Canad. Math. Bull. Vol. **55** (3), 2012 pp. 646–662
http://dx.doi.org/10.4153/CMB-2011-139-1

Marcinkiewicz Commutators with Lipschitz Functions in Non-homogeneous Spaces

Jiang Zhou and Bolin Ma

Abstract. Under the assumption that μ is a nondoubling measure, we study certain commutators generated by the Lipschitz function and the Marcinkiewicz integral whose kernel satisfies a Hörmander-type condition. We establish the boundedness of these commutators on the Lebesgue spaces, Lipschitz spaces, and Hardy spaces. Our results are extensions of known theorems in the doubling case.

1 Introduction

As an analogue of the classical Littlewood–Paley g function, in 1938 Marcinkiewicz [13] introduced the operator

$$\mathcal{M}(f)(x) = \Big(\int_0^{2\pi} \frac{|F(x+t) + F(x-t) - 2F(x)|^2}{t^3}\, dt \Big)^{1/2}, \quad x \in [0, 2\pi],$$

where $F(x) = \int_0^x f(t)\, dt$. This operator is now called the Marcinkiewicz integral. Zygmund proved that the operator $\mathcal{M}$ is bounded on the Lebesgue space $L^p([0, 2\pi])$ for $p \in (1, \infty)$. Stein [17] generalized the above Marcinkiewicz integral to the following higher-dimensional case. Let Ω be homogeneous of degree zero in $\mathbb{R}^d$ for $d \geq 2$, integrable and have mean value zero on the unit sphere S^{d-1}. The higher-dimensional Marcinkiewicz integral is then defined by

$$\mathcal{M}_\Omega(f)(x) = \Big(\int_0^\infty \Big| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{d-1}} f(y)\, dy \Big|^2 \frac{dt}{t^3} \Big)^{1/2}, \quad x \in \mathbb{R}^d.$$

The Marcinkiewicz integral and its related topics are important in harmonic analysis and are still the focus of active research. The reader can refer to [1–4, 23] and the references therein. Particularly, we want to mention the work by Torchinsky and Wang [18], where they introduced the commutator generated by the Marcinkiewicz integral $\mathcal{M}_\Omega$ and the classical $\mathrm{BMO}(\mathbb{R}^d)$ function and established its $L^p(\mathbb{R}^d)$-boundedness for all $p \in (1, \infty)$ if $\Omega \in \mathrm{Lip}_\alpha(S^{d-1})$ for some $\alpha \in (0, 1]$. It is also worth mentioning that another commutator generated by the Marcinkiewicz integral $\mathcal{M}_\Omega$ and the Lipschitz function was recently studied by Mo and Lu (see [15]) when Ω is homogeneous of degree zero and satisfies the cancellation condition, and they obtained its boundedness from $L^p(\mathbb{R}^d)$ into $L^s(\mathbb{R}^d)$ for $1 < p < n/\beta$ and $1/s = 1/p - \beta/n$.

Received by the editors September 15, 2009; revised March 17, 2010.
Published electronically July 4, 2011.
This work was supported by NSFC (No.10771054 and 10861010).
AMS subject classification: **42B25**, 47B47, 42B20, 47A30.
Keywords: non doubling measure, Marcinkiewicz integral, commutator, $\mathrm{Lip}_\beta(\mu)$, $H^1(\mu)$.

On the other hand, in recent years, harmonic analysis on spaces of nondoubling measure has become a very active research topic. Among a long list of research papers, one of them [9] is on the Marcinkiewicz integral related to the nondoubling measure μ, where μ is a positive Radon measure on $\mathbb{R}^d$ that only satisfies the growth condition that for all $x \in \mathbb{R}^d$ and all $r > 0$,

$$\mu(B(x,\, r)) \le C_0\, r^n, \tag{1.1}$$

where C_0 and n are some positive constants and $0 < n \le d$, and $B(x, r)$ is the open ball centred at x and having radius r. In [9], Hu, Lin, and Yang established its boundedness, respectively, from the Lebesgue space $L^1(\mu)$ to the weak Lebesgue space $L^{1,\infty}(\mu)$, from the Hardy space $H^1(\mu)$ to $L^1(\mu)$, and from the Lebesgue space $L^\infty(\mu)$ to the space $\mathrm{RBLO}(\mu)$. Also, they obtained the boundedness of the Marcinkiewicz integral in the Lebesgue space $L^p(\mu)$ with $p \in (1, \infty)$. Moreover, they also obtained the boundedness of the commutator generated by the $\mathrm{RBMO}(\mu)$ function and the Marcinkiewicz integral with kernel satisfying certain slightly stronger Hörmander-type condition (2.1), respectively, from $L^p(\mu)$ with $p \in (1, \infty)$ to itself, from the space $L \log L(\mu)$ to $L^{1,\infty}(\mu)$ and from $H^1(\mu)$ to $L^{1,\infty}(\mu)$.

We recall that μ is said to be a doubling measure if there is a positive constant C such that for any $x \in \operatorname{supp}(\mu)$ and $r > 0$, $\mu(B(x, 2r)) \le C\mu(B(x, r))$, and that the doubling condition is a key assumption in the classical theory of harmonic analysis. Recently, many classical results concerning the theory of Calderón–Zygmund operators and function spaces have been proved still valid if the Lebesgue measure is substituted by a measure μ as in (1.1); see [5, 7, 10, 14, 19, 20, 22] and their references. We mention that the analysis on non-homogeneous spaces played an essential role in solving the long-standing open Painlevé problem by Tolsa [22].

Let K be a locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\}$. Assume that there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^d$ with $x \ne y$,

$$|K(x, y)| \le C|x - y|^{-(n-1)}, \tag{1.2}$$

$$\int_{|x-y|\ge 2|y-y'|} \Big[|K(x,\, y) - K(x,\, y')| + |K(y,\, x) - K(y',\, x)| \Big] \times \frac{1}{|x-y|}\, d\mu(x) \le C \tag{1.3}$$

for any $y, y' \in \mathbb{R}^d$. The Marcinkiewicz integral $\mathcal{M}(f)$ associated with the above kernel K and the measure μ as in (1.1) is defined by

$$\mathcal{M}(f)(x) = \Big(\int_0^\infty \Big| \int_{|x-y|\le t} K(x,\, y) f(y)\, d\mu(y) \Big|^2 \frac{dt}{t^3} \Big)^{1/2}, \quad x \in \mathbb{R}^d. \tag{1.4}$$

Throughout this paper, we always assume that $\mathcal{M}$ is bounded on $L^2(\mu)$. Obviously, if μ is the d-dimensional Lebesgue measure in $\mathbb{R}^d$ with $d \ge 2$, and $K(x, y) = \Omega(x - y)/|x - y|^{d-1}$ with Ω homogeneous of degree zero and $\Omega \in \mathrm{Lip}_\beta(S^{d-1})$ for some $\beta \in (0, 1]$ $\int_{S^{d-1}} \Omega = 0$, then it is easy to verify that K satisfies (1.2) and (1.3), and $\mathcal{M}$ in (1.4) is just the Marcinkiewicz integral $\mathcal{M}_\Omega$ introduced by Stein [17]. Thus, $\mathcal{M}$ in (1.4) is a natural generalization of the classical Marcinkiewicz integral in the current setting.

By a cube $Q \subset \mathbb{R}^d$ we mean a closed cube whose sides are parallel to the coordinate axes, and we denote its centre and its side length by x_Q and $\ell(Q)$, respectively. For $\alpha > 1$,let αQ denote the cube with the same centre as Q and $\ell(\alpha Q) = \alpha \ell(Q)$.

Given two cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$S_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{[\ell(2^k Q)]^n},$$

where $N_{Q,R}$ is the smallest positive integer k such that $\ell(2^k Q) \geq \ell(R)$. The number of $S_{Q,R}$ first appeared in [19], where some useful properties of $S_{Q,R}$ could be found.

Now we define multilinear commutators generated by Marcinkiewicz integral and Lipschitz functions. First we recall the following definition of Lipschitz functions [5].

Definition 1.1 Let $\beta > 0$ and b be a μ-locally integrable function on $\mathbb{R}^d$. We say b belongs to the space $\mathrm{Lip}_\beta(\mu)$ if there is a constant $C > 0$ such that

$$|b(x) - b(y)| \leq C|x - y|^\beta \tag{1.5}$$

for $\mu-$almost every x and y in the support of μ. The minimal constant C in (1.5) is the $\mathrm{Lip}_\beta(\mu)$ norm of b and is denoted simply by $\|b\|_{\mathrm{Lip}_{(\beta)}}$.

Let $\mathcal{M}$ be the Marcinkiewicz integral operators as in (1.4), $m \in \mathbb{N}$ and $b_i \in \mathrm{Lip}\,\beta_i(\mu)$, $i = 1, 2, \dots, m$. The multilinear commutator $\mathcal{M}_{\vec{b}}$ is formally defined for $x \in \mathbb{R}^d$ by

$$\mathcal{M}_{\vec{b}}(f)(x) = \Big(\int_0^\infty \Big| \int_{|x-y|\leq t} \prod_{i=1}^m [b_i(x) - b_i(y)] K(x,\, y) f(y)\, d\mu(y) \Big|^2 \frac{dt}{t^3} \Big)^{1/2}. \tag{1.6}$$

In what follows, if $m = 1$ and $\vec{b} = b$, then we denote $\mathcal{M}_{\vec{b}}$ simply by $\mathcal{M}_b$. When $b_1 = \cdots = b_m$, $\mathcal{M}_{\vec{b}}$ is the higher commutator of the Marcinkiewicz integrals denoted by $\mathcal{M}_b^m$. In this paper, we will study the behaviours of the multilinear commutator defined by (1.6) on the Lebesgue space and the Hardy space.

In Section 2, we focus on the boundedness on Lebesgue spaces. Meng and Yang [14] obtained the $(L^p(\mu),\, L^q(\mu))$ boundedness of multilinear commutators defined by Calderón–Zygmund operators and Lipschitz functions for $1 < p < n/(\sum_{i=1}^m \beta_i)$ and $1/q = 1/p - (\sum_{i=1}^m \beta_i)/n$ and as well as their weak type $(L^1(\mu), L^{n/(n-\sum_{i=1}^m \beta_i)}(\mu))$ boundedness where $0 < \sum_{i=1}^m \beta_i < n$. When $m = 1$, they also considered the boundedness in the case that $n/\beta < p < \infty$ and the endpoint case that $p = n/\beta$. Here the author obtains the same bounded estimates for $\mathcal{M}_{\vec{b}}$.

Similar to the result in [14], in Section 3 we will prove that the multilinear commutator defined by (1.6) is bounded from the Hardy space $H^1(\mu)$ to some Lebesgue space with nondoubling measures.

Throughout this paper, we use the constant C with subscripts to indicate its dependence on the parameters in the subscripts. For a μ-measurable set E, χ_E denotes its characteristic function. For any $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$.

2 Boundedness on Lebesgue Spaces

This section is devoted to the behaviour of commutators on Lebesgue spaces.

Theorem 2.1 *Let $m \in \mathbb{N}$ and for $i = 1, 2, \dots, m$, $b_i \in \mathrm{Lip}(\beta_i, \mu)$ with $0 < \beta_i \le 1$. Assume K satisfies (1.2) and (1.3), and let $\mathcal{M}_{\vec{b}}$ be as in (1.6). Suppose $0 < \sum_{i=1}^m \beta_i < n$. Then there exists a positive constant $C > 0$ such that*

(i) *for all bounded functions f with compact support,*

$$\|\mathcal{M}_{\vec{b}}(f)\|_{L^q(\mu)} \le C \prod_{i=1}^m \|b_i\|_{\mathrm{Lip}_{(\beta_i)}} \|f\|_{L^p(\mu)},$$

where $1 < p < n/(\sum_{i=1}^m \beta_i)$ and $1/q = 1/p - (\sum_{i=1}^m \beta_i)/n$;

(ii) *for all bounded functions f with compact support and all $\lambda > 0$,*

$$\mu(\{x \in \mathbb{R}^d : \mathcal{M}_{\vec{b}}(f)(x) > \lambda\}) \le C \Big(\frac{\prod_{i=1}^m \|b_i\|_{\mathrm{Lip}_{(\beta_i)}} \|f\|_{L^1(\mu)}}{\lambda} \Big)^{n/(n-\sum_{i=1}^m \beta_i)}.$$

To prove Theorem 2.1, we need the following lemma about fractional integral operators on Lebesgue space with nondoubling measures.

Recall that for $0 < \alpha < n$ and all $x \in \mathrm{supp}(\mu)$, the fractional integral operator I_α is defined by

$$I_\alpha(f)(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^{n-\alpha}} f(y)\, d\mu(y).$$

García-Cuerva and Gatto [6] obtained the boundedness of I_α as follows.

Lemma 2.2 *Let $0 < \alpha < n$, $1 \le p < n/\alpha$ and $1/q = 1/p - \alpha/n$.Then there exists a positive constant $C > 0$ such that for all bounded functions f with compact support and all $\lambda > 0$,*

$$\|I_\alpha(f)\|_{L^q(\mu)} \le C\|f\|_{L^p(\mu)},$$

$$\mu(\{x \in \mathbb{R}^d : I_\alpha(f)(x) > \lambda\}) \le C \Big(\frac{\|f\|_{L^1(\mu)}}{\lambda} \Big)^{\frac{n}{n-\alpha}}.$$

Proof of Theorem 2.1 By the Minkowski inequality and the condition (1.4), we have

$$\begin{aligned}
\mathcal{M}_{\vec{b}}(f)(x) &= \Big(\int_0^\infty \Big|\int_{|x-y|\le t}\prod_{i=1}^m [b_i(x)-b_i(y)]K(x,y)f(y)\,d\mu(y)\Big|^2 \frac{dt}{t^3}\Big)^{1/2}\\
&= \int_{\mathbb{R}^d} |K(x,y)|\prod_{i=1}^m |b_i(x)-b_i(y)||f(y)|\Big(\int_{|x-y|\le t}^\infty \frac{dt}{t^3}\Big)^{1/2} d\mu(y)\\
&\le C\prod_{i=1}^m \|b_i\|_{\mathrm{Lip}_{(\beta_i)}}\int_{\mathbb{R}^d}\frac{|f(y)|}{|x-y|^{n-(\sum_{i=1}^m \beta_i)}}\,d\mu(y)\\
&\le C\prod_{i=1}^m \|b_i\|_{\mathrm{Lip}_{(\beta_i)}} I_{(\sum_{i=1}^m \beta_i)}(|f|)(x)
\end{aligned}$$

Then it is easy to deduce the result from Lemma 2.2. ■

By contrast with the endpoint estimate for the commutators generated by Marcinkiewicz integrals and RBMO(μ) functions ([9, Theorems 3.1, 3.5, 3.6]), we can see that the behaviour of commutators with Lipschitz functions is quite different from that of commutators with RBMO(μ) functions.

Now we assume $m = 1$. In the following, using Theorem 2.1, we consider the boundedness of commutators defined by (1.6) for $n/\beta < p < \infty$ and $p = n/\beta$.

Hu, Lin, and Yang [9]introduced a Hörmander condition

(2.1)
$$\sup_{\substack{l>0, y, y'\in\mathbb{R}^d\\ |y-y'|\le l}} \sum_{k=1}^\infty k\int_{2^k l<|x-y|\le 2^{k+1}l}\big[|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)|\big] \times \frac{1}{|x-y|}d\mu(x)\le C,$$

which is slightly stronger than (1.3). Actually they proved the boundedness of Marcinkiewicz commutator generated with $RBMO(\mu)$.

Here we will study the commutator $\mathcal{M}_b$ with kernel K satisfying (1.1) and the Hörmander condition defined as follows.

Definition 2.3 Given $1 \le s < \infty$, $0 < \varepsilon \le 1$, we say that the kernel K satisfies the L^s-Hörmander condition if there are numbers $c_s > 1$ and $C_s > 0$ such that for any $x \in \mathbb{R}^d$ and $l > c_s|x|$,

$$\sup_{\substack{l>0, y, y'\in\mathbb{R}^d\\ |y-y'|\le l}} \sum_{k=1}^\infty 2^{k\varepsilon}(2^k l)^n\Big(\frac{1}{(2^k l)^n}\int_{2^k l<|x-y|\le 2^{k+1}l}\Big[\Big(|K(x,y)-K(x,y')| + |K(y,x)-K(y',x)|\Big)\frac{1}{|x-y|}\Big]^s d\mu(x)\Big)^{1/s}\le C_s.$$

We will denote by $\mathscr{H}^s$ the class of kernels satisfying the L^s-Hörmander condition.

Observe that these classes are nested: by Hölder's inequality, it is easy to check that

$$\mathscr{H}^{s_2} \subset \mathscr{H}^{s_1} \subset \mathscr{H}^1, \quad 1 < s_1 < s_2 < \infty,$$

and $\mathscr{H}^1$ is obviously stronger than condition (2.1).

Theorem 2.4 *Let $n/\beta < p < \infty$, $0 < \varepsilon \leq 1$ and $b \in \mathrm{Lip}_\beta(\mu)$, $0 < \beta < \min\{1/2, \varepsilon\}$. If K satisfies* (1.2) *and the $\mathscr{H}^s (p \leq s < \infty)$ condition, then the commutator $\mathcal{M}_b$ in* (1.6) *is bounded from $L^p(\mu)$ into $\mathrm{Lip}_{\beta-n/p}(\mu)$, with bound no more than $C\|b\|_{\mathrm{Lip}_\beta(\mu)}$ such that $\|\mathcal{M}_b(f)\|_{\mathrm{Lip}_{\beta-n/p}(\mu)} \leq C\|b\|_{\mathrm{Lip}_\beta(\mu)}\|f\|_{L^p(\mu)}$.*

Remark 2.5 The method used in the proof of Theorem 2.4 is not applicable to multilinear commutators defined by (1.6) for $m \geq 2$.

Remark 2.6 When μ is the d-dimensional Lebesgue measure in $\mathbb{R}^d$ and $K(x, y) = \Omega(x - y)/|x - y|^{n-1}$ where $\Omega \in L^s(S^{d-1})$ satisfies the condition defined in [15, Theorem 1], by a straightforward computation using [12, Lemma 2.2.2], we know K must satisfy the condition $\mathscr{H}^s$ with some $0 < \varepsilon \leq 1$. In this sense, the condition $\mathscr{H}^s$ is weaker.

The following characterization of the space $\mathrm{Lip}_\beta(\mu)$ for $0 < \beta \leq 1$ in [5] plays a key role in the proof of Theorem 2.4.

Lemma 2.7 *For a function $b \in L^1_{\mathrm{loc}}(\mu)$, conditions* (i), (ii), *and* (iii) *below are equivalent.*

(i) *There is a constant $C_1 \geq 0$ such that $|b(x) - b(y)| \leq C_1|x - y|^\beta$, for μ-almost every x and y in the support of μ.*

(ii) *There exist some constant $C_2 \geq 0$ and a collection of numbers b_Q, one for each cube Q, such that these two properties hold: for any cube Q*

$$\frac{1}{\mu(2Q)} \int_Q |b(x) - b_Q|\, d\mu(x) \leq C_2 \ell(Q)^\beta, \tag{2.2}$$

and for any cube R such that $Q \subset R$ and $\ell(R) \leq 2\ell(Q)$,

$$|b_Q - b_R| \leq C_2 \ell(Q)^\beta. \tag{2.3}$$

(iii) *For any given p, $1 \leq p \leq \infty$, there is a constant $C(p) \geq 0$, such that for every cube Q, we have*

$$\Big[\frac{1}{\mu(Q)} \int_Q |b(x) - m_Q(b)|^p\, d\mu(x)\Big]^{1/p} \leq C(p)\ell(Q)^\beta,$$

where here and in the sequel, $m_Q(b) = \frac{1}{\mu(Q)} \int_Q b(y)\, d\mu(y)$, and also for any cube R such that $Q \subset R$ and $\ell(R) \leq 2\ell(Q)$, $|m_Q(b) - m_R(b)| \leq C(p)\ell(Q)^\beta$.

In addition, the quantities $\inf\{C_1\}$, $\inf\{C_2\}$, and $\inf\{C(p)\}$ with a fixed p are equivalent to $\|b\|_{\mathrm{Lip}_\beta(\mu)}$.

We remark that Lemma 2.7 is a slight variant of [5, Theorem 2.3]. To be precise, if we replace all balls in [5, Theorem 2.3] by cubes, we then obtain Lemma 2.7.

Remark 2.8 For $0 < \beta \leq 1$, (2.3) is equivalent to

$$|b_Q - b_R| \leq C_2' S_{Q,R} \ell(R)^{\beta} \tag{2.4}$$

for any two cubes $Q \subset R$ with $\ell(R) \leq 2\ell(Q)$; see [5, Remark 2.7]. Note that for $\beta = 0$, (2.2) and (2.4) are just the space RBMO(μ) of Tolsa; see [21]. Therefore, the space $\mathrm{Lip}_{\beta}(\mu)$ for $0 < \beta \leq 1$ can be seen as a member of a family containing RBMO(μ).

Proof of Theorem 2.4 For any cube Q in $\mathbb{R}^d$ and any cube R such that $Q \subset R$ satisfies $\ell(R) \leq 2\ell(Q)$, let $a_Q = m_Q[\mathcal{M}_b(f\chi_{\mathbb{R}^d \setminus \frac{3}{2}Q})r]$, and $a_R = m_R[\mathcal{M}_b(f\chi_{\mathbb{R}^d \setminus \frac{3}{2}R})]$. It is easy to see a_Q and a_R are real numbers. By Lemma 2.7, we need to show that there exists a constant $C > 0$ such that

$$\frac{1}{\mu(2Q)} \int_Q |\mathcal{M}_b(f)(x) - a_Q| \, d\mu(x) \leq C\|f\|_{L^p(\mu)} \, \ell(Q)^{\beta - n/p}, \tag{2.5}$$

$$|a_Q - a_R| \leq C\|f\|_{L^p(\mu)} \, \ell(Q)^{\beta - n/p}. \tag{2.6}$$

Let us first prove the estimate (2.5). For a fixed cube Q and $x \in Q$, decompose $f = f_1 + f_2$, where $f_1 = f\chi_{\frac{3}{2}Q}$ and $f_2 = f - f_1$. Write

$$\begin{aligned}\frac{1}{\mu(2Q)} \int_Q |\mathcal{M}_b(f)(x) - a_Q| \, d\mu(x) &\leq \frac{1}{\mu(2Q)} \int_Q |\mathcal{M}_b(f_1)(x)| \, d\mu(x) \\ &\quad + \frac{1}{\mu(2Q)} \int_Q |\mathcal{M}_b(f_2)(x) - a_Q| \, d\mu(x) \\ &= \mathrm{I}_1 + \mathrm{I}_2\end{aligned}$$

Choose $1 < p_1 < n/\beta < p$ and q_1 such that $1/q_1 = 1/p_1 - \beta/n$. From the Hölder inequality and Lemma 2.7, it follows that

$$\begin{aligned}\mathrm{I}_1 &\leq C \frac{1}{\mu(2Q)} \Big[\int_Q |\mathcal{M}_b(f_1)(x)|^{q_1} \, d\mu(x) \Big]^{1/q_1} \mu(Q)^{1-1/q_1} \\ &\leq C\|f\|_{L^p(\mu)} \, \|b\|_{\mathrm{Lip}_{\beta}(\mu)} \ell(Q)^{\beta - n/p}.\end{aligned}$$

To estimate the term I_2, set

$$\mathrm{D}_1(x, y) = \Big(\int_0^{\infty} \Big[\int_{|x-z| \leq t < |y-z|} |K(x, z)||b(z) - m_Q(b)||f_2(z)| \, d\mu(z) \Big]^2 \frac{dt}{t^3} \Big)^{1/2},$$

$$\mathrm{D}_2(x, y) = \Big(\int_0^{\infty} \Big[\int_{|y-z| \leq t < |x-z|} |K(y, z)||b(z) - m_Q(b)||f_2(z)| \, d\mu(z) \Big]^2 \frac{dt}{t^3} \Big)^{1/2},$$

$$\mathrm{D}_3(x,\ y) = \Big(\int_0^\infty \Big[\int_{\substack{|y-z|\le t\\|x-z|\le t}} |K(x,\ z)-K(y,\ z)||b(z)-m_Q(b)||f_2(z)|\,d\mu(z)\Big]^2\,\frac{dt}{t^3}\Big)^{1/2}.$$

It is easy to see that for any $x, y \in Q$,

$$\begin{aligned}
&|\mathcal{M}_b(f_2)(x)-\mathcal{M}_b(f_2)(y)|\\
&\quad = \Big|\Big(\int_0^\infty\Big|\int_{|x-z|\le t}[b(x)-b(z)]K(x,\ z)f_2(z)\,d\mu(z)\Big|^2\,\frac{dt}{t^3}\Big)^{1/2}\\
&\qquad -\Big(\int_0^\infty\Big|\int_{|y-z|\le t}[b(y)-b(z)]K(y,\ z)f_2(z)\,d\mu(z)\Big|^2\,\frac{dt}{t^3}\Big)^{1/2}\Big|\\
&\quad \le \sum_{j=1}^{3}\mathrm{D}_j(x,\ y).
\end{aligned}$$

To estimate $\mathrm{D}_1(x,\ y)$, for $x, y \in Q$, $z \in (\frac{3Q}{2})^c$, we have

$$\begin{aligned}
\mathrm{D}_1 &\le \Big(\int_0^\infty\Big[\int_{|x-z|\le t<|y-z|}\frac{|b(z)-m_Q(b)|}{|x-z|^{n-1}}|f_2(z)|\,d\mu(z)\Big]^2\,\frac{dt}{t^3}\Big)^{1/2}\\
&\le C\ell(Q)^{1/2}\int_{\mathbb{R}^d\setminus\frac{3}{2}Q}\frac{|b(z)-m_Q(b)|}{|x-z|^{n+1/2}}|f(z)|\,d\mu(z)\\
&\le C\ell(Q)^{1/2}\sum_{k=1}^{\infty}\int_{\frac{3}{2}2^kQ\setminus\frac{3}{2}2^{k-1}Q}\frac{|b(z)-m_Q(b)|}{|x-z|^{n+1/2}}|f(z)|\,d\mu(z)\\
&\le C\sum_{k=1}^{\infty}2^{-k/2}\ell(2^kQ)^{\beta-n}\|b\|_{Lip_\beta(\mu)}\int_{\frac{3}{2}2^kQ}|f(z)|\,d\mu(z)\\
&\le C\|b\|_{Lip_\beta(\mu)}\|f\|_{L^p(\mu)}\sum_{k=1}^{\infty}2^{-k/2}\ell(2^kQ)^{\beta-n}\mu(\frac{3}{2}2^kQ)^{1-1/p}\\
&\le C\|b\|_{Lip_\beta(\mu)}\|f\|_{L^p(\mu)}\,\ell(Q)^{\beta-n/p}.
\end{aligned}$$

Here we used the Minkowski inequality, $0 < \beta < 1/2$ and condition (ii) of Lemma 2.7, *i.e.*,

$$|b(z)-m_Q(b)| \le C\ell(2^kQ)^\beta\|b\|_{Lip_\beta(\mu)}, \quad \text{for } z \in \mathbb{R}^d\setminus\frac{3}{2}Q.$$

Similarly, by symmetry, we have that for $x, y \in Q$,

$$\mathrm{D}_2 \le C\|b\|_{Lip_\beta(\mu)}\|f\|_{L^p(\mu)}\,\ell(Q)^{\beta-n/p}.$$

Finally, applying the Minkowski inequality and condition (1.2).

$$
\begin{aligned}
&\mathrm{D}_3(x,\ y)\\
&\quad= \Big(\int_0^\infty \Big[\int_{\substack{|y-z|\le t\\ |x-z|\le t}} |K(x,\ z)-K(y,z)||b(z)-m_Q(b)||f_2(z)|\,d\mu(z)\Big]^2 \frac{dt}{t^3}\Big)^{1/2}\\
&\quad\le C\sum_{k=1}^\infty \int_{\frac32 2^kQ\setminus\frac32 2^{k-1}Q} |K(x,\ z)-K(y,\ z)||b(z)-m_Q(b)|\frac{|f(z)|}{|y-z|}\,d\mu(z)\\
&\quad\le C\|b\|_{\mathrm{Lip}_\beta(\mu)}\|f\|_{L^p(\mu)}\sum_{k=1}^\infty \ell(2^kQ)^{\beta-n/p}\ell(2^kQ)^{n/p}\\
&\qquad\quad\times\Big(\int_{\frac32 2^kQ\setminus\frac32 2^{k-1}Q}\Big[|K(x,\ z)-K(y,\ z)|\frac{1}{|y-z|}\Big]^{p'} d\mu(z)\Big)^{1/p'}\\
&\quad\le C\|b\|_{\mathrm{Lip}_\beta(\mu)}\|f\|_{L^p(\mu)}\sum_{k=1}^\infty \ell(2^kQ)^{\beta-n/p}\ell(2^kQ)^{n}\\
&\qquad\quad\times\Big(\frac{1}{\ell(2^kQ)^n}\int_{\frac32 2^kQ\setminus\frac32 2^{k-1}Q}\Big(|K(x,z)-K(y,z)|\frac{1}{|x-y|}\Big)^{p'} d\mu(z)\Big)^{1/p'}\\
&\quad\le C\|b\|_{\mathrm{Lip}_\beta(\mu)}\|f\|_{L^p(\mu)}\ell(Q)^{\beta-n/p}.
\end{aligned}
$$

Here we used the condition $\mathscr{H}^{p'}$ that kernel K satisfies and the fact that $0<\beta<\min\{1/2,\varepsilon\}$.

Combining the estimates above, we obtain that

$$\mathrm{I}_2 \le C\|b\|_{\mathrm{Lip}_\beta(\mu)}\|f\|_{L^p(\mu)}\ell(Q)^{\beta-n/p}.$$

and the estimate (2.5) is proved.

We now verify (2.6). For any cubes $Q\subset R$ with $x\in Q$, where Q is an arbitrary and R is a doubling cube with $\ell(R)\ge\ell(Q)$, denote $N_{Q,R}+1$ simply by N. Write

$$
\begin{aligned}
|a_Q-a_R| \le{}& \big|m_R[\mathcal{M}_b(f\chi_{\mathbb{R}^d\setminus 2^NQ})]-m_Q[\mathcal{M}_b(f\chi_{\mathbb{R}^d\setminus 2^NQ})]\big|\\
&+|m_Q[\mathcal{M}_b(f\chi_{2^NQ\setminus\frac32 Q})]|+|m_R[\mathcal{M}_b(f\chi_{2^NQ\setminus\frac32 R})]| \quad = \mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3\,.
\end{aligned}
$$

As in the estimate for the term II_2, we have

$$\mathrm{E}_1 \le C\|b\|_{\mathrm{Lip}_\beta(\mu)}\|f\|_{L^p(\mu)}\ell(Q)^{\beta-n/p}.$$

On the other hand, via $y\in R$, $z\in 2^NQ\setminus\frac32 Q$, we have

$$
\begin{aligned}
\mathcal{M}_b\Big(f\chi_{2^NQ\setminus\frac32 R}\Big)(y) &\le C\int_{2^NQ\setminus\frac32 R}|K(y,\ z)(b(y)-b(z))f(z)|\Big(\int_{|y-z|}^\infty\frac{dt}{t^3}\Big)^{1/2} d\mu(z)\\
&\le C\|b\|_{\mathrm{Lip}_\beta(\mu)}\|f\|_{L^p(\mu)}\ell(Q)^{\beta-n/p}.
\end{aligned}
$$

Taking the mean over $y \in R$, we obtain

$$E_3 \leq C\|b\|_{\mathrm{Lip}_\beta(\mu)}\|f\|_{L^p(\mu)}\ell(Q)^{\beta-n/p}.$$

An argument similar to the estimate for E_2 tells us that

$$\mathrm{E}_2 \leq C\|b\|_{\mathrm{Lip}_\beta(\mu)}\|f\|_{L^p(\mu)}\ell(Q)^{\beta-n/p}. \qquad \blacksquare$$

For the endpoint case that $p = n/\beta$, we have the following result.

Theorem 2.9 *Let K satisfy (1.2) and the $\mathscr{H}^{p'}$ condition for $p \in (n/\beta, \infty)$ and $p' = \frac{p}{p-1}$. Let $\mathcal{M}_b$ be defined as in (1.6). Then for any $0 < \beta \leq 1$ and $b \in \mathrm{Lip}_\beta(\mu)$, there is a constant $C > 0$ such that for all bounded functions f with compact support,*

$$\|\mathcal{M}_b(f)\|_{\mathrm{RBMO}(\mu)} \leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^{n/\beta}(\mu)}.$$

Here we will not give the details of the proof of Theorem 2.9, since we can prove it similarly to Theorem 2.4.

3 Boundedness on Hardy Spaces $H^1(\mu)$

In order to consider the boundedness of multilinear commutators generated by the Marcinkiewicz integrals with Lipschitz functions on the Hardy space $H^1(\mu)$ of Tolsa [19,20], we first recall the definition of the grand maximal operator M_Φ of Tolsa [20].

Definition 3.1 Given $f \in L^1_{\mathrm{loc}}(\mu)$, we define

$$M_\Phi f(x) = \sup_{\varphi\sim x}\Big|\int_{\mathbb{R}^d} f\varphi\, d\mu\Big|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)} \leq 1$,
(ii) $0 \leq \varphi(y) \leq \frac{1}{|y-x|^n}$ for all $y \in \mathbb{R}^d$,
(iii) $|\varphi'(y)| \leq \frac{1}{|y-x|^{n+1}}$ for all $y \in \mathbb{R}^d$.

Based on Tolsa ([20, Theorem 1.2]), we can define the Hardy space $H^1(\mu)$ as follows; see also [19].

Definition 3.2 The Hardy space $H^1(\mu)$ is the set of all functions $f \in L^1(\mu)$ satisfying $\int_{\mathbb{R}^d} f\, d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Moreover, we define the norm of $f \in H^1(\mu)$ by

$$\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}.$$

Using Theorem 2.1, we can obtain the following boundedness of multilinear commutators in the Hardy space $H^1(\mu)$.

https://doi.org/10.4153/CMB-2011-139-1 Published online by Cambridge University Press

Theorem 3.3 *Let $m \in \mathbb{N}$ and for $i = 1, 2, \dots, m$, $b_i \in \mathrm{Lip}(\beta_i, \mu)$ and $0 < \beta_i \leq 1$. Suppose that $0 < \sum_{i=1}^{m} \beta_i < n$ and $1/q = 1 - (\sum_{i=1}^{m} \beta_i)/n$. Let K satisfy (1.2) and the $\mathscr{H}^q$ condition and let $\mathcal{M}_{\vec{b}}$ be as in (1.6). Then $\mathcal{M}_{\vec{b}}$ is bounded from $H^1(\mu)$ to $L^q(\mu)$ with the operator norm at most $C\|b_1\|_{\mathrm{Lip}(\beta_1)} \cdots \|b_m\|_{\mathrm{Lip}(\beta_m)}$.*

Remark 3.4 In [19], Tolsa showed that the space $\mathrm{RBMO}(\mu)$ is the predual of the Hardy space $H^1(\mu)$, as in the doubling case. By this fact together with the fact that $L^{n/\beta}(\mu)$ is the dual of $L^{n/(n-\beta)}(\mu)$, we can deduce Theorem 2.9 from Theorem 3.3. We omit the details.

To prove Theorem 3.3, we first recall the atomic Hardy space $H^{1,\infty}_{\mathrm{atb}}(\mu)$, which has been proved to be the same space as the Hardy space $H^1(\mu)$; see [19, 20].

Definition 3.5 Let $\rho > 1$. A function $h \in L^1_{\mathrm{loc}}(\mu)$ is called an atomic block if

(i) there exists some cube R such that $\mathrm{supp}(h) \subset R$,

(ii) $\int_{\mathbb{R}^d} h(x)\, d\mu(x) = 0$,

(iii) for $i = 1, 2$, there are functions a_i supported on cubes $Q_i \subset R$ and numbers $\lambda_i \in \mathbb{R}$ such that $h = \lambda_1 a_1 + \lambda_2 a_2$, and $\|a_i\|_{L^\infty(\mu)} \leq [\mu(\rho Q_i) S_{Q_i, R}]^{-1}$.

Then we define $|h|_{H^{1,\infty}_{\mathrm{atb}}(\mu)} = |\lambda_1| + |\lambda_2|$. We say that $f \in H^{1,\infty}_{\mathrm{atb}}(\mu)$ if there are atomic blocks $\{h_j\}_{j\in\mathbb{N}}$ such that

$$f = \sum_{j=1}^{\infty} h_j$$

with $\sum_{j=1}^{\infty} |h_j|_{H^{1,\infty}_{\mathrm{atb}}(\mu)} < \infty$. The $H^{1,\infty}_{\mathrm{atb}}(\mu)$ norm of f is defined by

$$\|f\|_{H^{1,\infty}_{\mathrm{atb}}(\mu)} = \inf\Big\{ \sum_j |h_j|_{H^{1,\infty}_{\mathrm{atb}}(\mu)} \Big\},$$

where the infimum is taken over all possible decompositions of f in atomic blocks.

The definition of $H^{1,\infty}_{\mathrm{atb}}(\mu)$ does not depend on the constant $\rho > 1$, which was proved in [19].

Proof of Theorem 3.3 For simplicity, set

$$\beta = \sum_{i=1}^{m} \beta_i \quad \text{and} \quad \|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} = \prod_{i=1}^{m} \|b_i\|_{\mathrm{Lip}_{(\beta_i)}}.$$

It is easy to see that we only need to prove the theorem for atomic blocks h as in Definition 3.5 with $\rho = 4$. Let R be a cube such that $\mathrm{supp}(h) \subset R$, $\int_{\mathbb{R}^d} h(x)\, d\mu(x) = 0$, and

$$h(x) = \lambda_1 a_1(x) + \lambda_2 a_2(x), \tag{3.1}$$

where λ_i for $i = 1, 2$ are real numbers, $|h|_{H^{1,\infty}_{\mathrm{atb}}(\mu)} = |\lambda_1| + |\lambda_2|$, a_i for $i = 1, 2$ are bounded functions supported on some cube $Q_i \subset R$ and satisfy

$$\|a_i\|_{L^\infty(\mu)} \leq [\mu(4Q_i) S_{Q_i,R}]^{-1}. \tag{3.2}$$

https://doi.org/10.4153/CMB-2011-139-1 Published online by Cambridge University Press

Write

$$
\begin{aligned}
\|\mathcal{M}_{\vec{b}}(h)\|_{L^q(\mu)} &\le C\Big(\int_{2R}|\mathcal{M}_{\vec{b}}(h)(x)|^q\,d\mu(x)\Big)^{1/q}+\Big(\int_{\mathbb{R}^d\setminus(2R)}|\mathcal{M}_{\vec{b}}(h)(x)|^q\,d\mu(x)\Big)^{1/q}\\
&\le C\Big(\int_{2R}|\mathcal{M}_{\vec{b}}(h)(x)|^q\,d\mu(x)\Big)^{1/q}\\
&\quad+\Big\{\int_{\mathbb{R}^d\setminus(2R)}\Big(\int_0^{|x-x_R|+2\ell(R)}\Big|\int_{|x-y|\le t}K(x,y)\\
&\qquad\times\prod_{i=1}^m[b_i(x)-b_i(y)]h(y)\,d\mu(y)\Big|^2\frac{dt}{t^3}\Big)^{q/2}d\mu(x)\Big\}^{1/q}\\
&\quad+\Big\{\int_{\mathbb{R}^d\setminus(2R)}\Big(\int_{|x-x_R|+2\ell(R)}^{\infty}\Big|\int_{|x-y|\le t}K(x,y)\\
&\qquad\times\prod_{i=1}^m[b_i(x)-b_i(y)]h(y)\,d\mu(y)\Big|^2\frac{dt}{t^3}\Big)^{q/2}d\mu(x)\Big\}^{1/q}\\
&=\mathrm{I}+\mathrm{II}+\mathrm{III}\,.
\end{aligned}
$$

By (3.1), we can further decompose

$$
\mathrm{I}\le|\lambda_1|\Big(\int_{2R}|\mathcal{M}_{\vec{b}}(a_1)(x)|^q\,d\mu(x)\Big)^{1/q}+|\lambda_2|\Big(\int_{2R}|\mathcal{M}_{\vec{b}}(a_2)(x)|^q\,d\mu(x)\Big)^{1/q}=\mathrm{I}_1+\mathrm{I}_2\,.
$$

To estimate I_1, we write,

$$
\begin{aligned}
\mathrm{I}_1&\le|\lambda_1|\Big(\int_{2Q_1}|\mathcal{M}_{\vec{b}}(a_1)(x)|^q\,d\mu(x)\Big)^{1/q}+|\lambda_1|\Big(\int_{2R\setminus 2Q_1}|\mathcal{M}_{\vec{b}}(a_1)(x)|^q\,d\mu(x)\Big)^{1/q}\\
&=\mathrm{I}_{11}+\mathrm{I}_{12}\,.
\end{aligned}
$$

Choose p_1 and q_1 such that $1<p_1<n/\beta$ and $1/q_1=1/p_1-\beta/n$. It is obvious that $1<q<q_1$. The Hölder inequality, the fact that $S_{Q_1,R}\ge 1$, and the $(L^{p_1}(\mu),L^{q_1}(\mu))$-boundedness of $\mathcal{M}_{\vec{b}}$ by Theorem 2.1 in Section 2 tell us that

$$
\begin{aligned}
\mathrm{I}_{11}&\le|\lambda_1|\Big[\int_{2Q_1}|\mathcal{M}_{\vec{b}}(a_1)(x)|^{q_1}\,d\mu(x)\Big]^{1/q_1}\mu(2Q_1)^{1/q-1/q_1}\\
&\le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}}|\lambda_1|\|a_1\|_{L^{p_1}(\mu)}\mu(2Q_1)^{1/q-1/q_1}\\
&\le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}}|\lambda_1|.
\end{aligned}
$$

Denote $S_{2Q_1,2R}$ simply by N_1. Invoking the fact that $\|a_1\|_{L^\infty(\mu)}\le[\mu(4Q_1)S_{Q_1,R}]^{-1}$,

https://doi.org/10.4153/CMB-2011-139-1 Published online by Cambridge University Press

we have

$$\mathrm{I}_{12} \le C|\lambda_1|\Big\{ \sum_{k=1}^{N_1+1} \int_{2^{k+1}Q_1 \setminus 2^k Q_1} \Big[\int_0^\infty \Big| \int_{|x-y|\le t} \frac{\prod_{i=1}^m [b_i(x)-b_i(y)]}{|x-y|^{n-1}} a_1(y)\, d\mu(y) \Big|^2 \frac{dt}{t^3} \Big]^{q/2} d\mu(x) \Big\}^{1/q}$$

$$\le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} |\lambda_1| \Big\{ \sum_{k=1}^{N_1+1} \ell(2^k Q_1)^{q(\beta-n)} \int_{2^{k+1}Q_1\setminus 2^k Q_1} \Big[\int_{Q_1} |a_1(y)|\, d\mu(y) \Big]^q d\mu(x) \Big\}^{1/q}$$

$$\le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} |\lambda_1| \Big\{ \sum_{k=1}^{N_1+1} \ell(2^k Q_1)^{q(\beta-n)} \mu(4Q_1)^{-q} S_{Q_1,R}^{-q}\, \mu(2^{(k+1)}Q_1)\mu(Q_1)^q \Big\}^{1/q}$$

$$\le C\|\vec{b}\|_{Lip_{(\beta)}} |\lambda_1|.$$

Here we use the fact that

$$\sum_{k=2}^{N_1+2} \frac{\mu(2^k Q)}{l(2^k Q)^n} \le C S_{Q_1,R};$$

see [19, 20]. The estimates for I_{11} and I_{12} give the desired one for I_1. An argument similar to the estimate for I_1 tells us that $\mathrm{I}_2 \le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}}|\lambda_2|$. Combining the estimates for I_1 and I_2 yields the desired estimate for I.

For $i = 1, 2$, $y \in Q_i \subset R$, $x \in \mathbb{R}^d \setminus (2R)$, we have

$$|x-y| \sim |x-x_R| \sim |x-x_R| + 2\ell(R).$$

By the Minkowski inequality, we have

$$\mathrm{II} \le C\Big\{ \int_{\mathbb{R}^d\setminus(2R)} \Big[\int_{\mathbb{R}^d} \Big(\int_{|x-y|}^{|x-x_R|+2\ell(R)} \frac{dt}{t^3} \Big)^{1/2} \times \frac{|h(y)|}{|x-y|^{n-1}} \prod_{i=1}^m |b_i(x)-b_i(y)|\, d\mu(y) \Big]^q d\mu(x) \Big\}^{1/q}$$

$$\le C \int_R \Big\{ \int_{\mathbb{R}^d\setminus(2R)} \Big(\frac{\ell(R)^{1/2}}{|x-y|^{3/2}} \frac{|h(y)|}{|x-y|^{n-1}} \prod_{i=1}^m |b_i(x)-b_i(y)| \Big)^q d\mu(x) \Big\}^{1/q} d\mu(y)$$

$$\le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} \Big(\sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^1(\mu)} \Big) \Big\{ \sum_{k=1}^\infty \ell(R)^{1/2} \ell(2^k R)^{-n+\beta-1/2} \mu(2^{k+1}R)^{1/q} \Big\}$$

$$\le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} \Big(\sum_{j=1}^2 |\lambda_j| \Big).$$

Now we turn our attention to the estimate for III. For $1 \le i \le m$, we denote by C_i^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(i)\}$ of $\{1, 2, \dots, m\}$ with i different elements. For any $\sigma \in C_i^m$, the complementary sequence σ' is given by $\sigma' = \{1, 2, \dots, m\} \backslash \sigma$. For any $\sigma = \{\sigma(1), \sigma(2), \dots, \sigma(i)\} \in C_i^m$, set

$$\beta_\sigma = \beta_{\sigma(1)} + \cdots + \beta_{\sigma(i)} \quad \text{and} \quad \beta_{\sigma'} = \beta - \beta_\sigma.$$

For $1 \le i \le m$, all $\sigma \in C_i^m$, all $y \in \mathbb{R}^d$ and all cubes R, write

$$[b(y) - m_R(b)]_\sigma = [b_{\sigma(1)}(y) - m_R(b_{\sigma(1)})] \cdots [b_{\sigma(i)}(y) - m_R(b_{\sigma(i)})].$$

With the aid of the formula

$$\prod_{i=1}^m [b_i(x) - b_i(y)] = \sum_{i=0}^m \sum_{\sigma \in C_i^m} [b(x) - m_R(b)]_\sigma [m_R(b) - b(y)]_{\sigma'},$$

and for any $y \in R$, we have $t \ge |x - x_R| + 2\ell(R) \ge |x - x_R| + |y - x_R| \ge |x - y|$. So by the fact that $\int_R h(x)\, d\mu(x) = 0$, we obtain

$$\begin{aligned}
\mathrm{III} \le \Big\{ \int_{\mathbb{R}^d \backslash (2R)} \Big| \int_R K(x, y) \prod_{i=1}^m [b_i(x) - b_i(y)] h(y)\, d\mu(y) \\
\times \Big(\int_{|x - x_R| + 2\ell(R)}^\infty \frac{dt}{t^3} \Big)^{1/2} \Big|^q d\mu(x) \Big\}^{1/q} \\
\le C \Big\{ \int_{\mathbb{R}^d \backslash (2R)} \Big| \int_R \frac{K(x, y) h(y)}{|x - x_R| + 2\ell(R)} \prod_{j=1}^m [b_j(x) - m_R(b_j)]\, d\mu(y) \Big|^q d\mu(x) \Big\}^{1/q} \\
+ C \Big\{ \int_{\mathbb{R}^d \backslash (2R)} \Big| \int_R \frac{K(x, y) h(y)}{|x - x_R| + 2\ell(R)} \\
\times \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} [b(x) - m_R(b)]_{\sigma'} [m_R(b) - b(y)]_\sigma\, d\mu(y) \Big|^q d\mu(x) \Big\}^{1/q} \\
+ C \Big\{ \int_{\mathbb{R}^d \backslash (2R)} \Big| \int_R \frac{K(x, y) h(y)}{|x - x_R| + 2\ell(R)} \prod_{j=1}^m [m_R(b_j) - b_j(y)]\, d\mu(y) \Big|^q d\mu(x) \Big\}^{1/q} \\
= C\{\mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3\}.
\end{aligned}$$

For III_1, by the Minkowski inequality, the vanishing condition of h and (3.2), we have

$$\begin{aligned}
\mathrm{III}_1 &= \Big\{ \int_{\mathbb{R}^d\setminus(2R)} \Big| \prod_{j=1}^m [b_j(x) - m_R(b_j)] \int_R \frac{K(x,y)-K(x,x_R)}{|x-x_R|+2\ell(R)} h(y)\,d\mu(y) \Big|^q d\mu(x) \Big\}^{1/q} \\
&\le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} \\
&\quad \times \int_R |h(y)| \sum_{k=1}^\infty \Big(\int_{2^{k+1}R\setminus 2^kR} \Big[\ell(2^kR)^\beta \frac{|K(x,y)-K(x,x_R)|}{|x-y|} \Big]^q d\mu(x) \Big)^{1/q} d\mu(y) \\
&\le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} \sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

Here we use the fact that $1/q = 1 - \beta/n$ and $0 < \varepsilon \le 1$.

Similar to the estimate of III_1, we obtain

$$\begin{aligned}
\mathrm{III}_2 &\le C\Big\{ \int_{\mathbb{R}^d\setminus(2R)} \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \Big| \int_R \frac{K(x,y)h(y)}{|x-x_R|+2\ell(R)} \\
&\qquad \times [b(x)-m_R(b)]_{\sigma'} [m_R(b)-b(y)]_\sigma \,d\mu(y) \Big|^q d\mu(x) \Big\}^{1/q} \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \int_R \sum_{k=1}^\infty \|\vec{b}\|_{\mathrm{Lip}_{(\beta_\sigma)}} \ell(R)^{\beta_\sigma} \\
&\quad \times \Big(\int_{2^{k+1}R\setminus 2^kR} \Big[\frac{|K(x,y)-K(x,x_R)|}{|x-y|} |b(x)-m_R(b)|_{\sigma'} \Big]^q d\mu(x) \Big)^{1/q} |h(y)|\,d\mu(y) \\
&\le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} \sum_{j=1}^2 |\lambda_j| \|a_j\|_{L^1(\mu)} \le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} \sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

Let us now estimate III_3. Note that for any $y \in R, x \in \mathbb{R}^d\setminus 2R$, we have $|x-y| \sim |x-x_R| + 2\ell(R)$, so by the Minkowski inequality,

$$\begin{aligned}
\mathrm{III}_3 &\le \int_R \sum_{k=1}^\infty \Big(\int_{2^{k+1}R\setminus 2^kR} \Big[\frac{|K(x,y)|}{|x-y|} \prod_{j=1}^m |m_R(b_j) - b_j(y)| \Big]^q d\mu(x) \Big)^{1/q} |h(y)|\,d\mu(y) \\
&\le C \int_R \sum_{k=1}^\infty \|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} \ell(R)^\beta \ell(2^kR)^{-n} \mu(2^{k+1}R)^{1/q} \sum_{j=1}^2 |\lambda_j| |a_j(y)|\,d\mu(y) \\
&\le C\|\vec{b}\|_{\mathrm{Lip}_{(\beta)}} \sum_{j=1}^2 |\lambda_j|.
\end{aligned}$$

So III $\leq C\|\vec{b}\|_{Lip_{(\beta)}} \sum_{j=1}^{2} |\lambda_j|$. Combining the estimates for I, II, and III yields that $\|\mathcal{M}_{\vec{b}}(h)\|_{L^q(\mu)} \leq C|h|_{H^{1,\infty}_{\mathrm{atb}}(\mu)}$. ∎

Acknowledgement We thank the editors and anonymous referees for thoughtful comments that helped us improve the presentation of this paper. The first author would like to thank Professor Li Liang for invaluable discussions and suggestions.

References

[1] A. Al-Salman, H. Al-Qassem, L. C. Cheng, and Y. Pan, *L^p bounds for the function of Marcinkiewicz.* Math. Res. Lett. **9**(2002), no. 5-6, 697–700.

[2] Y. Ding, S. Lu, and Q. Xue, *Marcinkiewicz integral on Hardy spaces.* Integral Equations Operator Theory, **42**(2002), no. 2, 174–182. http://dx.doi.org/10.1007/BF01275514

[3] Y. Ding, S. Lu, and P. Zhang, *Weighted weak type estimates for commutators of the Marcinkiewicz integrals.* Sci. China Ser. A., **47**(2004), no. 1, 83–95. http://dx.doi.org/10.1360/03ys0084

[4] D. Fan and S. Sato, *Weak type $(1,1)$ estimates for Marcinkiewicz integrals with rough kernels.* Tohoku Math. J. **53**(2001), no. 2, 265–284. http://dx.doi.org/10.2748/tmj/1178207481

[5] J. García-Cuerva and A. E. Gatto, *Boundedness properties of fractional integral operators associated to non-doubling measures.* Studia Math. **162**(2004), no. 3, 245–261 http://dx.doi.org/10.4064/sm162-3-5

[6] ———, *Lipschitz spaces and Calderón-Zygmund operators associated to non-doubling measures.* Publ. Mat. **49**(2005), no. 2, 285–296.

[7] G. Hu, Y. Meng, and D. Yang, *Multilinear commutators of singular integrals with non doubling measures* Integral Equations Operator Theory **51**(2005), no. 2, 235–255. http://dx.doi.org/10.1007/s00020-003-1251-y

[8] G. Hu and D. Yan, *On the commutator of the Marcinkiewicz integral.* J. Math. Anal. Appl. **283**(2003), no. 2, 351–361 (2003) http://dx.doi.org/10.1016/S0022-247X(02)00498-5

[9] G. Hu, H. Lin, and D. Yang, *Marcinkiewicz integrals with non-doubling measures.* Integral Equations Operator Theory, **58**(2007), no. 2, 205–238, http://dx.doi.org/10.1007/s00020-007-1481-5

[10] L. Li and Y.-S. Jiang, *Estimates for maximal multilinear commutators on non-homogeneous spaces.* J. Math. Anal. Appl. **355**(2009), no. 1, 243–257. http://dx.doi.org/10.1016/j.jmaa.2009.01.022

[11] M. Lorente, M. S. Riveros, and A. de la Torre, *Weighted estimates for singular integral operators satisfying Hörmander's conditions of Young type.* J. Fourier Anal. Appl. **11**(2005), no. 5, 497–509. http://dx.doi.org/10.1007/s00041-005-4039-4

[12] S. Lu, Y. Ding, and D. Yan, *Singular Integral and Related Topics.* World Scientific Publishing Company, Hackensak, NJ, 2007.

[13] J. Marcinkiewicz, *Sur quelques intégrales du type de Dini.* Ann. Soc. Polon. Math. **17**(1938), 42–50.

[14] Y. Meng and D. Yang, *Boundedness of commutators with Lipschitz functions in non-homogeneous spaces.* Taiwanese J Math. **10**(2006), no. 6, 1443–1464.

[15] H. Mo and S. Lu, *Boundedness of generalized higher commutators of Marcinkiewicz integrals.* Acta Math. Sci. Ser. B Engl. Ed. **27**(2007) no. 4, 852–866.

[16] N. Sakamoto and K. Yabuta, *Boundedness of Marcinkiewicz functions.* Studia. Math. **135**(1999), no. 2, 103–142.

[17] E. M. Stein, *On the functions of Littlewood–Paley, Lusin, and Marcinkiewicz.* Trans. Amer. Math. Soc. **88**(1958), 430–466. http://dx.doi.org/10.1090/S0002-9947-1958-0112932-2

[18] A. Torchinsky and S. Wang, *A note on the Marcinkiewicz integral.* Colloq. Math., **60/61**(1990), no. 1, 235–243.

[19] X. Tolsa, *BMO, H^1 and Calderón-Zygmund operators for non doubling measures.* Math. Ann. **319**(2001), no. 1, 89–149. http://dx.doi.org/10.1007/PL00004432

[20] ———, *The space H^1 for nondoubling measures in terms of a grand maximal operator.* Trans. Amer. Math. Soc. **355**(2003), no. 1, 315–348. http://dx.doi.org/10.1090/S0002-9947-02-03131-8

[21] ———, *Littlewood-Paley theory and the $T(1)$ theorem with non-doubling measures.* Adv. Math. **164**(2001), no. 1, 57–116. http://dx.doi.org/10.1006/aima.2001.2011

[22] ———, *Painlevé's problem and the semiadditivity of analytic capacity.* Acta Math. **190**(2003), no. 1, 105–149. http://dx.doi.org/10.1007/BF02393237

[23] H. Wu, *On Marcinkiewicz integral operators with rough kernels.* Integral Equations Operator Theory **52**(2005), no. 2, 285–298. http://dx.doi.org/10.1007/s00020-004-1339-z

College of Mathematics and Econometrics, Hunan University, ChangSha, 410082, P.R. China

and

College of Mathematics and System Sciences, Xinjiang University, Urumqi, 830046, P.R. China
e-mail: zhoujiangshuxue@126.com

College of Mathematics and Econometrics, Hunan University, ChangSha, 410082, P.R. China

and

College of Mathematics and Engineering, Jiaxing University, Jiaxing, 314001, P.R. China
e-mail: blma3030@yahoo.com.cn