Bull. Aust. Math. Soc. **94** (2016), 70–79
doi:10.1017/S0004972715001458

RESOLVABILITY OF MEASURABLE SPACES

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(Received 31 August 2015; accepted 16 October 2015; first published online 11 January 2016)

Abstract

We consider a special kind of structure resolvability and irresolvability for measurable spaces and discuss analogues of the criteria for topological resolvability and irresolvability.

2010 *Mathematics subject classification*: primary 28A05; secondary 54A10, 06E25.

Keywords and phrases: resolvability, irresolvability, algebra of sets, ideal of sets, Marczewski–Burstin representation.

1. Introduction

In 1943, Hewitt [8] introduced the concept of a resolvable space, that is, a topological space containing two disjoint dense subsets. More generally, adopting terminology introduced subsequently by Ceder [6], we say that a topological space (X, τ) is α-resolvable for a cardinal number α greater than one if there is a family of α-many pairwise disjoint dense subsets of X. According to this usage 'resolvable' coincides with '2-resolvable'. If a space is not resolvable, it is called irresolvable.

Generalising the concepts of resolvability and irresolvability, Jiménez and Malykhin [11] introduced the general notion of structure resolvability and structure irresolvability. They recalled some fundamental results of the theory of resolvable topological spaces and considered their analogues for structure resolvability. Resolvability or irresolvability of a measurable space is a special case of structure resolvability (or irresolvability) but hardly mentioned in [11]. This particular case is especially interesting because there is a nice criterion for resolvability of a topological space based only on properties of a special measurable space. In this paper we restrict our attention to resolvability or irresolvability of measurable spaces and focus on deeper aspects of this notion. The main results of [11] do not apply directly to the case studied here.

We discuss analogues of the criteria for topological resolvability or irresolvability for measurable spaces. This leads us to consider Marczewski–Burstin operations and to the notion of weak resolvability. The main results are presented in the diagram in the final part of the paper.

2. Notions of resolvability

For any topological space (X, τ) the family of all nowhere dense sets, denoted here by $\mathcal{ND}(\tau)$ or shortly $\mathcal{ND}$, is an ideal and the family of all sets with nowhere dense boundary, denoted by $\mathcal{NB}(\tau)$ or $\mathcal{NB}$, is an algebra. Evidently $\mathcal{NB} = \tau \vartriangle \mathcal{ND}$, that is, the smallest algebra generated by the family of sets $\{G \vartriangle N : G \in \tau \wedge N \in \mathcal{ND}\}$, and $\mathcal{ND} \subset \mathcal{NB}$. It is obvious that there may exist different topologies (X, τ_1) and (X, τ_2) with $\mathcal{ND}(\tau_1) = \mathcal{ND}(\tau_2)$ and $\mathcal{NB}(\tau_1) = \mathcal{NB}(\tau_2)$. Topologies with these properties are called similar and they have already been studied in the context of resolvability in [5, 9, 12].

For an arbitrary algebra $\mathcal{A}$ we denote by $\mathcal{H}(\mathcal{A})$ the maximal hereditary subfamily of $\mathcal{A}$, that is,

$$\mathcal{H}(\mathcal{A}) := \{A \in \mathcal{A} : \forall_{B \subset A} B \in \mathcal{A}\}.$$

It is easy to see that $\mathcal{H}(\mathcal{A})$ is the greatest ideal contained in $\mathcal{A}$.

We can now formulate the criterion mentioned above for resolvability of a space based on properties of the families $\mathcal{NB}$ and $\mathcal{ND}$.

Theorem 2.1 [5, 12]. *A topological space (X, τ) is resolvable if and only if $\mathcal{H}(\mathcal{NB}) = \mathcal{ND}$.*

Since resolvability of a topological space can be described by using only notions of a special algebra and a special ideal, it is interesting to consider a notion of resolvability for a set $X \neq \emptyset$ with an algebra $\mathcal{A} \subset 2^X$ and an ideal $\mathcal{I} \subset \mathcal{A}$. We assume here that $X \notin \mathcal{I}$.

We say that $A \subset X$ is *positive* if $A \in \mathcal{A} \backslash \mathcal{I}$, and we denote the family $\mathcal{A} \backslash \mathcal{I}$ by $\mathcal{I}^+$, but only in cases when it does not cause confusion about the algebra under consideration. We say that a set $A \subset X$ is $(\mathcal{A}, \mathcal{I})$-*dense* if it meets any positive set. The family of all $(\mathcal{A}, \mathcal{I})$-dense sets is denoted by $\mathcal{D}(\mathcal{A}, \mathcal{I})$. We say that a measurable space $(X, \mathcal{A}, \mathcal{I})$ is *resolvable* if X contains two disjoint $(\mathcal{A}, \mathcal{I})$-dense subsets, otherwise X is *irresolvable*.

For a family $\mathcal{F} \subset 2^X$ and a set $Y \subset X$ we write $\mathcal{F}_{|Y} := \{Y \cap F : F \in \mathcal{F}\}$. For a measurable space $(X, \mathcal{A}, \mathcal{I})$ and a set $Y \subset X$ we say that Y is resolvable if the space $(Y, \mathcal{A}_{|Y}, \mathcal{I}_{|Y})$ is resolvable. Observe that every set $I \in \mathcal{I}$ is resolvable, and that resolvability of $(X, \mathcal{A}, \mathcal{I})$ implies resolvability of every element of $\mathcal{A}$. We say that the space $(X, \mathcal{A}, \mathcal{I})$ is *strongly irresolvable* if every element of $\mathcal{I}^+$ is irresolvable.

It is an easy observation that if for two measurable structures $(X, \mathcal{A}_1, \mathcal{I}_1)$ and $(X, \mathcal{A}_2, \mathcal{I}_2)$ on X, we have the inclusion $\mathcal{I}_1^+ \subset \mathcal{I}_2^+$, then $\mathcal{D}(\mathcal{A}_2, \mathcal{I}_2) \subset \mathcal{D}(\mathcal{A}_1, \mathcal{I}_1)$. Thus, if $(X, \mathcal{A}_2, \mathcal{I}_2)$ is resolvable then $(X, \mathcal{A}_1, \mathcal{I}_1)$ is also resolvable.

The assumption of the previous observation can be weakened. It is enough to assume that $\mathcal{I}_2^+$ is coinitial to $\mathcal{I}_1^+$, which means that for every set $A \in \mathcal{I}_1^+$ there exists a set $B \in \mathcal{I}_2^+$ such that $B \subset A$. Indeed, if $G \in \mathcal{D}(\mathcal{A}_2, \mathcal{I}_2)$ and for any $A \in \mathcal{I}_1^+$ there exists a set $B \in \mathcal{I}_2^+$ such that $B \subset A$, then $G \cap A \neq \emptyset$, so $G \in \mathcal{D}(\mathcal{A}_1, \mathcal{I}_1)$.

The following example shows that in a measurable space $(X, \mathcal{A}, \mathcal{I})$ the union of a family of resolvable sets from $\mathcal{I}^+$ need not be resolvable.

EXAMPLE 2.2. Let $X = [0, 2]$. Let *Count* denote the σ-ideal of all countable subsets of X and $\mathcal{B}$ denote the σ-algebra of Borel sets. Let

$$\mathcal{A} = \{A \subset X : A \cap [0, 1] \in \mathcal{B}\}.$$

Then $\mathcal{A}$ forms an algebra of sets and $(X, \mathcal{A}, \mathrm{Count})$ is irresolvable, since for any subset of X this set or its complement contains an uncountable subset of $(1, 2]$ which is a positive set. However, X can be represented as a union of resolvable sets from Count^+ as follows: $X = \bigcup_{x\in(1,2]}[0, 1] \cup \{x\}$.

It is worth mentioning that in the topological case the union of an arbitrary family of resolvable subspaces of X is resolvable. This leads to a classical result of Hewitt [8]: every topological space has its unique Hewitt decomposition $X = F \cup G$, where F is closed and resolvable, G is hereditarily irresolvable and $F \cap G = \emptyset$. Of course it is possible that one of the sets F, G is empty.

It is not difficult to see that the families of dense sets in a topological sense and the families of $(\mathcal{NB}(\tau), \mathcal{ND}(\tau))$-dense sets coincide, so (X, τ) is resolvable if and only if $(X, \mathcal{NB}(\tau), \mathcal{ND}(\tau))$ is resolvable.

A triple $(X, \mathcal{A}, \mathcal{I})$ is called topological if there exists a topology τ on X such that $\mathcal{A} = \mathcal{NB}(\tau)$ and $\mathcal{I} = \mathcal{ND}(\tau)$. The really interesting questions arise for triples $(X, \mathcal{A}, \mathcal{I})$ which are not topological.

We note some consequences of being a topological triple. First we need some definitions. Let $\mathcal{I}$ be an ideal on X and $\mathcal{A}$ an algebra of subsets of X such that $\mathcal{I} \subset \mathcal{A}$.

- $(\mathcal{A}, \mathcal{I})$ is *Marczewski–Burstin inner representable (abbreviated MB)* if there exists a nonempty family $\mathcal{F} \subset \mathcal{A}$ of nonempty subsets of X such that $S(\mathcal{F}) = \mathcal{A}$ and $S_0(\mathcal{F}) = \mathcal{I}$, where

$$S(\mathcal{F}) := \{E \subset X : \forall_{A\in\mathcal{F}} \exists_{B\in\mathcal{F}}\ (B \subset A \cap E \vee B \subset A \backslash E)\}$$

  and

$$S_0(\mathcal{F}) := \{E \subset X : \forall_{A\in\mathcal{F}} \exists_{B\in\mathcal{F}}\ B \subset A \backslash E\}.$$

  (It is easy to see that if $(\mathcal{A}, \mathcal{I})$ is MB then $S(\mathcal{I}^+) = \mathcal{A}$ and $S_0(\mathcal{I}^+) = \mathcal{I}$ [1].)
- $\mathcal{I}$ is *small* in $\mathcal{A}$ if $\mathcal{H}(\mathcal{A}) \backslash \mathcal{I}$ is coinitial to $\mathcal{A} \backslash \mathcal{H}(\mathcal{A})$.
- $(\mathcal{A}, \mathcal{I})$ has the *hull* property if for every $A \subset X$ there is a set $Y \in \mathcal{A}$ such that $A \subset Y$, and whenever $Z \in \mathcal{A}$ is such that $A \subset Z$, we have $Y \backslash Z \in \mathcal{I}$.

Observe that the hull property is trivially equivalent to what might be called the 'dual hull property', that is, if $A \subset X$, then there is a set $Y \in \mathcal{A}$ such that $Y \subset A$ and, if $Z \in \mathcal{A}$ is such that $Z \subset A$, then $Z \backslash Y \in \mathcal{I}$. We use the notation $\ker(A)$ for the sets Y with this property and we use the notation $h(A)$ for the sets Y occurring in the definition of the 'hull property'.

If $(X, \mathcal{A}, \mathcal{I})$ is topological, then $(\mathcal{A}, \mathcal{I})$ has the hull property [1], which implies that $(\mathcal{A}, \mathcal{I})$ is MB [1, 4]. The inverse is not true, that is, there are pairs of an algebra and an ideal without the hull property which are MB and having the hull property is not a sufficient condition for being topological [4].

Another consequence of being topological is the existence of a set $G \subset X$ such that

$$\mathcal{H}(\mathcal{A}) = \{A \cup B : A \subset G \wedge B \in \mathcal{I}\}.$$

This is a straightforward consequence of [12, Theorem 6], which says that for any topological space

$$\mathcal{H}(\mathcal{NB}) = \{A \cup B : A \subset G \wedge B \in \mathcal{ND}\},$$

where G is the hereditarily irresolvable part of the Hewitt decomposition of X. If we additionally assume that $\mathcal{I}$ is small in $\mathcal{A}$, then for a topological triple $(X, \mathcal{A}, \mathcal{I})$ we have $\mathcal{A} = 2^X$ [12].

Now we can give examples of triples which are not topological and are either resolvable or not.

Example 2.3. Consider $(\mathbb{R}, \mathcal{B}, \text{Count})$. This triple is not topological since it does not have the hull property. Indeed, let A be an analytic but not a Borel set. For any Borel set $B \subset A$, the difference $A \backslash B$ is analytic but not a Borel set, so it is an uncountable analytic set and, by Luzin's theorem, it contains a perfect set, which is an uncountable Borel set. This gives $A \backslash B \notin \text{Count}$. Note that $(\mathbb{R}, \mathcal{B}, \text{Count})$ is not even MB [1].

Observe that $(\mathbb{R}, \mathcal{B}, \text{Count})$ is resolvable since any Bernstein set is $(\mathcal{B}, \text{Count})$-dense and, of course, the complement of a Berstein set is Bernstein.

Example 2.4. Consider $(\mathbb{R}, \mathcal{B}, Fin)$, where Fin denotes the ideal of finite subsets of the real line. It is not topological for the same reason as in Example 2.3.

This triple is not resolvable since any subset of $\mathbb{R}$ or its complement contains a countable infinite set, which is a positive set in our context.

3. The Hewitt decomposition

Analogously to the case of topological spaces, we consider a Hewitt decomposition for measurable spaces.

Definition 3.1. We say that $(X, \mathcal{A}, \mathcal{I})$ has a *Hewitt decomposition* if there exist sets $F, G \subset X$ such that $X = F \cup G$, $F \cap G = \emptyset$, $F, G \in \mathcal{A}$, F is resolvable and G is strongly irresolvable, that is, it does not contain a resolvable subset $A \in \mathcal{I}^+$.

Obviously a Hewitt decomposition of the measurable space can be determined at most mod $\mathcal{I}$.

If $(X, \mathcal{A}, \mathcal{I})$ is topological then it has a Hewitt decomposition. The spaces presented in the previous examples have Hewitt decompositions but this is not always the case in our context. In the sequel we shall give an example of an 'indecomposable' measure space (Example 3.7). Under rather strong assumptions we can obtain a Hewitt decomposition of a measurable space.

Theorem 3.2. *Suppose that $\mathcal{A}$ is a σ-algebra, $\mathcal{I}$ is a σ-ideal and the space $(X, \mathcal{A}, \mathcal{I})$ satisfies the countable chain condition (c.c.c.), that is, the condition that there are only countably many pairwise disjoint elements in the quotient algebra $\mathcal{A}/\mathcal{I}$. Then $(X, \mathcal{A}, \mathcal{I})$ has a Hewitt decomposition.*

https://doi.org/10.1017/S0004972715001458 Published online by Cambridge University Press

Proof. Let $\mathcal{D}$ be the family of all positive resolvable sets. Let $\tilde{\mathcal{D}}$ be a maximal pairwise disjoint family of $\mathcal{D}$. From c.c.c., it follows that $\tilde{\mathcal{D}}$ is countable and may be expressed as $\tilde{\mathcal{D}} = (D_n)_{n\in\mathbb{N}}$. Let P_n, Q_n denote disjoint sets dense in D_n. Let $F = \bigcup_{n\in\mathbb{N}} D_n$. By virtue of the fact that $\mathcal{A}$ is a σ-algebra, we obtain $F \in \mathcal{A}$. Observe that $P = \bigcup_{n\in\mathbb{N}} P_n$ and $Q = \bigcup_{n\in\mathbb{N}} Q_n$ are disjoint and dense in F. Indeed, let $A \subset F$, $A \in \mathcal{A}\backslash\mathcal{I}$. Since $\mathcal{I}$ is a σ-ideal, there exists $n_0 \in \mathbb{N}$ such that $A \cap D_{n_0} \notin \mathcal{I}$. Hence, $A \cap P$ and $A \cap Q$ are not empty.

Now let $A \subset X\backslash F$, $A \in \mathcal{A}\backslash\mathcal{I}$. Then A is irresolvable, since $\tilde{\mathcal{D}}$ was maximal. □

One of the consequences of the Hewitt decomposition of a topological space is the following characterisation of irresolvable spaces: a topological space is irresolvable if and only if it contains an open hereditarily irresolvable subset. It remains an open question whether for any measurable space irresolvability is equivalent to the existence of a positive strongly irresolvable set. That is why we introduce the following notion.

We say that a space $(X, \mathcal{A}, \mathcal{I})$ is *weakly resolvable* if it contains no strongly irresolvable set belonging to $\mathcal{I}^+$.

Obviously every resolvable space is weakly resolvable and, if $(X, \mathcal{A}, \mathcal{I})$ has a Hewitt decomposition, then its weak resolvability implies resolvability.

Theorem 3.3. *If $(X, \mathcal{A}, \mathcal{I})$ is weakly resolvable, then $\mathcal{H}(\mathcal{A}) = \mathcal{I}$.*

Proof. Assume that there exists $B \in \mathcal{H}(\mathcal{A})\backslash\mathcal{I}$. Since $(X, \mathcal{A}, \mathcal{I})$ is weakly resolvable, there exists a set $V \subset B$, $V \in \mathcal{I}^+$, such that V is resolvable. Hence, we can find a set $A \subset V$ such that A and $V\backslash A$ are $(\mathcal{A}_{|V}, \mathcal{I}_{|V})$-dense. Then both sets A and $V\backslash A$ are in $\mathcal{A}$ being subsets of B and at least one of them does not belong to $\mathcal{I}$, so is positive. Hence, the other one cannot be $(\mathcal{A}_{|V}, \mathcal{I}_{|V})$-dense, which gives a contradiction. □

Example 3.4. Consider $(\mathbb{R}, \mathcal{L}, \mathcal{H}_0)$, where $\mathcal{L}$ denotes the σ-algebra of Lebesgue measurable sets and $\mathcal{H}_0$ denotes the σ-ideal of sets of Hausdorff dimension zero. Then $(\mathbb{R}, \mathcal{L}, \mathcal{H}_0)$ is not topological since $\mathcal{H}_0$ is small in $\mathcal{L}$ and $\mathcal{L} \neq 2^R$. The last theorem shows that $(\mathbb{R}, \mathcal{L}, \mathcal{H}_0)$ is not weakly resolvable since $\mathcal{H}(\mathcal{L}) = \mathcal{N}$, the σ-ideal of Lebesgue null sets, and $\mathcal{H}_0 \neq \mathcal{N}$.

Example 3.5. The same reasoning as in the previous example shows that $(\mathbb{R}, \mathcal{L}, \mathcal{M})$, where $\mathcal{M}$ denotes the σ-ideal of microscopic sets, is not topological and it is not weakly resolvable. For the definition and properties of microscopic sets, see [10].

Using Theorem 3.3, we can find conditions under which a measurable space does not have a Hewitt decomposition.

Theorem 3.6. *Let $(X, \mathcal{A}, \mathcal{I})$ be resolvable. Let $\mathcal{K}$ be an ideal of subsets of X satisfying the following conditions:*

(1) *the families $(\mathcal{A} \cap \mathcal{K})\backslash\mathcal{I}$ and $\mathcal{I}\backslash\mathcal{K}$ are coinitial to the family $\mathcal{A}\backslash(\mathcal{K} \cup \mathcal{I})$;*
(2) *the ideals $\mathcal{K}$ and $\mathcal{I}$ are not orthogonal, that is, there is no decomposition of the space $X = A \cup B$ such that $A \in \mathcal{K}$ and $B \in \mathcal{I}$.*

Then the measurable space $(X, \mathcal{A}, \mathcal{I} \cap \mathcal{K})$ does not have a Hewitt decomposition.

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Proof. First observe that every set P from the family $(\mathcal{A} \cap \mathcal{K})\backslash\mathcal{I}$ is resolvable, since the space $(X, \mathcal{A}, \mathcal{I})$ is resolvable and the family of positive subsets of P is the same in both spaces. At the same time, every set Q from the family $\mathcal{I}\backslash\mathcal{K}$ is irresolvable. By the assumption about the resolvability of $(X, \mathcal{A}, \mathcal{I})$ and Theorem 3.3, we have $\mathcal{I} = \mathcal{H}(\mathcal{A})$. Hence, for every partition $Q = A \cup B$, at least one part is positive in the space $(X, \mathcal{A}, \mathcal{I} \cap \mathcal{K})$.

Observe now that if $Z \in (\mathcal{I} \cap \mathcal{K})^+$ is resolvable, then $Z \in \mathcal{K}$. In fact, by virtue of the first assumption every set from the family $\mathcal{A}\backslash\mathcal{K}$ contains a set from $\mathcal{I}\backslash\mathcal{K}$, which is irresolvable. On the other hand every positive strongly irresolvable set belongs to $\mathcal{I}$. Indeed, by the first assumption, every set from the family $\mathcal{A}\backslash\mathcal{I}$ contains a set from $(\mathcal{A} \cap \mathcal{K})\backslash\mathcal{I}$, which is resolvable.

Suppose that $X = F \cup G$, where $F, G \in \mathcal{A}$, F is resolvable and G is strongly irresolvable in $(X, \mathcal{A}, \mathcal{I} \cap \mathcal{K})$. Then $F \in \mathcal{K}$ and $G \in \mathcal{I}$, contrary to our second assumption. □

Example 3.7. Consider the space $(\mathbb{R}, \mathcal{B}, \mathrm{Count})$ from Example 2.3. Let $\mathcal{K}$ denote the ideal of bounded subsets of the real line. It is easily observed that $\mathcal{K}$ satisfies the conditions of Theorem 3.6. Therefore, $(\mathbb{R}, \mathcal{B}, \mathcal{K} \cap \mathrm{Count})$ has no Hewitt decomposition.

Observe that Theorem 3.6 does not settle the question of whether the Hewitt characterisation of irresolvability is valid for measurable spaces. In fact, spaces constructed by using this theorem always contain a strongly irresolvable positive subset.

4. Resolvability and irresolvability

To prove that the converse to Theorem 3.3 is not true we need an auxiliary lemma.

Lemma 4.1. *A measurable space $(X, \mathcal{A}, \mathcal{I})$ is strongly irresolvable if and only if $S(\mathcal{I}^+) = 2^X$.*

Proof. The existence of a set $E \subset X$ which is not in $S(\mathcal{I}^+)$ is, by the definition of $S(\mathcal{I}^+)$, equivalent to the existence of a positive set $A \subset X$ such that no positive B is contained in either $A \cap E$ or $A\backslash E$. This implies the existence of a positive resolvable subset of X. □

Corollary 4.2. *Under the assumption that $(\mathcal{A}, \mathcal{I})$ is MB, a measurable space $(X, \mathcal{A}, \mathcal{I})$ is strongly irresolvable if and only if $\mathcal{A} = 2^X$.*

Proposition 4.3. *There exists a measurable space $(X, \mathcal{A}, \mathcal{I})$ which is not weakly resolvable but has $\mathcal{H}(\mathcal{A}) = \mathcal{I}$.*

Proof. Balcerzak *et al.* [2] proved the existence of a measurable space $(X, \mathcal{A}, \mathcal{I})$ such that $\mathcal{H}(\mathcal{A}) = \mathcal{I}$ and $\mathcal{A}$ satisfies the condition

$$\forall_{B \in 2^X \backslash \mathcal{H}(\mathcal{A})} \exists_{A \in \mathcal{A} \backslash \mathcal{H}(\mathcal{A})} \quad A \subset B. \tag{$*$}$$

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If X is countable, such an algebra and an ideal exist in ZFC. If the cardinality of X is equal to $\mathfrak{c}$, we need to assume MA (Martin's Axiom) or CH (the Continuum Hypothesis).

For our purpose it suffices to prove that $(*)$ implies $S(\mathcal{I}^+) = 2^X$ and use Lemma 4.1. Suppose, contrary to our claim, that there is a subset $E \subset X$ not belonging to $S(\mathcal{I}^+)$. Then obviously $E \notin \mathcal{I}$ and there exists a positive set $A \subset X$ such that no positive set is contained in either $A \cap E$ or $A \backslash E$. Therefore, $E \cap A \notin \mathcal{I}$. By $(*)$ and $\mathcal{H}(\mathcal{A}) = \mathcal{I}$, for the set $E \cap A$ there exists a set $U \in \mathcal{A} \backslash \mathcal{H}(\mathcal{A})$ such that $U \subset E \cap A$, which is impossible. □

Studying Examples 3.4 and 3.5 leads us to the next, more general, theorem, which shows in particular that the spaces from these examples do not have the hull property.

Theorem 4.4. *Let $(X, \mathcal{A}, \mathcal{I})$ be a measurable space such that $\mathcal{A} \neq 2^X$ and let $\mathcal{I}$ be small in $\mathcal{H}(\mathcal{A})$. Then $(X, \mathcal{A}, \mathcal{I})$ does not have the hull property.*

Proof. Suppose, contrary to our claim, that $(X, \mathcal{A}, \mathcal{I})$ has the hull property. Let $A \notin \mathcal{A}$. Then $P := h(A) \backslash \ker(A) \in \mathcal{A}$. Observe that $P \notin \mathcal{H}(\mathcal{A})$. In fact, the set $A \backslash \ker(A) \subset P$ and $A \backslash \ker(A) \notin \mathcal{A}$. Then $P \in \mathcal{A} \backslash \mathcal{H}(\mathcal{A})$ and, by virtue of the fact that $\mathcal{I}$ is small in $\mathcal{H}(\mathcal{A})$, there exists a set $Q \subset P$ such that $Q \in \mathcal{H}(\mathcal{A}) \backslash \mathcal{I}$. Now both sets $Q \cap A$ and $Q \backslash A$ belong to $\mathcal{A}$, but one of them does not belong to $\mathcal{I}$. This contradicts the definition of $h(A)$ and $\ker(A)$. □

Due to the fact that having the hull property is a necessary condition for $(\mathcal{A}, \mathcal{I})$ to be topological, the last theorem is a strengthening of [12, Proposition 20].

For a topological space (with the hereditarily irresolvable part in the Hewitt decomposition denoted by G), we have $\mathcal{H}(\mathcal{NB}) = \{A \cup N : A \subset G \wedge N \in \mathcal{ND}\}$ (see [12, Theorem 6]) and consequently a topological space is resolvable if and only if $\mathcal{H}(\mathcal{NB}) = \mathcal{ND}$ (Theorem 2.1). We consider this property for a measurable space. One implication has already been proved in Theorem 3.3.

Proposition 4.5. *If $(X, \mathcal{A}, \mathcal{I})$ has a Hewitt decomposition (with a strongly irresolvable part denoted by G), then*

$$\mathcal{H}(\mathcal{A}) \subset \{A \cup N : A \subset G \wedge N \in \mathcal{I}\}.$$

If, in addition, $(X, \mathcal{A}, \mathcal{I})$ has the hull property, then

$$\mathcal{H}(\mathcal{A}) = \{A \cup N : A \subset G \wedge N \in \mathcal{I}\}.$$

Proof. Let $B \in \mathcal{H}(\mathcal{A})$ and $N := B \backslash G$. It is enough to show that $N \in \mathcal{I}$. Since $N \subset B$, we have $N \in \mathcal{A}$. If $N \notin \mathcal{I}$, it is a positive subset of the resolvable part F in the Hewitt decomposition $X = G \cup F$ and so it contains a nonmeasurable subset. But this is impossible since $N \in \mathcal{H}(\mathcal{A})$.

Now we assume also the hull property. Let $A \subset G$. It is enough to show that $A \in \mathcal{A}$, because then $G \in \mathcal{H}(\mathcal{A}) = \{A \in \mathcal{A} : \forall_{B \subset A} B \in \mathcal{A}\}$ and $\mathcal{I} \subset \mathcal{H}(\mathcal{A})$, so $\{A \cup N : A \subset G \wedge N \in \mathcal{I}\} \subset \mathcal{H}(\mathcal{A})$.

Suppose, contrary to our claim, that $A \notin \mathcal{A}$. Introduce the sets $B := G \backslash A$ and $U := G \backslash \ker(A) \backslash \ker(B)$. Then U is positive and $U \cap A$, $U \cap B$ are disjoint $(\mathcal{A}, \mathcal{I})$-dense subsets of U, which means that G has a positive resolvable subset, which is a contradiction. □

Corollary 4.6. *If $(X, \mathcal{A}, \mathcal{I})$ has a Hewitt decomposition and has the hull property, then $\mathcal{H}(\mathcal{A}) = \mathcal{I}$ implies resolvability of $(X, \mathcal{A}, \mathcal{I})$.*

In the next theorem we weaken the assumptions about the measurable space.

Theorem 4.7. *If $(\mathcal{A}, \mathcal{I})$ is MB and $\mathcal{H}(\mathcal{A}) = \mathcal{I}$, then $(X, \mathcal{A}, \mathcal{I})$ is weakly resolvable.*

Proof. Let $G \in \mathcal{I}^+$. Since $\mathcal{H}(\mathcal{A}) = \mathcal{I}$, there exists a subset $E \subset G$, $E \notin \mathcal{A}$, and, by the assumption MB, $E \notin S(\mathcal{I}^+)$. Consequently, by the definition of $S(\mathcal{I}^+)$, there exists a set $A \in \mathcal{I}^+$ such that no positive B is contained in either $A \cap E$ or $A \backslash E$. Let $C := A \cap G$. If $C \in \mathcal{I}$, then $A \backslash G$ is a positive set contained in $A \backslash E$, which is impossible. Thus, C is a positive subset of G and $(C, \mathcal{A}_{|C}, \mathcal{I}_{|C})$ is resolvable, since $C \backslash E$ and $C \cap E$ are $(\mathcal{A}, \mathcal{I})$-dense on C. □

Corollary 4.8. *Suppose that $(\mathcal{A}, \mathcal{I})$ is MB and so has a Hewitt decomposition. If $\mathcal{H}(\mathcal{A}) = \mathcal{I}$, then $(X, \mathcal{A}, \mathcal{I})$ is resolvable.*

Corollary 4.9. *For every measurable space $(X, \mathcal{A}, \mathcal{I})$, $\mathcal{H}(S(\mathcal{I}^+)) = S_0(\mathcal{I}^+)$ is equivalent to weak resolvability of $(X, S(\mathcal{I}^+), S_0(\mathcal{I}^+))$.*

Proof. The corollary follows from the fact that $(X, S(\mathcal{I}^+), S_0(\mathcal{I}^+))$ is MB by the following properties of $S(\mathcal{I}^+) \backslash S_0(\mathcal{I}^+)$. If $\mathcal{F}_1, \mathcal{F}_2 \subset 2^X$ are mutually coinitial, then $S(\mathcal{F}_1) = S(\mathcal{F}_2)$ and $S_0(\mathcal{F}_1) = S_0(\mathcal{F}_2)$ (see [3, Proposition 1.2]) and, moreover, if $\mathcal{A} \backslash \mathcal{I} \subset S(\mathcal{I}^+) \backslash S_0(\mathcal{I}^+)$, then $\mathcal{A} \backslash \mathcal{I}$ is coinitial to $S(\mathcal{I}^+) \backslash S_0(\mathcal{I}^+)$. □

The mutual coinitiality of $\mathcal{A} \backslash \mathcal{I}$ and $S(\mathcal{I}^+) \backslash S_0(\mathcal{I}^+)$ obviously implies $\mathcal{D}(\mathcal{A}, \mathcal{I}) = \mathcal{D}(S(\mathcal{I}^+), S_0(\mathcal{I}^+))$.

Corollary 4.10. *$(X, \mathcal{A}, \mathcal{I})$ is resolvable if and only if $(X, S(\mathcal{I}^+), S_0(\mathcal{I}^+))$ is resolvable.*

Theorem 4.11. *$(X, \mathcal{A}, \mathcal{I})$ is weakly resolvable if and only if $(X, S(\mathcal{I}^+), S_0(\mathcal{I}^+))$ is weakly resolvable.*

Proof. Assume that $(X, S(\mathcal{I}^+), S_0(\mathcal{I}^+))$ is weakly resolvable. Let $G \in \mathcal{I}^+$. Then $G \in S_0(\mathcal{I}^+)^+$. By assumption there exists a set $H \in S_0(\mathcal{I}^+)^+$, $H \subset G$, which is resolvable as $(H, S(\mathcal{I}^+)_{|H}, S_0(\mathcal{I}^+)_{|H})$. By coinitiality of $\mathcal{I}^+$ to $S_0(\mathcal{I}^+)^+$, there exists a set $K \in \mathcal{I}^+$, $K \subset H$, which is resolvable as $(K, S(\mathcal{I}^+)_{|K}, S_0(\mathcal{I}^+)_{|K})$. Since the families of $(\mathcal{A}, \mathcal{I})$-dense sets and the family of $(S(\mathcal{I}^+), S_0(\mathcal{I}^+))$-dense sets coincide, the set K is resolvable as $(K, \mathcal{A}_{|K}, \mathcal{I}_{|K})$.

Analogously, suppose that $(X, \mathcal{A}, \mathcal{I})$ is weakly resolvable. For an arbitrary set $A \in S_0(\mathcal{I}^+)^+$, by coinitiality of $\mathcal{I}^+$ to $S_0(\mathcal{I}^+)^+$, there exists a set $B \in \mathcal{I}^+$, $B \subset A$. By assumption there is a set $C \in \mathcal{I}^+$, $C \subset B$, which is resolvable as $(C, \mathcal{A}_{|C}, \mathcal{I}_{|C})$. Hence, $C \in S_0(\mathcal{I}^+)^+$ and it is also resolvable as $(C, S(\mathcal{I}^+)_{|C}, S_0(\mathcal{I}^+)_{|C})$. □

The following diagram summarises our results (see Figure 1).

In [11], Elkin's criterion of topological irresolvability (compare [7]) was studied in the context of Borel resolvability. This criterion says that a topological space (X, τ) is irresolvable if and only if τ contains a base of some ultrafilter on X. In our setting one implication is easy to prove.

https://doi.org/10.1017/S0004972715001458 Published online by Cambridge University Press

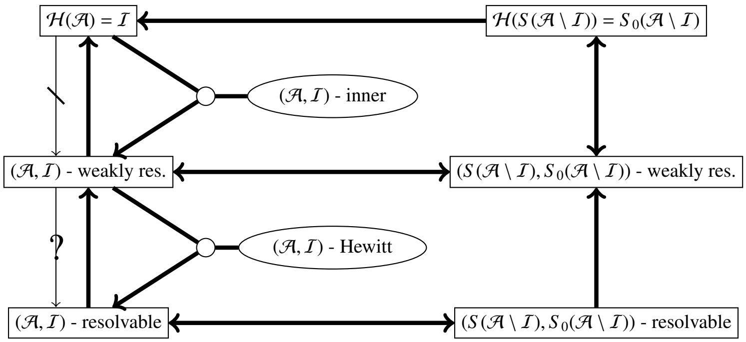

FIGURE 1. Diagram summarising the main results.

PROPOSITION 4.12. *If $\mathcal{I}^+$ contains a base of some ultrafilter, then $(X, \mathcal{A}, \mathcal{I})$ is irresolvable.*

PROOF. Consider a decomposition of X, $X = A \cup B$. Let $\tilde{\mathcal{B}} \subset \mathcal{I}^+$ be a base of some ultrafilter $\tilde{\mathcal{F}}$. Then $A \in \tilde{\mathcal{F}}$ or $B \in \tilde{\mathcal{F}}$. But the set belonging to $\tilde{\mathcal{F}}$ contains a positive set from the base $\tilde{\mathcal{B}}$, so the second one cannot be dense. □

Under an additional assumption, the converse is a consequence of Corollary 4.2.

PROPOSITION 4.13. *If $(X, \mathcal{A}, \mathcal{I})$ is strongly irresolvable and $(\mathcal{A}, \mathcal{I})$ is MB, then $\mathcal{I}^+$ contains a base of some ultrafilter.*

PROOF. Let $\mathcal{P}$ be an arbitrary maximal subfamily of $\mathcal{I}^+$ closed under finite intersections. Then $\mathcal{F} := \{F \subset X : \exists_{A \in \mathcal{P}} A \subset F\}$ is an ultrafilter with the base $\mathcal{P}$. Indeed, sets containing subsets from $\mathcal{F}$ obviously belong to $\mathcal{F}$. If $F_1, F_2 \in \mathcal{F}$, then there exist sets $A_1, A_2 \in \mathcal{P}$ such that $A_1 \subset F_1$ and $A_2 \subset F_2$. Then $A_1 \cap A_2 \in \mathcal{P}$ and $A_1 \cap A_2 \subset F_1 \cap F_2$. Let $B \subset X$; then $B, X \backslash B \in \mathcal{A}$ by Corollary 4.2. Suppose that neither B nor $X \backslash B$ contains a set from $\mathcal{P}$. Then, for every $P \in \mathcal{P}$, we have $B \cap P \neq \emptyset$, which contradicts the maximality of the family $\mathcal{P}$. As a result, one of the sets B or $X \backslash B$ belongs to $\mathcal{F}$. □

COROLLARY 4.14. *If $(X, \mathcal{A}, \mathcal{I})$ is not weakly resolvable and $(\mathcal{A}, \mathcal{I})$ is MB, then $\mathcal{I}^+$ contains a base of some ultrafilter.* □

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