arrows between the vertices) and the commutativity relations:

$$A = \ 0 \underset{x_d}{\overset{x_0}{\underset{\vdots}{\rightrightarrows}}} 1 \underset{x_d}{\overset{x_0}{\underset{\vdots}{\rightrightarrows}}} \cdots \underset{x_d}{\overset{x_0}{\underset{\vdots}{\rightrightarrows}}} d\,, \qquad x_i x_j = x_j x_i.$$

The category $\operatorname{add}\{R(-i) \mid i \in \mathbb{Z}\} = \operatorname{add}\{\mathcal{O}(i) \mid i \in \mathbb{Z}\} = \operatorname{add}\{\nu_d^{-i} A \mid i \in \mathbb{Z}\}$ (which are identified via the equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{coh} \mathbb{P}^d) \simeq \mathcal{D}^b(\operatorname{mod} A)$) is presented by the following quiver:



The autoequivalence $\nu_d^{-1/(d+1)}$ on $\mathcal{D}^b(\operatorname{mod} A)$ acts on this subcategory by 'moving one place' along the d-fold arrows. On the other hand, the Iwanaga-Gorenstein algebra $B = A \oplus U$ is presented by the following quiver with relations:

$$\begin{tikzcd} 0 \arrow[r, shift left, "x_0"] \arrow[r, shift right, "x_d"'] & 1 \arrow[l, bend right=50, "u"'] \arrow[r, shift left, "x_0"] \arrow[r, shift right, "x_d"'] & \cdots \arrow[l, bend right=50, "u"'] \arrow[r, shift left, "x_0"] \arrow[r, shift right, "x_d"'] & d \arrow[l, bend right=50, "u"'] \end{tikzcd}, \qquad u x_i = x_i u, \quad u^2 = 0.$$

This is a truncation of $T_A U = A \oplus U \oplus U^2 \oplus \cdots$, which is the endomorphism ring of a tilting object $\pi^* T$ over the total space of the line bundle $\pi\colon \mathcal{O}(-1) \to \mathbb{P}^d$ (see, e.g., the computation in [W, Example 4.12], which carries over to $\mathbb{P}^d$). Applying equation (8.3), we have an embedding and an equivalence

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_d^{-1/(d+1)}[1] \hookrightarrow \mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}(R^{\mathrm{dg}}).$$

11. Examples: Jacobian algebras arising from dimer models

A *dimer model* is a finite bipartite graph Γ on a real 2-torus inducing a polygonal cell decomposition. Being bipartite, we can color the vertices of Γ black and white so that each edge connects a black vertex to a white vertex. We denote by Γ_0, (respectively, Γ_1, Γ_2) the set of vertices (respectively, edges, faces) of Γ. It gives rise to a quiver with potential (Q, W) in the sense of [DWZ] in the following way. Let Q denote the dual quiver of Γ; thus the set of vertices Q_0 (respectively, arrows Q_1) corresponds bijectively to Γ_2 (respectively, Γ_1). By convention, the arrows of Q see white vertices of Γ on the right. Then for each vertex v of Γ, there is a unique cycle c_v of Q consisting of arrows corresponding to the edges of Γ that are adjacent to v. Now define the potential by $W = \sum_{v\colon \text{white}} c_v - \sum_{v\colon \text{black}} c_v$. We then obtain the *Jacobian algebra* associated to the quiver with potential (Q, W): that is, the algebra $kQ/(\partial_a W \mid a \in Q_1)$, where ∂_a denotes the cyclic derivative [DWZ, Definition 3.1] with respect to the

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

arrow a. Thus for each edge in Γ connecting vertices x and y with the corresponding arrow $a \in Q_1$, we have a relation $\partial_a c_x = \partial_a c_y$ (see [Bro, Remark 2.3]).

We assume that Γ is *consistent* in the sense that there exists a map $\mathrm{R}\colon \Gamma_1 \to \mathbb{R}_{>0}$ such that

- $\sum_{v \in \partial a} \mathrm{R}(a) = 2$ for all $v \in \Gamma_0$, where the sum runs over $a \in \Gamma_1$ adjacent to v;
- $\sum_{a \in \partial f} (1 - \mathrm{R}(a)) = 2$ for all $f \in \Gamma_2$, where the sum runs over $a \in \Gamma_1$ in the boundary of f.

We refer to [Boc2] for equivalence of various consistency conditions.

Fix a map

$$d\colon Q_1 = \Gamma_1 \longrightarrow \mathbb{Z} \tag{11.1}$$

such that $\sum_{v \in \partial a} d(a)$ is a constant l for all $v \in \Gamma_0$. Such maps are typically given by perfect matchings on Γ. Recall that a *perfect matching* on a graph is a set of its edges such that each vertex is contained in precisely one edge in the set. It is known that the consistency condition ensures the existence of perfect matchings [Bro, Section 2.3]. We can identify a perfect matching P on Γ with a map $d\colon \Gamma_1 \to \{0, 1\}$ such that $\sum_{v \in \partial a} d(a) = 1$ for all $v \in \Gamma_0$ by setting $d(a) = 1$ if and only if $a \in P$. Consequently, any $\mathbb{Z}$-linear combination of perfect matchings gives a function satisfying equation (11.1).

Proposition 11.1 (See [Bro, Theorem 7.7], [AIR, Proposition 6.1]). *Let Γ be a consistent dimer model, and let d be a map in equation (11.1) such that $\sum_{v \in \partial a} d(a) = l$ for all $v \in \Gamma_0$. Then d gives a grading on the Jacobian algebra making it into a bimodule 3-CY algebra of a-invariant $-l$.*

Proof. Give a grading on the quiver Q by setting $\deg a = d(a)$ for $a \in Q_1$. Then the potential W is homogeneous of degree l; thus d induces a grading on the Jacobian algebra R. Consider the complex

$$P\colon \bigoplus_{i \in Q_0} Re_i \otimes e_i R \xrightarrow{d_3} \bigoplus_{a \in Q_1} Re_{s(a)} \otimes e_{t(a)} R \xrightarrow{d_2} \bigoplus_{a \in Q_1} Re_{t(a)} \otimes e_{s(a)} R \xrightarrow{d_1} \bigoplus_{i \in Q_0} Re_i \otimes e_i R$$

with maps

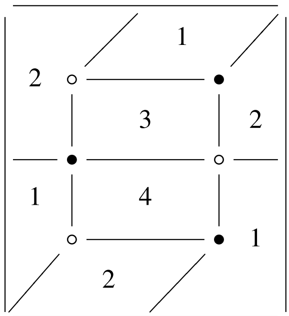
$$\begin{aligned} d_1(e_{t(a)} \otimes e_{s(a)}) &= a \otimes e_{s(a)} - e_{t(a)} \otimes a \\ d_2(e_{s(a)} \otimes e_{t(a)}) &= \sum_{b \in Q_1} p \otimes q \text{ for each cycle } apbq \text{ in } W \\ d_3(e_i \otimes e_i) &= \sum_{t(a)=i} a \otimes e_i - \sum_{s(a)=i} e_i \otimes a. \end{aligned}$$

By [Bro, Theorem 7.7], this complex P together with the multiplication map $\bigoplus_{i \in Q_0} Re_i \otimes e_i R \to R$ gives a bimodule projective resolution of R such that $\mathcal{H}\mathrm{om}_R(P, R)[3] \simeq P$ in $\mathcal{C}^b(\operatorname{proj} R^e)$, making it into a 3-CY algebra. Now, since each of the summands

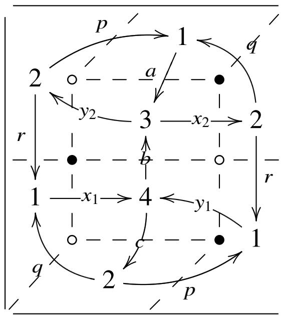
- $Re_{t(a)} \otimes e_{s(a)} R \to Re_i \otimes e_i R$ of d_1 has degree $d(a)$,
- $Re_{t(b)} \otimes e_{s(b)} R \to Re_{t(a)} \otimes e_{s(a)} R$ of d_2 has degree $l - d(a) - d(b)$ since the potential is homogeneous of degree l, and
- $Re_i \otimes e_i R \to Re_{t(b)} \otimes e_{s(b)} R$ of d_3 has degree $d(b)$,

we deduce that R is graded bimodule 3-CY of a-invariant $-l$. ◻

Example 11.2. Let Γ be a dimer model as in the left picture below, where the vertical and horizontal ends are identified so that it has four faces that are labeled by 1, 2, 3 and 4. It gives a 3-CY algebra R presented by the quiver in the right-hand picture:

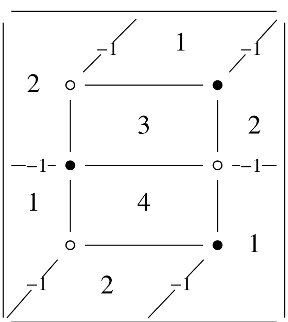


,

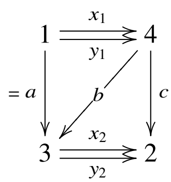


Now we consider the grading d in equation (11.1). We discuss two variations.

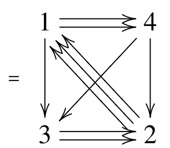
(1) First we consider the grading below. The labels on the edges show the values under d, and unlabeled ones have degree 0; thus it is a perfect matching. This grading makes R into a bimodule 3-CY algebra of a-invariant 1; thus $A = R_0$ and $B = R_{\geq -1}$ are given by quivers below. The relations for B are given by that for R (induced by the potential) and $R_{-1}^2 = 0$.



,

A = 

$y_2a = cy_1,\ x_2a = cx_1,\ x_2by_1 = y_2bx_1.$

B = 

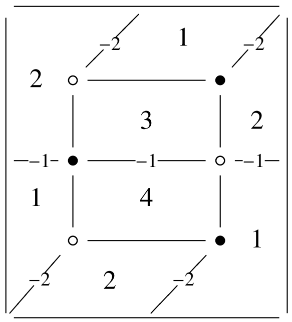
The algebra A is 2-representation infinite by Proposition 4.9. It is the endomorphism ring of a tilting bundle T on the Hirzebruch surface $\Sigma_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, which is the blow-up of $\mathbb{P}^2$ at one point as well (see [Ki, Section 6, Section 8]). Also, R is the 3-preprojective algebra of A, which is the endomorphism ring of a tilting bundle π^*T on the total space of the canonical bundle $\pi\colon \omega \to \Sigma_1$ over Σ_1 (see [BH, Theorem 3]). Now the DG algebra R^{dg} is 4-CY by Theorem 5.2, and applying Corollary 9.8, we have triangle equivalences

$$\mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}_3(A) \simeq \mathcal{C}(R^{\mathrm{dg}}).$$

(2) We next consider the grading below; again the non-zero degree of each edge is labeled. This is not given by a perfect matching (but by a sum of two perfect matchings) and makes R into a bimodule 3-CY algebra of a-invariant 2. In this case, the 2-representation infinite algebra $A = \begin{pmatrix} R_0 & 0 \\ R_{-1} & R_0 \end{pmatrix}$ and the

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

2-Iwanaga-Gorenstein algebra $B = A \oplus U$ with $U = \begin{pmatrix} R_{-1} & R_0 \\ R_{-2} & R_{-1} \end{pmatrix}$ are presented as follows:



$$A = \begin{tikzcd} 1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & 4 \arrow[d] \arrow[dr] & 1' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & 4' \arrow[d] \\ 3 \arrow[r, shift left] \arrow[r, shift right] & 2 \arrow[ur] & 3' \arrow[r, shift left] \arrow[r, shift right] & 2' \end{tikzcd}$$

$$B = \begin{tikzcd} 1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & 4 \arrow[d] \arrow[dr] & 1' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[ll, bend right] & 4' \arrow[d] \arrow[ll, bend right] \\ 3 \arrow[r, shift left] \arrow[r, shift right] & 2 \arrow[ur] \arrow[ur, shift left=2] \arrow[ur, shift right=2] & 3' \arrow[r, shift left] \arrow[r, shift right] \arrow[ll, bend left] & 2' \arrow[ll, bend left] \end{tikzcd}$$

,

Here, the arrows $4 \longrightarrow 3'$ and $2 \longrightarrow 1'$ in the quiver of A are given by a and r, respectively. Also, in the quiver of B, the two additional arrows $2 \longrightarrow 1'$ are $p, q \in R_2$, and $i' \longrightarrow i$ for $1 \le i \le 4$ are the idempotents from R_0.

Applying our results, the DG algebra R^{dg} is sign twisted 5-CY, and there exist an embedding and a triangle equivalence

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_2^{-1/2}[1] \hookrightarrow \mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}(R^{\mathrm{dg}}).$$

A. Multiplying gradings

Let R be a graded ring. For a fixed integer $n \ge 1$, define the graded ring ${}^nR$ by

$$({}^nR)_i = \begin{cases} R_{i/n} & \text{if } n \mid i \\ 0 & \text{if } n \nmid i \end{cases}.$$

If R is twisted bimodule $(d+1)$-CY of a-invariant a, then clearly ${}^nR$ is twisted bimodule $(d+1)$-CY of a-invariant na. Although the category $\operatorname{qper}^{\mathbb{Z}}{}^nR$ just splits as a direct product of n copies of $\operatorname{qper}^{\mathbb{Z}}R$ and yields nothing new, the cluster category $\mathcal{C}({}^nR^{\mathrm{dg}})$, being a triangulated hull of $\operatorname{qper}^{\mathbb{Z}}{}^nR/(-1)[1]$, becomes 'connected' by the action of the automorphism $(-1)[1]$, which yields something new.

The aim of this section is to describe the category $\mathcal{C}({}^nR^{\mathrm{dg}})$ in terms of the relevant objects from R. Although this can be regarded as a special case of our main results, we shall obtain a better presentation of orbit categories.

Let $d \ge 0$, and let R be a negatively graded twisted bimodule $(d+1)$-CY algebra of a-invariant a with Nakayama automorphism α, such that each R_i is finite dimensional. Recall the definitions of the d-representation infinite algebra $A = A(R)$, the cotilting bimodule $U = U(R)$ and the d-Iwanaga-Gorenstein algebra $B = B(R)$ from equation (8.2). Assuming $(-)_\alpha \simeq 1$ on $\operatorname{qper}^{\mathbb{Z}}R$, we have the following description of $\mathcal{C}({}^nR^{\mathrm{dg}})$ in terms of A, which generalises Theorem 6.1 and Corollary 6.2.

Theorem A.1. *There exists a fully faithful functor*

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_d^{-1/a}[n] = \operatorname{qper}^{\mathbb{Z}}R/(-1)[n] \longrightarrow \mathcal{C}({}^nR^{\mathrm{dg}})$$

whose image generates $\mathcal{C}({}^nR^{\mathrm{dg}})$ as a triangulated subcategory.

Let the d-representation infinite algebra $\widetilde{A} = A({}^nR)$, the cotilting $(\widetilde{A}, \widetilde{A})$-bimodule $\widetilde{U} = U({}^nR)$ and the d-Iwanaga-Gorenstein algebra $\widetilde{B} = B({}^nR)$ be as given in equation (8.2) for ${}^nR$; thus we have

$$\widetilde{A} = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix} \simeq A \times \cdots \times A, \quad \widetilde{U} = \begin{pmatrix} 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \\ U & 0 & \cdots & 0 \end{pmatrix}, \quad \widetilde{B} = \widetilde{A} \oplus \widetilde{U}. \tag{A.1}$$

Then we also have the consequence in terms of singularity category of $\widetilde{B}$.

Corollary A.2. *There exists a fully faithful functor*

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_d^{-1/a}[n] \longrightarrow \mathcal{D}_{\mathrm{sg}}(\widetilde{B})$$

whose image generates $\mathcal{D}_{\mathrm{sg}}(\widetilde{B})$ as a triangulated subcategory.

Now we start the proof. The first step is to apply our results in Section 6 for the twisted CY algebra ${}^nR$. The assumption $(-)_\alpha \simeq 1$ on $\operatorname{qper}^{\mathbb{Z}} R$ implies that the same isomorphism $(-)_\alpha \simeq 1$ holds on $\operatorname{qper}^{\mathbb{Z}}{}^nR$, so by Corollary 4.10, the d-AR translation ν_d on $\mathcal{D}^b(\operatorname{mod} \widetilde{A})$ has an nath root, and by Theorem 6.1 and Corollary 6.2, we have an equivalence and an embedding

$$\mathcal{D}^b(\operatorname{mod} \widetilde{A})/\nu_d^{-1/na}[1] \simeq \operatorname{qper}^{\mathbb{Z}}{}^nR/(-1)[1] \hookrightarrow \mathcal{C}({}^nR^{\mathrm{dg}}). \tag{A.2}$$

We next compare the derived orbit categories of ${}^nR$ (respectively, $\widetilde{A}$) and R (respectively, A). Obviously there is a diagram of equivalences and compatible autoequivalences

$$\begin{tikzcd} (-1) \circlearrowright & \operatorname{qper}^{\mathbb{Z}}{}^nR \arrow[r, equal] \arrow[d, "\wr"'] & \operatorname{qper}^{\mathbb{Z}} R \times \cdots \times \operatorname{qper}^{\mathbb{Z}} R \arrow[d, "\wr"] \\ \nu_d^{-1/na} \circlearrowright & \mathcal{D}^b(\operatorname{mod} \widetilde{A}) \arrow[r, equal] & \mathcal{D}^b(\operatorname{mod} A) \times \cdots \times \mathcal{D}^b(\operatorname{mod} A). \end{tikzcd}$$

We describe the action of these autoequivalences on the right-hand side.

Lemma A.3.

1. *The action of* (-1) *on* $\operatorname{qper}^{\mathbb{Z}}{}^nR$ *becomes* $(X_1, \dots, X_n) \mapsto (X_n(-1), X_1, \dots, X_{n-1})$ *on* $\operatorname{qper}^{\mathbb{Z}} R \times \cdots \times \operatorname{qper}^{\mathbb{Z}} R$.
2. *The action of* $\nu_d^{-1/na}$ *on* $\mathcal{D}^b(\operatorname{mod} \widetilde{A})$ *is* $(X_1, \dots, X_n) \mapsto (\nu_d^{-1/a} X_n, X_1, \dots, X_{n-1})$ *on* $\mathcal{D}^b(\operatorname{mod} A) \times \cdots \times \mathcal{D}^b(\operatorname{mod} A)$.

Proof. We only prove (2); the proof of (1) is similar. By the equation (A.1) form of $\widetilde{U}$, we see that $- \otimes^{\mathrm{L}}_{\widetilde{A}} \widetilde{U}$ maps $(X_1, \dots, X_n)$ to $(X_n \otimes^{\mathrm{L}}_A U, X_1, \dots, X_{n-1})$. □

We next relate the orbit categories arising from R and ${}^nR$.

Lemma A.4.

1. *The functor* $\operatorname{qper}^{\mathbb{Z}} R \to \operatorname{qper}^{\mathbb{Z}}{}^nR$ *given by* $X \mapsto (X, 0, \dots, 0)$ *induces an equivalence*

$$\operatorname{qper}^{\mathbb{Z}} R/(-1)[n] \xrightarrow{\simeq} \operatorname{qper}^{\mathbb{Z}}{}^nR/(-1)[1].$$

2. *The functor* $\mathcal{D}^b(\operatorname{mod} A) \to \mathcal{D}^b(\operatorname{mod} \widetilde{A})$ *given by* $X \mapsto (X, 0, \dots, 0)$ *induces an equivalence*

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_d^{-1/a}[n] \xrightarrow{\simeq} \mathcal{D}^b(\operatorname{mod} \widetilde{A})/\nu_d^{-1/na}[1].$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

Proof. Again, we only prove (2). We first want to show that there is a natural isomorphism

$$\bigoplus_{l\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}(A)}(X, (\nu_d^{-1/a}[n])^l Y) \longrightarrow \bigoplus_{l\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}(\widetilde{A})}(FX, (\nu_d^{-1/na}[1])^l FY)\ ,$$

where $FX = (X, 0, \ldots, 0)$. Since $F(\nu_d^{-1/a} Y[n]) = (\nu_d^{-1/na}[1])^n F(Y)$ and $\operatorname{Hom}_{\mathcal{D}(\widetilde{A})}(FX, (\nu_d^{-1/na}[1])^l FY) = 0$ unless $n \mid l$ by Lemma A.3, we have a natural bijection. We next verify that the functor is dense. Note that $(0, \ldots, X_i, \ldots, 0) \simeq (X_i[-i+1], 0 \ldots, 0)$ in the orbit category $\mathcal{D}^b(\widetilde{A})/\nu_d^{-1/na}[1]$. Therefore, $\bigoplus_{i=1}^n X_i[-i+1] \in \mathcal{D}^b(A)$ is mapped to $(X_1, \ldots, X_n)$. □

We now have our desired results.

Proof of Theorem A.1 and Corollary A.2. We have Theorem A.1 by equation (A.2) and Lemma A.4. Then Corollary A.2 follows by Theorem 6.4. □

Let us demonstrate the difference of $\mathcal{C}(R^{\mathrm{dg}})$ and $\mathcal{C}({}^nR^{\mathrm{dg}})$.

Example A.5. This is a generalization of Example 6.5, which is still almost trivial. Let

$$R = k[x], \quad \deg x = -1,$$

which is bimodule 1-CY of a-invariant 1. Then we have $A = k$ and $U = k$. Now fix $n \geq 1$, and consider the graded algebra ${}^nR$. We have

$$\widetilde{A} = \begin{pmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{pmatrix}, \quad \widetilde{U} = \begin{pmatrix} 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \\ k & 0 & \cdots & 0 \end{pmatrix},$$

thus $\widetilde{B} = \widetilde{A} \oplus \widetilde{U}$ is the self-injective Nakayama algebra with n vertices and of Loewy length 2. By Theorem A.1 and Corollary A.2, we have equivalences of triangulated categories

$$\mathcal{D}^b(\operatorname{mod} k)/[n] \simeq \mathcal{C}({}^nR^{\mathrm{dg}}) \simeq \mathcal{D}_{\mathrm{sg}}(\widetilde{B}),$$

which is the n-cluster category of k.

Example A.6. This is a generalization of Example 6.6. As in Examples 4.13, 5.10 and 6.6, let

$$R = k\langle x_1, \ldots, x_m\rangle/(x_1^2 + \cdots + x_m^2), \quad \deg x_i = -1,$$

which is twisted 2-CY of a-invariant 2, the DG algebra R^{dg} is 4-CY, and the 1-representation infinite algebra A is the path algebra kQ_m of the m-Kronecker quiver Q_m.

Now we consider the cluster category $\mathcal{C}({}^nR^{\mathrm{dg}})$. The algebra $\widetilde{A}$ is just the n copies of $A = kQ_m$, and the 1-Iwanaga-Gorenstein algebra $\widetilde{B}$ is presented by the following quiver with relations:

$$\begin{tikzcd}
\circ \arrow[d, Rightarrow, "m"'] & \circ \arrow[l, "v"'] \arrow[d, Rightarrow, "m"'] & \cdots \arrow[l, "v"'] & \circ \arrow[l, "v"'] \arrow[d, Rightarrow, "m"] \\
\circ \arrow[urrr, "u"] & \circ \arrow[l, "v"] & \cdots \arrow[l, "v"] & \circ \arrow[l, "v"]
\end{tikzcd}, \qquad \begin{gathered} x_i v = v x_i \ (1 \leq i \leq m),\ v^2 = 0,\ uv = vu = 0 \\ \textstyle\sum_{i=1}^m x_i u x_i = 0, \end{gathered}$$

where we have denoted by $x_1, \ldots, x_m$ the m-fold arrows. By Theorem A.1, we obtain triangle equivalences

$$\mathcal{D}^b(\operatorname{mod} kQ_m)/\nu_1^{-1/2}[n] \simeq \mathcal{D}_{\mathrm{sg}}(\widetilde{B}) \simeq \mathcal{C}({}^nR^{\mathrm{dg}})$$

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

since the orbit category is already triangulated [Ke2, Theorem 1]. Similarly to Example 6.6 and Remark 6.7, these are precisely the $(2n+1)$-CY triangulated category in [KMV, Remark 3.4.5].

B. t-structure in $\mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$

We give a version of Theorem 3.1 for the derived category $\mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$ for graded coherent rings, as announced in Remark 3.2. Let R be a negatively graded, graded coherent ring. Then the category $\mathrm{mod}^{\mathbb{Z}} R$ of finitely presented graded R-modules is abelian. We impose the following technical assumption.

(R3) The ideal $R_{\leq i}$ is finitely generated as a right R-module for each $i \leq 0$.

Note that this is automatic when R is Noetherian.

Lemma B.1. *Let R be a negatively graded ring satisfying (R3), and let X be a finitely presented graded R-module. Then the truncation $X_{>i}$ is finitely presented for each $i \in \mathbb{Z}$.*

Proof. Let $P_1 \to P_0 \to X \to 0$ be a finite presentation of X, and consider its truncation $(-)_{>i}$. Since the $(P_0)_{>i}$ and $(P_1)_{>i}$ are finitely presented by the assumption (R3), so is $X_{>i}$. □

Theorem B.2 (compare Theorem 3.1). *Let R be a negatively graded, graded coherent ring satisfying (R3). Set*

$$t^{\leq 0} = \{X \in \mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R) \mid H^i(X) \in \mathrm{mod}^{\leq -i} R \textit{ for all } i \in \mathbb{Z}\},$$
$$t^{\geq 0} = \{X \in \mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R) \mid H^i(X) \in \mathrm{mod}^{\geq -i} R \textit{ for all } i \in \mathbb{Z}\}.$$

Then $(t^{\leq 0}, t^{\geq 0})$ is a t-structure in $\mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$.

We give two independent proofs. The first one is a short proof using silting theory and DG categories. For the sake of readers who are not familiar with these, we include the second direct proof.

B.1. *The first proof*

Recall from Lemma 3.4 that we have a t-structure $(\mathcal{D}^{\leq 0}_{\mathcal{M}}, \mathcal{D}^{\geq 0}_{\mathcal{M}})$ in the big derived category $\mathcal{D} := \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}} R)$, which is given by

$$\mathcal{D}^{\leq 0}_{\mathcal{M}} = \{X \in \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}} R) \mid H^i(X) \in \mathrm{Mod}^{\leq -i} R \text{ for all } i \in \mathbb{Z}\},$$
$$\mathcal{D}^{\geq 0}_{\mathcal{M}} = \{X \in \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}} R) \mid H^i(X) \in \mathrm{Mod}^{\geq -i} R \text{ for all } i \in \mathbb{Z}\}.$$

As in Section 3, we show that the t-structure $(\mathcal{D}^{\leq 0}_{\mathcal{M}}, \mathcal{D}^{\geq 0}_{\mathcal{M}})$ above on $\mathcal{D}$ restricts to that on $\mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$.

Proof of Theorem B.2. Since R is right-graded coherent, the small derived category $\mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$ identifies with the thick subcategory of $\mathcal{D}$ consisting of complexes with bounded and finitely presented cohomologies. Let $X \in \mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$, and consider the truncation triangle $X' \to X \to X'' \to X'[1]$ in $\mathcal{D}$. Since X has bounded cohomology, so do X' and X'' by Lemma 3.5(1). Moreover, since each $H^i X$ is finitely presented, so are $H^i X'$ and $H^i X''$ by Lemma 3.5(2) and Lemma B.1. Therefore, the t-structure in the big derived category restricts to that of the small one, which is precisely $(t^{\leq 0}, t^{\geq 0})$. □

B.2. *The second proof*

We turn to the second direct proof. In this subsection we will use $\mathcal{D}$ for the small derived category $\mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$. We need several lemmas for the proof. Put, as usual, $t^{\leq n} = t^{\leq 0}[-n]$ and $t^{\geq n} = t^{\geq 0}[-n]$. The first one is obvious.

Proposition B.3. *We have $t^{\leq -1} \subset t^{\leq 0}$ and $t^{\geq 1} \subset t^{\geq 0}$.*

The following easy observations will be useful.

Lemma B.4. *Let $\mathcal{A}$ be an abelian category with enough projectives $\mathcal{P}$. Let $P \in \mathcal{K}^-(\mathcal{P})$, $X \in \mathcal{D}^b(\mathcal{A})$, and suppose that $\operatorname{Hom}_{\mathcal{A}}(P^i, H^i(X)) = 0$ for all $i \in \mathbb{Z}$. Then $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(P, X) = 0$.*

Proof. We may assume by induction on the length of X that $X \in \mathcal{A}$. Then we have $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(P, X) = \operatorname{Hom}_{\mathcal{K}(\mathcal{A})}(P, X) \twoheadleftarrow \operatorname{Hom}_{\mathcal{C}(\mathcal{A})}(P, X) \subset \operatorname{Hom}_{\mathcal{A}}(P^0, X) = 0$. ◻

Lemma B.5. *Let $X \in t^{\leq 0}$. Then there exists a quasi-isomorphism $P \to X$ such that each term $P^i \in \operatorname{add}\{R(j) \mid j \geq i\}$ for each $i \in \mathbb{Z}$.*

Proof. This can be seen by recalling the construction of a quasi-isomorphism $P \to X$. We denote $B^n = \operatorname{Im}(X^{n-1} \to X^n)$ and $C^n = \operatorname{Coker}(X^{n-1} \to X^n)$. We start with $C = C^m$ for the largest m such that $X^m \neq 0$, $P^m \to C$ a surjection from a projective, and lift it to $P^m \to X^m$. Suppose we have constructed such P for degree $\geq n$. As in the diagram below, let $B = \operatorname{Ker}(P^n \to C)$, A the pull-back, $P^{n-1} \to A$ a surjection from a projective, and lift it to $P^{n-1} \to X^{n-1}$:

$$\begin{tikzcd}
P^{n-1} \arrow[rrr, dashed] \arrow[dr, dashed] \arrow[dd, dashed] & & & P^n \arrow[rr] \arrow[dr] \arrow[dd] & & {} \\
& A \arrow[r, dashed, two heads] \arrow[dd, dashed] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[ur, dashed] \arrow[dd, dashed] & & C \arrow[r, two heads] \arrow[dd] & {} \\
X^{n-1} \arrow[rrr] \arrow[dr] & & & X^n \arrow[rr] \arrow[dr] & & {} \\
& C^{n-1} \arrow[r, two heads] & B^n \arrow[ur] & & C^n \arrow[r, two heads] & .
\end{tikzcd}$$

Then $B \in \operatorname{mod}^{\leq -n} R$ since it is a subset of P^n and $P^n \in \operatorname{add}\{R(i) \mid i \geq n\}$. Also, since there exists an exact sequence $0 \to H^{n-1}(X) \to A \to B \to 0$ and $H^{n-1}(X) \in \operatorname{mod}^{\leq -n+1} R$ by $X \in t^{\leq 0}$, we have $A \in \operatorname{mod}^{\leq -n+1} R$. Therefore, we can take its projective cover $P^{n-1} \in \operatorname{add}\{R(i) \mid i \geq n-1\}$. ◻

These observations yield the following:

Proposition B.6. *We have $\operatorname{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $X \in t^{\leq 0}$ and $Y \in t^{\geq 1}$.*

Proof. Take a projective resolution $P \to X$ in Lemma B.5. On the other hand, we have $H^i(Y) \in \operatorname{mod}^{> -i} R$ for each $i \in \mathbb{Z}$. Therefore, we deduce $\operatorname{Hom}_{\mathcal{D}}(X, Y) = \operatorname{Hom}_{\mathcal{D}}(P, Y) = 0$ by Lemma B.4. ◻

We now give a final observation.

Proposition B.7. *For any $X \in \mathcal{D}$, there exists a triangle $X' \to X \to X'' \to X'[1]$ in $\mathcal{D}$ with $X' \in t^{\leq 0}$ and $X'' \in t^{\geq 1}$.*

Proof. We proceed by induction on $w(X) = \max\{i \in \mathbb{Z} \mid H^i(X) \neq 0\} - \min\{i \in \mathbb{Z} \mid H^i(X) \neq 0\}$.

If $w(X) = 0$, then $X \simeq Y[n]$ for some $Y \in \operatorname{mod}^{\mathbb{Z}} R$ and $n \in \mathbb{Z}$. In this case, truncating the graded module Y as $0 \to Y_{\leq n} \to Y \to Y_{>n} \to 0$ in $\operatorname{mod}^{\mathbb{Z}} R$ and shifting by $[-n]$ yields a desired triangle by Lemma B.1.

If $w(X) > 0$, there exists $n \in \mathbb{Z}$ such that in the truncation $X^{\le n} \to X \to X^{>n} \to X^{\le n}[1]$ of X with respect to (the shift of) the standard t-structure $(\mathcal{D}^{\le n}, \mathcal{D}^{\ge n})$, one has $w(Y), w(Z) < w(X)$, where $Y := X^{\le n}$ and $Z := X^{>n}$. By induction hypothesis, there exist triangles $Y' \to Y \to Y'' \to Y'[1]$ and $Z' \to Z \to Z'' \to Z'[1]$ such that $Y', Z' \in t^{\le 0}$ and $Y'', Z'' \in t^{\ge 1}$; thus the diagram below:

$$\begin{tikzcd}
Z'[-1] \arrow[r] \arrow[d, dashed] & Z[-1] \arrow[r] \arrow[d] & Z''[-1] \arrow[r] \arrow[d, dashed] & Z' \arrow[d, dashed] \\
Y' \arrow[r] & Y \arrow[r] \arrow[d] & Y'' \arrow[r] & Y'[1] \\
 & X \arrow[d] & & \\
Z' \arrow[r] & Z \arrow[r] & Z'' \arrow[r] & Z'[1].
\end{tikzcd}$$

We claim that $\mathrm{Hom}_{\mathcal{D}}(Z'[-1], Y'') = 0$. This allows us to complete the morphism $Z[-1] \to Y$ to a morphism of triangles as in the dashed line above, thus the diagram above to a 3×3 diagram of triangles by [BBD, Proposition 1.1.11]. We then have a triangle $X' \to X \to X'' \to X'[1]$ in the third row, which is a desired one since the first and the third column are triangles and $t^{\le 0}$ and $t^{\ge 0}$ are extension-closed.

We now prove the claim. By Lemma 3.5, any triangle $W' \to W \to W'' \to W'[1]$ in $\mathcal{D}$ with $W' \in t^{\le 0}$ and $W'' \in t^{\ge 1}$ yields a short exact sequence

$$0 \longrightarrow H^i(W') \longrightarrow H^i(W) \longrightarrow H^i(W'') \longrightarrow 0$$

in $\mathrm{mod}^{\mathbb{Z}} R$ for all $i \in \mathbb{Z}$. In particular, $W', W'' \in \mathcal{D}^{\le n}$ (respectively, $\in \mathcal{D}^{\ge n}$) if and only if $W \in \mathcal{D}^{\le n}$ (respectively, $\in \mathcal{D}^{\ge n}$). Now apply the above argument to $Z' \to Z \to Z'' \to Z'[1]$, which shows $Z' \in \mathcal{D}^{\ge n+1}$, hence $Z'[-1] \in t^{\le 1} \cap \mathcal{D}^{\ge n+2}$. Therefore, $H^i(Z'[-1]) \in \mathrm{mod}^{\le -n-1} R$ for all $i \in \mathbb{Z}$ since it is 0 for $i \le n+1$ and is in $\mathrm{mod}^{\le -i+1} R$ for $i \ge n+2$. Similarly, we have $H^i(Y'') \in \mathrm{mod}^{\ge -n+1} R$ for all i since $Y'' \in \mathcal{D}^{\le n} \cap t^{\ge 1}$. Now R is negatively graded, so there is a quasi-isomorphism $P \to Z'[-1]$ with P consisting of projective modules such that each term is concentrated in degree $\le -n-1$. Therefore, $\mathrm{Hom}_{\mathcal{D}}(Z'[-1], Y'') = \mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}^{\mathbb{Z}} R)}(P, Y'') = 0$, as desired. □

Now Theorem B.2 is a consequence of Propositions B.3, B.6 and B.7.

C. Proof of Proposition 4.9

In this section, we give a proof of Minamoto–Mori's equivalence [MM] based on Theorem 4.6. The main tool is the realization of the Verdier quotient as a subcategory given in Theorem 4.6(3).

We need the following computation of morphism in $\mathrm{qper}^{\mathbb{Z}} R$, which is not covered by Theorem 4.6(3). Note that when R is graded coherent, this is clear from the standard t-structure on $\mathrm{qper}^{\mathbb{Z}} R = \mathcal{D}^b(\mathrm{qgr}\, R)$.

Lemma C.1. *Let $X, Y \in \mathrm{Mod}^{\mathbb{Z}} R$, which are perfect considered as stalk complexes, and $\mathrm{Hom}_{\mathrm{Mod}^{\mathbb{Z}} R}(L, Y) = 0$ for all $L \in \mathrm{fl}^{\mathbb{Z}} R$. Then we have $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(X, Y[<0]) = 0$.*

Proof. Let $l < 0$, and let a morphism $X \to Y[l]$ in $\mathrm{qper}^{\mathbb{Z}} R$ be presented by a diagram $X \xleftarrow{s} Z \to Y[l]$ in $\mathrm{per}^{\mathbb{Z}} R$ with $L := \mathrm{cone}\, s \in \mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R)$. We claim that we can replace s by a morphism whose cone lies in $\mathrm{fl}^{\mathbb{Z}} R$:

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

$$
\begin{tikzcd}[column sep=small]
 & & Z' \arrow[dl, dashed, "s'"'] \arrow[dr] & & \\
 & X \arrow[dl, dashed] \arrow[dr, equal] & & Z \arrow[dl, "s"'] \arrow[dr] & \\
L^{\leq 0} \arrow[dr] & & X \arrow[dl] & & Y[l] \\
 & L & & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & & Z' \arrow[dl] \arrow[dr, equal] & & \\
 & & Z'' \arrow[dl] \arrow[dr, "s''"'] \arrow[rrrd, dashed] & & Z' \arrow[dl, "s'"] \arrow[dr] & \\
 & M^{<0} \arrow[dr] & & X \arrow[dl] \arrow[dr] & & Y[l] \\
 & & M \arrow[dr] & & H^0M \arrow[dl, equal] & \\
 & & & H^0M & &
\end{tikzcd}
.
$$

First complete s to a triangle $Z \xrightarrow{s} X \to L \to Z[1]$, and consider the truncation $L^{\leq 0} \to L \to L^{>0} \to L^{\leq 0}[1]$ with respect to the standard t-structure. Since $\operatorname{Hom}_{\operatorname{per}^{\mathbb{Z}} R}(X, L^{>0}) = 0$, there is a map $X \to L^{\leq 0}$; and setting $Z' := \operatorname{cocone}(X \to L^{\leq 0})$ as in the left diagram above, the original morphism equals the morphism $X \xleftarrow{s'} Z' \to Z \to Y[l]$ with cone $s' = L^{\leq 0}$ concentrated in (cohomological) degree ≤ 0.

Next consider the truncation of $M := L^{\leq 0}$ along the standard t-structure: $M^{<0} \to M \to H^0 M \to M^{<0}[1]$. By the octahedral axiom, we find a commutative diagram in the above right. Now, since we have $\operatorname{Hom}_{\operatorname{per}^{\mathbb{Z}} R}(M^{<0}[-1], Y[l]) = 0$, the morphism $Z' \to Y[l]$ factors through Z''. Then the diagram $X \xleftarrow{s''} Z'' \to Y[l]$ with cone $s'' = H^0 M \in \operatorname{fl}^{\mathbb{Z}} R$ gives the same morphism in $\operatorname{qper}^{\mathbb{Z}} R$ as the original one, which establishes our claim.

Now let $X \to Y[l]$ be a morphism in $\operatorname{qper}^{\mathbb{Z}} R$ presented by the diagram $X \xleftarrow{s} Z \to Y[l]$ in $\operatorname{per}^{\mathbb{Z}} R$ with $L := \operatorname{cone} s \in \operatorname{fl}^{\mathbb{Z}} R$. Since $\operatorname{Hom}_{\operatorname{per}^{\mathbb{Z}} R}(L[-1], Y[l]) = 0$ by the assumption on Y, the map $Z \to Y[l]$ factors through s, hence we have $\operatorname{Hom}_{\operatorname{qper}^{\mathbb{Z}} R}(X, Y[l]) \twoheadleftarrow \operatorname{Hom}_{\operatorname{per}^{\mathbb{Z}} R}(X, Y[l]) = 0$. $\square$

Now we are ready to give our proof.

Proof of Proposition 4.9. (1) We first show the vanishing of extensions: that is, $\operatorname{Hom}_{\operatorname{qper}^{\mathbb{Z}} R}(T, T[i]) = 0$ for all $i \neq 0$. Since we have $\operatorname{Hom}_{\operatorname{Mod}^{\mathbb{Z}} R}(L, T) = D \operatorname{Ext}^{d+1}_{\operatorname{Mod}^{\mathbb{Z}} R}(T, L_\alpha(a)) = 0$ for all $L \in \operatorname{fl}^{\mathbb{Z}} R$ by relative Serre duality, the case $i < 0$ follows from Lemma C.1. Also, when $i > d$, we have $\operatorname{Hom}_{\operatorname{qper}^{\mathbb{Z}} R}(T, T[i]) = D \operatorname{Hom}_{\operatorname{qper}^{\mathbb{Z}} R}(T, T(a)[d-i])$ by Serre duality, thus 0 again by Lemma C.1. Therefore, it remains to consider the case $0 < i \leq d$. Note that in $\operatorname{per}^{\mathbb{Z}} R$, we have $R(l) = (R(l)[-l])[l] \in \mathcal{M}[l]$, thus $T \in \mathcal{M}[-a+1] * \cdots * \mathcal{M}$. Therefore, $T[i]$ lies in the shifted fundamental domain $\mathcal{M}[-a+1] * \cdots * \mathcal{M}[d]$ for all $0 \leq i \leq d$. This shows $\operatorname{Hom}_{\operatorname{qper}^{\mathbb{Z}} R}(T, T[i]) = \operatorname{Hom}_{\operatorname{per}^{\mathbb{Z}} R}(T, T[i]) = 0$ for $0 < i \leq d$. We next have to show that T generates $\operatorname{qper}^{\mathbb{Z}} R$, but this follows from [MM, Proposition 4.3]. Indeed, they actually prove that all the shifts $R(l)$ for $l \in \mathbb{Z}$ lie in the thick subcategory of $\mathcal{D}(\operatorname{Mod}^{\mathbb{Z}} R)$ generated by T and the finite length modules.

(2) We deduce by (1) that there exists a triangle equivalence $\operatorname{qper}^{\mathbb{Z}} R \simeq \operatorname{per} A$ with

$$
A = \begin{pmatrix} R_0 & 0 & \cdots & 0 \\ R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \end{pmatrix},
$$

the lower triangular matrix algebra with diagonal entries R_0. Since R_0 has finite global dimension by Lemma 4.3, so does A, hence $\operatorname{qper}^{\mathbb{Z}} R \simeq \mathcal{D}^b(\operatorname{mod} A)$. Comparing the Serre functor of these categories, we have $(-)_\alpha(a)[d] \leftrightarrow \nu := - \otimes^{\mathrm{L}}_A DA$; thus $(-)_\alpha(a) \leftrightarrow \nu_d := - \otimes^{\mathrm{L}}_A DA[-d]$.

To prove that A is d-representation infinite, we first show $\nu_d^{-i} A \in \operatorname{mod} A$ for all $i \geq 0$. For this we have to verify $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(A, \nu_d^{-i} A[l]) = 0$ for all $l \neq 0$ and $i \geq 0$. By the triangle equivalence, this is to show $\operatorname{Hom}_{\operatorname{qper}^{\mathbb{Z}} R}(T, T(-ia)[l]) = 0$ since $R_\alpha \simeq R$, or $\operatorname{Hom}_{\operatorname{qper}^{\mathbb{Z}} R}(T(ia), T[l]) = 0$. By Lemma C.1 and the Serre duality, we may assume $0 < l \leq d$. To prove this, we apply Theorem 4.6(3). Note that

$T(ia) \in \mathcal{M}[-(a-1)] * \cdots$ for all $i \geq 0$ and $T[l] \in \cdots * \mathcal{M}[d]$ for all $l \leq d$. By Theorem 4.6(3), we deduce $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(T(ia), T[l]) = \mathrm{Hom}_{\mathrm{per}^{\mathbb{Z}} R}(T(ia), T[l])$, which is zero for $l \neq 0$.

Finally, we prove $\mathrm{gl.dim}\, A \leq d$. Since we have seen that $\mathrm{gl.dim}\, A$ is finite, it is sufficient to show $\mathrm{Ext}^l_A(DA, A) = 0$ for $l > d$. For any $i > 0$, we have $\mathrm{Ext}^{d+i}_A(DA, A) = \mathrm{Hom}_{\mathcal{D}^b(A)}(\nu A, A[d+i]) = \mathrm{Hom}_{\mathcal{D}^b(A)}(A, \nu_d^{-1} A[i])$, which is 0 by the previous claim. □

Acknowledgments. The author is deeply grateful to his supervisor, Osamu Iyama, for his constant and valuable discussions. He also wishes to thank Steffen Oppermann for stimulating discussions. Finally, he would like to thank the anonymous referee for carefully reading the manuscript and for numerous suggestions to improve readability.

Financial Support. This work is supported by JSPS KAKENHI Grant Numbers JP19J21165 and JP22J00649.

Conflicts of interest. None.

References

[AI] T. Aihara and O. Iyama, Silting mutation in triangulated categories, *J. London Math. Soc.* **85** (2012) no.3, 633-668.

[Am] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potentional, *Ann. Inst. Fourier, Grenoble* **59**, no.6 (2009) 2525-2590.

[AIR] C. Amiot, O. Iyama, and I. Reiten, Stable categories of Cohen-Macaulay modules and cluster categories, *Amer. J. Math*, **137** (2015) no.3, 813-857.

[AIRT] C. Amiot, O. Iyama, I. Reiten, and G. Todorov, Preprojective algebras and c-sortable words, *Proc. Lond. Math. Soc. (3)* **104** (2012) no. 3, 513-539.

[AO] C. Amiot and S. Oppermann, Higher preprojective algebras and stably Calabi-Yau properties, *Math. Res. Lett.* **21** (2014) no. 4, 617-647.

[ART] C. Amiot, I. Reiten, and G. Todorov, The ubiquity of generalized cluster categories, *Adv. Math.* **226** (2011) no. 4, 3813-3849.

[AS] M. Artin and W. Schelter, Graded algebras of global dimension 3, *Adv. Math.* **66** (1987), no. 2, 171-216.

[AZ] M. Artin and J. J. Zhang, Noncommutative projective schemes, *Adv. Math.* **109** (1994) 228-287.

[BBD] A. A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, in: Analysis and topology on singular spaces I, Luminy, 1981, *Astérisque* **100** (1982) 5-171.

[Boc1] R. Bocklandt, *Graded Calabi Yau algebras of dimension*3, *J. Pure Appl. Algebra* **212** (2008), no. 1, 14-32.

[Boc2] R. Bocklandt, Consistency conditions for dimer models, *Glasgow Math. J.* **54** (2012) 429-447.

[BSW] R. Bocklandt, T. Schedler, and M. Wemyss, Superpotentials and higher order derivations, *J. Pure Appl. Algebra* **214** (2010), no. 9, 1501–1522.

[Bon] M. V. Bondarko, Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general), *J. K-Theory* **6** (2010) no.3, 387-504.

[BS] T. Bridgeland and D. Stern, Helices on del Pezzo surfaces and tilting Calabi-Yau algebras, *Adv. Math.* **224** (2010), no. 4, 1672-1716.

[Bri] J. Brightbill, The differential graded stable category of a self-injective algebra, arXiv:1811.08992.

[Bro] N. Broomhead, Dimer models and Calabi-Yau algebras, *Mem. Amer. Math. Soc.* **215** (2012) no.1011, viii+86.

[Bu] R. O. Buchweitz, Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings, unpublished manuscript.

[BH] R. O. Buchweitz and L. Hille, Higher Representation-Infinite Algebras from Geometry, *Oberwolfach Rep.* **11** (2014), 466-469.

[BIRS] A. B. Buan, O. Iyama, I. Reiten, and J. Scott, Cluster structures for 2-Calabi-Yau categories and unipotent groups, *Compos. Math.* **145** (2009) 1035-1079.

[BMRRT] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, Tilting theory and cluster combinatorics, *Adv. Math.* **204** (2006) 572-618.

[DWZ] H. Derksen, J. Weyman, and A. Zelevinsky, Quivers with potentials and their representations I. Mutations, *Selecta Math. (N.S.)* **14** (2008), no. 1, 59-119.

[FZ] S. Fomin and A. Zelevinsky, Cluster algebras. I. Foundations, *J. Amer. Math. Soc.* **15** (2002), no. 2, 497-529.

[GLS] C. Geiss, B. Leclerc, and J. Schröer, Rigid modules over preprojective algebras, *Invent. math.* **165** (2006) 589-632.

[Gi] V. Ginzburg, Calabi-Yau algebras, arXiv:0612139.

[Gu] L. Guo, Cluster tilting objects in generalized higher cluster categories, *J. Pure Appl. Algebra* **215** (2011), no. 9, 2055-2071.

[HM] J. He and X. Mao, *Connected cochain DG algebras of Calabi-Yau dimension*0, *Proc. Amer. Math. Soc.* **145** (2017), no. 3, 937-953.

[HIO] M. Herschend, O. Iyama, and S. Oppermann, n-representation infinite algebras, *Adv. Math.* **252** (2014) 292-342.

[HJ] T. Holm and P. Jørgensen, Realizing higher cluster categories of Dynkin type as stable module categories, *Q. J. Math.* **64** (2013), no. 2, 409-435.

[Hu] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[I1] O. Iyama, Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories, *Adv. Math.* **210** (2007) 22-50.

[I2] O. Iyama, d-Calabi-Yau algebras and d-cluster tilting subcategories, unpublished manuscript, https://www.math.nagoya-u.ac.jp/~iyama/ctilt2.pdf.

[IO] O. Iyama and S. Oppermann, Stable categories of higher preprojective algebras, *Adv. Math.* **244** (2013), 23-68.

[IR] O. Iyama and I. Reiten, Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras, *Amer. J. Math.* **130** (2008), no. 4, 1087-1149.

[IYa1] O. Iyama and D. Yang, Silting reduction and Calabi-Yau reduction of triangulated categories, *Trans. Amer. Math. Soc.* **370** (2018) no.11, 7861-7898.

[IYa2] O. Iyama and D. Yang, Quotients of triangulated categories and equivalences of Buchweitz, Orlov and Amiot–Guo–Keller, *Amer. J. Math.* **142** (2020), no. 5, 1641–1659.

[IYo] O. Iyama and Y. Yoshino, Mutation in triangulated categories and rigid Cohen-Macaulay modules, *Invent. math.* **172**, 117-168 (2008)

[KY] M. Kalck and D. Yang, Derived categories of graded gentle one-cycle algebras, *J. Pure Appl. Algebra* **222** (2018) 3005-3035.

[Ke1] B. Keller, Deriving DG categories, *Ann. scient. Éc. Norm. Sup. (4)* **27** (1) (1994) 63-102.

[Ke2] B. Keller, On triangulated orbit categories, *Doc. Math.* **10** (2005) 551-581. Corrections available at https://webusers.imj-prg.fr/~bernhard.keller/publ/corrTriaOrbit.pdf

[Ke3] B. Keller, *On differential graded categories, Proceedings of the International Congress of Mathematicians*, vol. **2**, Eur. Math. Soc, 2006, 151-190.

[Ke4] B. Keller, Calabi-Yau triangulated categories, in: *Trends in representation theory of algebras and related topics, EMS series of congress reports*, European Mathematical Society, Zürich, 2008.

[Ke5] B. Keller, Cluster algebras, quiver representations and triangulated categories, in: *Triangulated categories, London Math. Soc. Lecture Note Ser.* **375**, Cambridge Univ. Press, Cambridge, 2010.

[Ke6] B. Keller, Deformed Calabi-Yau completions, with an appendix by M. Van den Bergh, *J. Reine Angew. Math.* **654** (2011) 125-180.

[KMV] B. Keller, D. Murfet, and M. Van den Bergh, On two examples of Iyama and Yoshino, *Compos. Math.* **147** (2011) 591-612.

[KR1] B. Keller and I. Reiten, Cluster tilted algebras are Gorenstein and stably Calabi-Yau, *Adv. Math.* **211** (2007) 123-151.

[KR2] B. Keller and I. Reiten, Acyclic Calabi-Yau categories, with an appendix by M. Van den Bergh, *Compos. Math.* **144** (2008) 1332-1348.

[Ki] A. King, Tilting bundles on some rational surfaces, unpublished manuscript, https://people.bath.ac.uk/masadk/papers/tilt.pdf.

[KS] M. Kontsevich and Y. Soibelman, Stability structures, Donaldson–Thomas invariants and cluster transformations, arXiv:0811.2435.

[MGYC] X. Mao, X. Gao, Y. Yang, and J. Chen, DG polynomial algebras and their homological properties, *Sci. China Math.* **62** (2019), no. 4, 629-648.

[Min1] H. Minamoto, A noncommutative version of Beilinson's theorem, *J. Algebra* **320** (2008), no. 1, 238–252.

[Min2] H. Minamoto, Ampleness of two-sided tilting complexes, *Int. Math. Res. Not.* (2012) **no. 1**, 67-101.

[MM] H. Minamoto and I. Mori, The structure of AS-Gorenstein algebras, *Adv. Math.* **226** (2011) 4061-4095.

[MY] H. Minamoto and K. Yamaura, The Happel functor and homologically well-graded Iwanaga-Gorenstein algebras, *J. Algebra* **565** (2021), 441-488.

[Miy] J. Miyachi, Localization of triangulated categories and derived categories, *J. Algebra* **141** (1991), no. 2, 463–483.

[P] D. Pauksztello, Compact corigid objects in triangulated categories and co-t-structures, *Cent. Eur. J. Math.* **6** (2008), no. 1, 25-42.

[RR] M. L. Reyes and D. Rogalski, Graded twisted Calabi-Yau algebras are generalized Artin-Schelter regular, *Nagoya Math. J.*, (2021), **1–54**.

[TV] L. de Thanhoffer de Völcsey and M. Van den Bergh, Explicit models for some stable categories of maximal Cohen-Macaulay modules, *Math. Res. Lett.* **23** (2016), no. 5, 1507-1526.

[VdB1] M. Van den Bergh, Noncommutative homology of some three-dimensional quantum spaces, Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992). *K-Theory* **8** (1994), no. 3, 213–230.

[VdB2] M. Van den Bergh, Calabi-Yau algebras and superpotentials, *Selecta Math. (N.S.)* **21** (2015), no. 2, 555-603.

[Ve] J.-L. Verdier, Des categories derivees des categories abeliennes, *Asterisque* **239** (1996).

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press

[W] M. Wemyss, Lectures on noncommutative resolutions, arXiv:1210.2564.

[Y] A. Yekutieli, Derived categories, *Cambridge Studies in Advanced Mathematics*, **183**. Cambridge University Press, Cambridge, 2020.

[YZ] A. Yekutieli and J. J. Zhang, Homological transcendence degree, *Proc. London Math. Soc. (3)* **93** (2006), no. 1, 105-137.

[Z] J. Zhang, Non-Noetherian regular rings of dimension 2, *Proc. Amer. Math. Soc.* **126** (1998), no. 6, 1645-1653.

https://doi.org/10.1017/fms.2022.30 Published online by Cambridge University Press