ON THE WARING-SIEGEL THEOREM

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1. Introduction. The Waring problem deals with the decomposition of integers into sums of kth powers. Consider

$$\nu = \lambda_1^k + \ldots + \lambda_s^k. \tag{1}$$

Waring conjectured and Hilbert [**2**] first proved the existence of s depending on k only, such that every rational integer could be expressed as a sum of s kth powers.

It was Hardy and Littlewood [**1**] using the now classical "circle" method who obtained a bound for s as a function of k and at the same time derived an asymptotic formula for the number of solutions of (1). They proved the following theorem:

Let $C(\nu)$ be the number of solutions of (1) and let $s \geqslant (k-2)2^k + 5$, then

$$C(\nu) = \sigma_{\nu,k,s}\, \nu^{-1+s/k} \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)} + o(\nu^{-1+s/k}),$$

where $\sigma_{\nu,k,s}$, the so-called singular series, is proved positive. It was then of interest to find the best possible result for the bound on s and at the same time to make the summands more general replacing in (1) kth powers by polynomial summands.

It was not until Hecke had developed the theory of theta functions in algebraic fields that Siegel [**8**; **9**; **10**; **11**] envisaged the possibility of extending the problem to algebraic fields. He proved a result (to be stated later) which corresponds to the above result of Hardy and Littlewood. It is our object to give two natural extensions of Siegel's theorem, namely to replace the kth powers by polynomial summands and to give a slight improvement of the lower bound for s. We rely for the most part on the methods of Siegel referring frequently as well to the methods of Landau [**7**] and Hua [**3**; **4**, **5**; **6**].

2. Notations, definitions, and formulation of the problem. Let F be an algebraic extension of the rationals of degree n and suppose that F is totally real, i.e. that all the conjugates of F are real. Let J be the ring of integers of F and suppose that $\omega_1, \ldots, \omega_n$ form a basis for J. If $\mathfrak{d}$ be the different (ramification ideal) for F and $(\omega_j^{(i)})^{-1} = (\rho_j^{(i)})$, then $\rho_1, \ldots, \rho_n$ is a basis for $\mathfrak{d}^{-1}$. The fundamental property of $\mathfrak{d}^{-1}$ used here is that if α is in $\mathfrak{d}^{-1}$, then $S(\lambda\alpha)$ is a rational integer for every λ in J. $S(\alpha)$ and $N(\alpha)$ denote as usual the trace and norm of α

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respectively. An inequality between elements of F means that it holds for all the corresponding conjugates, e.g. $\alpha < \beta$ means

$$\alpha^{(i)} < \beta^{(i)} \qquad i = 1, \ldots, n.$$

The Waring problem in algebraic fields has a somewhat different character from that in the rational field as shown by the following simple example. Let $R(\sqrt{d})$ be a quadratic field with $d \equiv 2, 3 \pmod 4$. The integers of such a field are of the form $a + b\sqrt{d}$, with a and b rational integers. The square of such an integer has even second coefficient; therefore an integer with odd second coefficient is never a sum of squares. This leads Siegel to the following construction; J_k is the ring generated by kth powers of elements of J. Finally let D be the discriminant of F. Consider now equation (1) as an equation in F with ν and λ_i totally positive. Let $B(\nu)$ be the number of solutions of (1) in F. Siegel [**10**] proved the following theorem:

If $s \geqslant kn(2^{k-1} + n) + 1$ then

$$B(\nu) = D^{\frac{1}{2}(1-s)}\, \sigma_{\nu,k,s} \left(\frac{\Gamma^s(1 + 1/k)}{\Gamma(s/k)}\right)^n N(\nu)^{-1+s/k} + o(N(\nu)^{-1+s/k}),$$

where $\sigma_{\nu,k,s} > 0$ if ν belongs to J_k and $\sigma_{\nu,k,s} = 0$ otherwise.

Consider now the polynomial

(2) $$\phi(\xi) = \alpha\xi^k + \alpha_1\xi^{k-1} + \ldots + \alpha_k$$

with α, $\alpha_i > 0$, $\nu > 0$ and

(3) $$\nu = \phi(\xi_1) + \ldots + \phi(\xi_s).$$

It is our object to prove the following

MAIN THEOREM. *Let $A(\nu)$ be the number of solutions of* (3) *and $s \geqslant n(2^k + n) + 1$, then*

$$A(\nu) = D^{\frac{1}{2}(1-s)}\, \sigma'_{\nu,k,s} \left(\frac{\Gamma^s(1 + 1/k)}{\Gamma(s/k)}\right)^n N(\alpha)^{-s/k} N(\nu)^{-1+s/k} + o(N(\nu)^{-1+s/k}).$$

We shall defer the discussion of the singular series to a further paper. It will be noted that the bound on s is a slight improvement over the previously known one but is far from the desirable bound which is independent of the degree of the field. For $k = 2$ Siegel has shown that such indeed is the case.

3. The generalized Farey dissection. Let X denote n-dimensional Euclidean space, then α in F is represented in X by the point $(\alpha^{(1)}, \ldots, \alpha^{(n)})$. If $(x_1, \ldots, x_n)$ be a point of X we put

(4) $$\xi^{(j)} = \rho_1^{(j)} x_1 + \ldots + \rho_n^{(j)} x_n.$$

For γ in F, denote by $\mathfrak{a}_\gamma$ (dropping the subscript when it is clear what is meant) the denominator of $\gamma\mathfrak{d}$. Let $k > 1$ (no restriction) and $a = (2^{k-1} + n)$; let T satisfy $T^{2a} > 2D^{1/n}$, and put $t = T^{1-a}$, $h = T^{k-a-1}$. The O and o processes refer to $T \to \infty$. For γ in F define B_γ as follows: B_γ is the set of points of X satisfying

$$(5) \qquad N(\max(h|\xi - \gamma|, t^{-1})) \leqslant N(\mathfrak{d})^{-1}.$$

The B_γ are generalizations of the so-called "major arcs" and evidently B_γ is empty if $N(\mathfrak{a}) > t^n$.

THEOREM 3.1. *If $\gamma \neq \delta$, then $B_\gamma \cap B_\delta$ is empty.*

Proof. Suppose $B_\gamma \cap B_\delta \neq 0$; put

$$\max(h|\xi - \gamma|, t^{-1}) = \sigma^{-1}, \qquad \max(h|\xi - \delta|, t^{-1}) = \tau^{-1};$$

then $\sigma \leqslant t$, $\tau \leqslant t$, $(N\mathfrak{d}_\gamma\, \mathfrak{d}_\delta) \leqslant N(\sigma\tau)$. Moreover

$$|\gamma - \delta| \leqslant |\xi - \gamma| + |\xi - \delta| \leqslant h^{-1}(\sigma^{-1} + \tau^{-1}) \leqslant h^{-1}(\sigma + \tau)\, \sigma^{-1}\, \tau^{-1} \leqslant 2t(h\sigma\tau)^{-1};$$

therefore $N((\gamma - \delta)\, \mathfrak{a}_\gamma\, \mathfrak{a}_\delta) \leqslant 2^n t^n h^{-n} = 2^n T^{n-na-kn-na+n} < D^{-1} < 1$. This is a contradiction since $(\gamma - \delta)\, \mathfrak{a}_\gamma\, \mathfrak{a}_\delta$ is an integral ideal.

THEOREM 3.2. *Let x be a point of X not in any B_γ, then there exist an integer α in F and a number β in $\mathfrak{d}^{-1}$ such that*

(i) $|\alpha\xi - \beta| < h^{-1}, \qquad 0 < |\alpha| < h,$

(ii) $\max(h|\alpha\xi - \beta|, |\alpha|) \geqslant D^{-\frac{1}{2}},$

(iii) $\max(|\alpha^{(1)}|, \ldots, |\alpha^{(n)}|) > t,$

(iv) $N(\alpha\beta\mathfrak{d})) \leqslant D^{\frac{1}{2}}.$

Proof. The proof may be found in Siegel [**10**]. This is the analogue of the usual theorem for the "minor arc" but is much more complicated. The proof is achieved by a multifold application of Minkowski's theorem on linear forms.

4. Analytical expression of $A(\nu)$. Let γ run over all numbers of F and let E denote the unit cube $0 \leqslant x_i < 1$ $(i = 1, \ldots, n)$. Let E_0 be the set of points of E which do not lie in any B_γ.

Choose now a complete system of modulo $\mathfrak{d}^{-1}$ incongruent numbers γ with $N(\mathfrak{a}_\gamma) \leqslant t^n$. Denote this set by Γ; henceforth the summation index γ will range over the set Γ. If G be any group of transformations of a space into itself, we say that two points x, y of the space are equivalent with respect to G if there is a transformation of G taking x into y. A subset M of the space is called a fundamental region if no two points of M are equivalent and if every point of the space is equivalent to a point of M. Two subsets are equivalent if every point of the one is equivalent to a point of the other and conversely. The set of translations $\xi \to \xi + \rho$, with ρ any number of $\mathfrak{d}^{-1}$ forms a group H; E is clearly a fundamental region with respect to H.

THEOREM 4.1. *The sum of all B_γ summed over the set Γ is under H equivalent to $E - E_0$.*

Proof. Let ξ be a point of $E - E_0$, then there exists a number β in $\mathfrak{d}^{-1}$ and a number γ in Γ such that $\xi - \beta = \eta$ lies in B_γ. The disjointness of the B_γ provides then the uniqueness of γ and β.

Let $\mathscr{T}$ be the set of integers λ of F satisfying

(6) $$0 < \lambda < T.$$

Let

(7) $$f(x) = \sum_{\lambda \in \mathscr{T}} e(S(\phi(\lambda)\xi)$$

where $e(x)$ is an abbreviation for $e^{2\pi ix}$. Consider the following integral:

$$I = \int_E f^s(x)\, e(-S(\nu\xi))\, dx = \int_E g(x)\, dx,$$

say. On writing $f^s(x)$ as a multiple sum and using the properties of $\mathfrak{d}^{-1}$ we conclude that $I = A(\nu)$. Since the integrand is invariant under the above group H of translations we get the fundamental equation

(8) $$A(\nu) = \sum_{\gamma} \int_{B_\gamma} g(x)\, dx + \int_{E_0} g(x)\, dx.$$

5. Estimate on the major arcs. Introduce a new variable $y = (y_1, \ldots, y_n)$ and set

$$\eta = (\omega_1 y_1 + \ldots + \omega_n y_n).$$

Let $Y(T)$ denote the domain in X where $0 < \eta < T$. Suppose furthermore that $(\alpha\gamma, \alpha_1\gamma, \ldots, \alpha_k\gamma) = \mathfrak{b}$ and let $\mathfrak{b}\mathfrak{d}$ have denominator $\mathfrak{a}$.

THEOREM 5.1. *If*

$$G(\gamma) = N(\mathfrak{a})^{-1} \sum_{\mu \bmod \mathfrak{a}} e(S(\phi(\mu)\,\gamma)),$$

then $G(\gamma) = O(N(\mathfrak{a})^{\epsilon - 1/k})$.

Proof. The proof of this result may be found in Hua [**5**].

THEOREM 5.2. *Let*

$$h(x) = \sum_{\lambda+\mu \in \mathscr{T}} e(S(\alpha(\lambda + \mu)^k \zeta)), \qquad \mathfrak{a}|\lambda,$$

where $\zeta = \xi - \gamma$, *then*

$$h(x) = N(\mathfrak{a})^{-1} \int_{Y(T)} e(S(\alpha(\eta + \mu)^k \zeta))\, dy + N(\mathfrak{a})^{-1} O(T^{n-a}).$$

Proof. The proof is almost identical with the corresponding result of Siegel; only the slightest modification is necessary.

THEOREM 5.3. *Let*

$$b(x) = \sum_{\lambda+\mu \in \mathscr{T}} e(S(\phi(\lambda + \mu)\zeta)), \qquad \mathfrak{a}|\lambda,$$

then

$$b(x) = h(x) + N(\mathfrak{a})^{-1}\, O(T^{n-a}).$$

Proof. Since $\lambda + \mu \in \mathscr{T}$, then

$$\phi(\lambda + \mu)\zeta - \alpha(\lambda + \mu)^k \zeta = \zeta\, O(T^{k-1}).$$

Therefore,

$$
\begin{aligned}
b(x) &= h(x) + O\Big(\sum_{\lambda+\mu \in \mathcal{T}} S(\phi(\lambda+\mu)\zeta - \alpha(\lambda+\mu)^k\zeta\Big) && (\mathfrak{a}|\lambda)\\
&= h(x) + O\Big(\sum_{\lambda+\mu\in\mathcal{T}} S(|\zeta|\, T^{k-1}\Big) && (\mathfrak{a}|\lambda)\\
&= h(x) + O(T^{k+n-1})\, N(\mathfrak{a})^{-1}\, h^{-1}\, N(\mathfrak{a})^{-1/n}\\
&= h(x) + O(T^{n-a})\, N(\mathfrak{a})^{-1-1/n}.
\end{aligned}
$$

THEOREM 5.4.

$$f(x) = G(\gamma) \int_{Y(T)} e(S(\alpha\,\eta^k\zeta))\,dy + O(T^{n-a}).$$

Proof. We have

$$
\begin{aligned}
f(x) &= \sum_{\lambda\in\mathcal{T}} e(S(\phi(\lambda)(\zeta+\gamma)))\\
&= \sum_{\mu \bmod \mathfrak{a}} e(S(\phi(\mu)\gamma)) \sum_{\lambda+\mu\in\mathcal{T}} e(S(\phi(\lambda+\mu)\zeta)) \qquad (\mathfrak{a}|\lambda)\\
&= \sum_{\mu \bmod \mathfrak{a}} e(S(\phi(\mu)\gamma))\Big\{N(\mathfrak{a})^{-1}\int_{Y(T)} e(S(\alpha\,\eta^k\zeta))\,dy + N(\mathfrak{a})^{-1}O(T^{n-a})\Big\}\\
&= N(\mathfrak{a})^{-1} \sum_{\mu \bmod \mathfrak{a}} e(S(\phi(\mu)\gamma)) \int_{Y(T)} e(S(\alpha\,\eta^k\zeta))\,dy + O(T^{n-a})\\
&= G(\gamma)\int_{Y(T)} e(S(\alpha\,\eta^k\zeta))\,dy + O(\iota^{n-a}).
\end{aligned}
$$

by Theorems 5.1, 5.2, and 5.3.

6. Estimate on the minor arc. We follow again in this section the procedure of Siegel [**10**; **11**] based on Weyl's method for estimating trigonometric sums. The presence of a polynomial in the exponent leads to no essential difficulty.

THEOREM 6.1. *Let*

$$
\begin{aligned}
&\psi(\lambda) = S(\phi(\lambda)\xi), \quad \psi(\lambda;\lambda_1) = \psi(\lambda+\lambda_1) - \psi(\lambda),\\
&\qquad \psi(\lambda;\lambda_1,\ldots,\lambda_m) = \psi(\lambda+\lambda_m;\lambda_1,\ldots,\lambda_{m-1}) - \psi(\lambda;\lambda_1,\ldots,\lambda_{m-1})
\end{aligned}
$$

and A_m be the number of systems of integers $\lambda_1, \ldots, \lambda_m$ such that the 2^m simultaneous conditions

(9) $\quad \lambda + \lambda_{p_1} + \ldots + \lambda_{p_g} \in \mathcal{T}(1 \leqslant p_1 < p_2 < \ldots p_g < m; g = 0, \ldots, m-1)$

have at least one solution λ. Then

$$|f(x)|^{2^m} \leqslant A_1^{2^{m-2}} \ldots A_{m-2}^2 A_{m-1} \sum_{\lambda_1,\ldots,\lambda_{m-1}} \Big|\sum_{\lambda} e(\psi(\lambda;\lambda_1,\ldots,\lambda_{m-1}))\Big|$$

for λ_i satisfying (9) *and* $m = 1, \ldots, k-1$.

Proof. The proof is by induction on m.

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THEOREM 6.2.

$$|f(x)|^{2^{k-1}} = O(T^{n(2^{k-1}-k)}) \sum_{\lambda_1, \dots, \lambda_k} e(k!\, S(\alpha\lambda\lambda_1 \dots \lambda_{k-1}\xi)).$$

Proof. We first observe that $A_m = O(T^{nm})$; moreover

$$\sum_{m=1}^{k-1} m\, 2^{k-m-1} = 2^k - k - 1.$$

Since $\psi(\lambda; \lambda_1, \dots, \lambda_{k-1}) = S(k!\, \alpha\lambda\lambda_1 \dots \lambda_{k-1}\, \xi) + \psi(0; \lambda_1, \dots, \lambda_{k-1})$, the result follows.

THEOREM 6.3. *If x is a point of E_0, then*

$$f(x) = O(T^{n-(2^{k-1}+n)^{-1}+\epsilon}) = O(T^{n-a+\epsilon}).$$

Proof. Let

(11) $$\mu = \alpha k!\, \lambda_1 \dots \lambda_{k-1}$$

then we deduce

(12) $$\begin{aligned} u &= \sum e(S(\lambda\mu\xi)) \\ &= \min\,(T, |e(S(\omega_1\, \mu\xi)) - 1|^{-1}, \dots, |e(S(\omega_n\mu\xi)) - 1|^{-1})\, O(T^{n-1}). \end{aligned}$$

Let

$$S(\omega_j\, \alpha\mu\xi) = a_j + d_j \qquad (j = 1, \dots, n)$$

with rational integers a_j and $-\frac{1}{2} \leqslant d_j < \frac{1}{2}$, and define

$$\sum_{j=1}^{n} a_j\, \rho_j = \theta, \quad \sum_{j=1}^{n} d_j\, \rho_j = \tau.$$

We have $\theta \in \mathfrak{d}^{-1}$ and $e(S(\omega_j\alpha\mu\xi)) = e(d_j)$. Also $S(\omega_j\, \tau) = d_j$ and $\theta + \tau = \mu\xi$. Determine now numbers η and β with η an integer and β in $\mathfrak{d}^{-1}$ satisfying the condition of Theorem 3.2. There is an index $b \leqslant n$ such that $|\eta^{(b)}| > t$; let v denote the number of indices p satisfying $|\eta^{(p)}| < D^{-\frac{1}{2}}$, then $0 \leqslant v < n - 1$ and $p \neq b$. Let

(13) $$q(\mu) = \min\,(T, |\tau^{(b)}|^{-1}),$$

then from (12), we deduce

$$u = O(T^{n-1}) \min\,(T, |\tau^{(b)}|^{-1}) = O(T^{n-1})\, q(\mu).$$

For given $\mu \neq 0$, the number of solutions of (11) subject to the condition $|\lambda_m| < 2T$ $(m = 1, \dots, k-1)$ is $O(T^{\epsilon})$ for arbitrarily small ϵ. On the other hand if $\mu = 0$, the number of solutions is $O(T^{n(k-2)})$. We conclude therefore

$$|f(x)|^{2^{k-1}} = O(T^{n(2^{k-1}-1)}) + O(T^{\epsilon+n(2^{k-1}-k+1)-1}) \sum_{\mu} q(\mu)$$

where μ runs over all integers satisfying

(14) $$|\mu| < 2^k k!\, T^{k-1}.$$

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With Siegel, we proceed to define $z_i = \tau^{(i)}$ and let $g_1, \ldots, g_n$ be rational integers; $W = W(g_1, \ldots, g_n)$ denotes the number of integers μ satisfying (14) and the further conditions

(15) $$g_i \leqslant 2D^{1/n} z_i \max(|\eta^{(i)}|, \quad D^{-\frac{1}{2}}) < g_{i+1} \qquad (i = 1, \ldots, n).$$

Let μ_0 be a fixed one of these μ and set $\mu_0 \xi = \theta_0 + \tau_0$, $\eta\xi - \beta = \delta$. We have

$$\delta(\mu - \mu_0) - \eta(\tau - \tau_0) = \eta(\theta - \theta_0) - \beta(\mu - \mu_0) = \kappa.$$

By observing that κ lies in $\mathfrak{d}^{-1}$, we deduce that $\kappa = 0$. It follows therefore that $\eta | \beta\mathfrak{d}(\mu - \mu_0)$, and since $N((\eta, \beta\mathfrak{d})) \leqslant D^{\frac{1}{2}}$ then $\eta | c(\mu - \mu_0)$ where c is a positive rational integer depending only on the field F. It follows that

$$(\mu - \mu_0)\,\eta^{-1} = \eta^{-1}\, O(T^{k-1})$$

and hence that

$$(\mu^{(p)} - \mu_0^{(p)})\,\eta^{(p)^{-1}} = (\tau^{(p)} - \tau_0^{(p)})\,\delta^{(p)^{-1}} = O(h).$$

Consequently the number of differences $\mu - \mu_0$ is

$$1 + O(h^v) \prod (|\eta^{(i)}|^{-1}\, T^{k-1}), \qquad i \neq p.$$

Therefore

(16) $$W = O(1) + O(T^{n(k-1)+av}) \prod |\eta^{(i)}|^{-1}, \qquad i \neq p.$$

If $W > 0$, then $g_p = O(1)$ and $g_1 = O(\eta^{(i)})$ if $i \neq p$. If then g_b is fixed, the number of systems $g_1, \ldots, g_n$ with $W > 0$ is

$$O(\prod \eta^{(i)}), \qquad i \neq p, b.$$

Then the number of integers μ in F satisfying (14) and the single condition

(17) $$g < 2D^{1/n} z_b |\eta^{(b)}| < g + 1$$

has value

$$W_g = \sum_{g_i} W(g_1, \ldots, g_n) = T^{(n-1)(k+a-1)}\, O(1 + T^{k-1}|\eta^{(b)}|^{-1}), \qquad i \neq b.$$

On the other hand,

$$L = \sum_{0 \leqslant g < |\eta^{(b)}|} \min(T, g^{-1}\eta^{(b)}) = T^{k+a-1}\, O(\log T).$$

Furthermore, $LT^{k-1}|\eta^{(b)}|^{-1} = O(T^{k+a-1})$, therefore

$$\begin{aligned}\sum_{\mu} q(\mu) &= \sum_{g} W_g\, O(\min(T, |g|^{-1}|\eta^{(b)}|, |g+1|^{-1}|\eta^{(b)}|))\\ &= L\, T^{(n-1)(k+a-1)}\, O(1 + T^{k-1}|\eta^{(b)}|^{-1})\\ &= O(T^{n(k+a-1)}).\end{aligned}$$

The estimate for $f(x)$ now follows.

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7. Further estimates.

THEOREM 7.1. *Let*

$$\Phi(\xi) = \Phi_T(\xi) = \left(\int_{Y(T)} e(S(\alpha\eta^k\xi))\, dy \right)^s;$$

then

$$\sum_\gamma \int_{B_\gamma} g(x)\, dx = \sum_\gamma G^s(\gamma)\, e(-S(\nu\gamma)) \int_B \Phi(\zeta)\, e(-S(\nu\zeta))\, dx + O(T^{n(s-k)}).$$

Proof. Let $u_i = T^{-1}\eta^{(i)}$ $(i = 1, \ldots, n)$ and $T^k\zeta = \tau$. Then

$$(18) \qquad \int_{Y(T)} e(S(\alpha\eta^k\zeta))\, dy = D^{-\frac{1}{2}}\, T^n N\left(\int_0^1 e(\tau^{(i)}\alpha^{(i)} u^k)\, du \right).$$

Also

$$(19) \qquad \int_0^1 e(\alpha^{(i)}\tau^{(i)} u^k)\, du = O(\min\,(1, |\tau^{(i)}|^{-1/k}))$$

and if x is a point of B_γ, then by Theorem 5.4,

$$(20) \qquad f^s(x) - \left(G(\gamma) \int_{Y(T)} e(S(\alpha\eta^k\zeta))\, dy \right)^s$$
$$= O(T^{n-a}) \max \left(f(x),\, G(\gamma) \int_{Y(T)} e(S(\alpha\eta^k\zeta))\, dy \right)^{s-1}.$$

By (19) and Theorem 5.1, we obtain

$$(21) \qquad f^s(x) = G^s(\gamma)\left(\int_{Y(T)} e(\alpha\eta^k\zeta)\, dy \right)^s$$
$$+ O(T^{ns-a})\, N(\mathfrak{a}^{\epsilon-(s-1)/k})\, N(\min\,(1, |\tau^{-1/k}|)^{s-1}).$$

On the other hand,

$$(22) \qquad \int_{B_\gamma} N(\min\,(1, |\tau^{-1/k}|)^{s-1})\, dx = O(T^{-kn}),$$

and since by partial summation $\sum_\gamma N(\mathfrak{a})^{-(s-1)/k} = O(1)$, we have

$$\sum_\gamma \int_{B_\gamma} g(x)\, dx = \sum_\gamma \int_{B_\gamma} f^s(x)\, e(-S(\nu\xi))\, dx$$
$$= \sum_\gamma G^s(\gamma)\, e(-S(\nu\gamma)) \int_{B_\gamma} \Phi(\zeta)\, e(-S(\nu\zeta))\, dx$$
$$+ \sum N(\mathfrak{a})^{-(s-1)/k} O(T^{sn-a})\, O(T^{-kn})$$
$$= \sum_\gamma G^s(\gamma)\, e(-S(\nu\gamma)) \int_{B_\gamma} \Phi(\zeta)\, e(-S(\nu\zeta))\, dx + o(T^{n(s-k)}).$$

THEOREM 7.2.

$$\sum_\gamma \int_{B_\gamma} g(x)\, dx = \sum_\gamma G^s(\gamma)\, e(-S(\nu\gamma)) \int_X \Phi(\xi)\, e(-S(\nu\xi))\, dx + o(T^{n(s-k)}).$$

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Proof. We replace here, it will be noted, B_γ by X. It is therefore sufficient to prove that

$$U = \sum_\gamma G^s(\gamma)\, e(-S(\nu\gamma)) \int_{X-B_\gamma} \Phi(\zeta)\, e(-S(\nu\zeta))\, dx = o(T^{n(s-k)}).$$

If x is a point of $X - B_\gamma$, then by (5) there is at least one index i such that $h|\zeta^{(i)}| > N(\mathfrak{a})^{-1/n}$. Therefore,

$$\begin{aligned}\int_{X-B_\gamma} \Phi(\zeta)\, e(-S(\nu\zeta))\, dx &= O(T^{ns}) \int_{X-B_\gamma} N(\min(1, \tau^{-s/k}))\, dx \\ &= O(T^{n(s-k)}) \int_{x_1 > tN(\mathfrak{a})^{-1/n}} x_1^{-s/k}\, dx_1 \\ &= O(T^{n(s-k)})(t\, N(\mathfrak{a})^{-1/n})^{1-s/k}.\end{aligned}$$

Consequently,

$$\begin{aligned}U &= O(T^{n(s-k)}) \sum_\gamma |G(\gamma)|^s\, t^{1-s/k}\, N(\mathfrak{a})^{-(1-s/k)/n} \\ &= O(T^{n(s-k)})\, t^{1-s/k} \sum_\gamma N(\mathfrak{a})^{-s/k-1/n+s/nk} \\ &= O(T^{n(s-k)})\, t^{1+n-s/k} \\ &= o(T^{n(s-k)}).\end{aligned}$$

THEOREM 7.3.

$$\int_E |f(x)|^{2^k}\, dx = O(T^{n(2^k-k)+\epsilon}).$$

Proof. This theorem was proved, for the rational field by Hua [**6**]; this is an extension to the present case. The proof proceeds by induction on k. For $k = 0$ the result is trivial, assume it true for $k - 1$. Then

$$\begin{aligned}\int_E |f(x)|^{2^k}\, dx &= \int_E |f(x)|^{2^{k-1}} |f(x)|^{2^{k-1}}\, dx \\ &= \int_E |f(x)|^{2^{k-1}} \Big\{ O(T^{n(2^{k-1}-1)}) \\ &\quad + O(T^{n(2^{k-1}-k)}) \sum_{\lambda_1} \dots \sum_{\lambda_{k-1}} \sum_{\lambda}{}^{*}\, e(S(k!\,\alpha\lambda_1 \dots \lambda_{k-1}\lambda\xi)) \Big\}\, dx,\end{aligned}$$

by Theorem 6.2, the asterisk indicating that the summation excludes the value 0 of $\lambda_1, \dots, \lambda_{k-1}, \lambda$.

By the inductive hypothesis however, we have

$$\begin{aligned}\int_E |f(x)|^{2^k}\, dx &= O(T^{n(2^{k-1}-1)}) T^{n(2^{k-1}-k+1)+\epsilon} \\ &\quad + O(T^{n(2^{k-1}-k)}) \int_E |f(x)|^{2^{k-1}} \sum_{\lambda_1} \dots \sum_{\lambda_{k-1}} \sum_{\lambda} e(S(k!\,\alpha\lambda_1 \dots \lambda_{k-1}\xi))\, dx \\ &= O(T^{n(2^{k-1}-k)+\epsilon}) - O(T^{n(2^{k-1}-k)}) \\ &\quad \cdot \int_E \Big\{ \sum_{\mu 1} \dots \sum_{\mu_{2^{k-1}}} e(S(\phi(\mu))) \sum_{\lambda_1} \dots \sum_{\lambda_{k-1}} \sum_{\lambda}{}^{*} e(S(\theta)) \Big\}\, dx\end{aligned}$$

where

$$\phi(\mu) = (\phi(\mu_1) - \phi(\mu_2) + \ldots - \phi(\mu_{2^{k-1}}))\xi$$

and

$$\theta = k!\,\alpha\lambda_1 \ldots \lambda_{k-1}\lambda\xi.$$

This follows by writing the square of the absolute value as the product of complex conjugates and noting therefore that

$$|f(x)|^{2^{k-1}} = \sum_{\mu 1} \ldots \sum_{\mu_{2^{k-1}}} e(S((\phi(\mu_1) - \phi(\mu_2) + \ldots - \phi(\mu_{2^{k-1}}))\xi))).$$

Therefore, using the properties of $\mathfrak{d}^{-1}$, we get

$$\int_E |f(x)|^{2^k} = O(T^{n(2^k-k)+\epsilon}) - O(T^{n(2^{k-1}-k)}).\, C,$$

where C is the number of solutions of the equation $\phi(\mu) = \theta$, the λ and μ being restricted by the conditions $|\lambda_i| < T$, $|\mu_i| < T$.

On the other hand, as in the rational case, it is proved that

$$C = O(T^{\epsilon}\, T^{n(2^{k-1})}).$$

We conclude therefore finally,

$$\int_E |f(x)|^{2^k} = O(T^{n(2^k-k)+\epsilon}) - O(T^{n(2^{k-1}-k)+n2^{k-1}+\epsilon})$$
$$= O(T^{n(2^k-k)+\epsilon}).$$

8. Proof of the asymptotic formula for $A(\nu)$. In the same way as in Siegel [**11**], we can prove the following

THEOREM 8.1.

$$I = \int_X e(-S(\nu\xi))\,\Phi_T(\xi)\,dx = D^{\frac{1}{2}(1-s)}\left(\frac{\Gamma^s(1+1/k)}{\Gamma(s/k)}\right)^n N(\alpha)^{-s/k}\,N(\nu)^{-1+s/k},$$

Again using Dirichlet's theorem on units, we could prove

THEOREM 8.2. *Let θ_0 be an algebraic integer, then there exists a totally positive unit η such that $\theta = \eta^k\theta_0$ fulfils the conditions*

$$c_1 N(\theta)^{1/n} \leqslant \theta \leqslant c_2 N(\theta)^{1/n}$$

with c_1 and c_2 real numbers.

We now show that the "singular series" converges.

THEOREM 8.3. *If γ runs over a complete system of modulo $\mathfrak{d}^{-1}$ incongruent numbers in F, then the "singular series"*

$$\sigma' = \sigma'_{\nu,k,s} = \sum_{\gamma} G^s(\gamma)\, e(-S(\nu\gamma))$$

is convergent for $s > 2k + 1$.

Proof. Suppose

$$H(\mathfrak{a}) = \sum G^s(\gamma)\, e(-S(\nu\gamma))$$

the summation being over a complete system of modulo $(\mathfrak{a}\mathfrak{d})^{-1}$ incongruent numbers γ such that the denominator of $\gamma\mathfrak{d}$ is $\mathfrak{a}$. Then

$$\sigma' = \sum_{\mathfrak{a}} \sum_{\gamma \bmod (\mathfrak{a}\mathfrak{d})^{-1}} G^s(\gamma)\, e(-S(\nu\gamma)) = \sum_{\mathfrak{a}} H(\mathfrak{a}).$$

Therefore by Theorem 5.1,

$$\sigma' = O(1) \sum_{\mathfrak{a}} |H(\mathfrak{a})| = O(1) \sum_{\mathfrak{a}} N(\mathfrak{a})^{1+\epsilon-s/k} = O(1).$$

COROLLARY.

$$\sigma' = \sum_{\gamma \epsilon \Gamma} G^s(\gamma)\, e(-S(\nu\gamma)) + o(1).$$

The proof of our main theorem is now merely a collection of the results established. It is clear that $A(\nu) = A(\nu\eta^k)$ where η is a unit. Put $N(\nu)^{1/n} = T^k$; then by Theorem 8.2, we may assume that

$$c_1 N(\nu)^{1/n} \leqslant \nu \leqslant c_2 N(\nu)^{1/n}.$$

By (8), we have

$$A(\nu) = \sum_{\gamma} \int_{B_\gamma} g(x)\, dx + \int_{E_0} g(x)\, dx.$$

By Theorems 7.2 and 8.1, we have,

$$\sum_{\gamma} \int_{B_\gamma} g(x)\, dx = \sigma' I + o(N(\nu)^{-1+s/k}).$$

On the other hand, using Theorems 6.2, and 7.3, we deduce

$$\begin{aligned}
\int_{E_0} g(x) &= O(1) \int_{E_0} |f(x)|^s\, dx \\
&= O(1) \int_{E_0} |f(x)|^{s-2^k+2^k}\, dx \\
&= O(T^{(n-(2^{k-1}+n)^{-1}+\epsilon)(s-2^k)}) \int_{E} |f(x)|^{2^k}\, dx \\
&= O(T^{(n-(2^{k-1}+n)^{-1}+\epsilon)(s-2^k)})\, O(T^{n(2^k-k)+\epsilon}) \\
&= o(N(\nu)^{-1+s/k}).
\end{aligned}$$

This completes the proof.

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References

1. G. H. Hardy and J. E. Littlewood, *Some problems of partitio numerorum* I: *A new solution of Waring's Problem*, Nachr. Ges. Wiss. Gottingen (1920), 33–54.
2. D. Hilbert, *Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl nter Potenzen*, Math. Ann., *67* (1909), 281–300.
3. L.-K. Hua, *On a generalized Waring problem*, Proc. London Math. Soc., *43* (1937), 161–182.
4. ———, *On a generalized Waring problem*, Chinese Math. Soc., *1* (1940), 23–61.
5. ———, *Exponential sums over algebraic fields*, Can. J. Math., *3* (1951), 44-51.
6. ———, *On Waring's problem*, Quarterly J. Math., *9* (1938), 199–201.
7. E. Landau, *Über die Winogradoffsche Behandlung des Waringsche Problem*, Math. Z., *31* (1930), 319–338.
8. C. L. Siegel, *Additive Theorie der Zahlkörper*, Math. Ann., *87* (1922), 1–35.
9. ———, *Additive Theorie der Zahlkörper*, Math. Ann., *88* (1923), 184–210.
10. ———, *Generalization of Waring's problem to algebraic number fields*, Amer. J. Math., *66* (1944), 122–136.
11. ———, *Sums of mth powers of algebraic integers*, Ann. Math., *46* (1945), 313–339.

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