ON THE GROUP RING OF A FREE PRODUCT WITH AMALGAMATION

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1. Introduction. Let $G = A *_H B$ be the free product of the groups A and B amalgamating the proper subgroup H and let R be a ring with 1. If H is finite and G is not finitely generated we show that any non-zero ideal I of $R(G)$ intersects non-trivially with the group ring $R(M)$, where $M = M(I)$ is a subgroup of G which is a free product amalgamating a finite normal subgroup. This result compares with A. I. Lichtman's results in [**6**] but is not a direct generalisation of these.

We then apply this theorem together with results in [**4**] and [**1**] to obtain the following theorems on $JR(G)$, the Jacobson radical of $R(G)$, and on $ZR(G)$, the right singular ideal of $R(G)$. We denote by $NR(\Delta^+(G))$ the nilpotent radical of $R(\Delta^+(G))$.

THEOREM. *Let $G = A *_H B$, where H is a finite group, and let R be a right noetherian ring with* 1. *If G is not finitely generated then*

(i) *$R(G)$ is semiprimitive if and only if $R(G)$ is semiprime,*

(ii) *if R is a field, $JR(G) = NR(\Delta^+(G))R(G)$.*

THEOREM. *Let $G = A *_H B$, where H is a finite group, and let K be a field. If G is not finitely generated then $ZK(G) = NK(G)$.*

Our notation will be that usually employed. In particular, $A *_H B$ will denote the free product of groups A, B amalgamating the subgroup H; $|A : H|$ will denote the number of cosets of H in A. If we choose right transversals S, T, respectively, for A, B modulo H then every element $g \in G = A *_H B$ can be written uniquely in the form

$$g = ha_1b_1a_2b_2 \dots a_nb_n, \tag{1}$$

where $h \in H$, $a_i \in S$, $b_j \in T$, $a_i \neq 1$ if $i \neq 1$ and $b_j \neq 1$ if $j \neq n$. This is called the normal form of g[**7**, p. 205]. If $a_1 \neq 1 \neq b_n$ we say that g has AB form. We define similarly AA, BA and BB form for elements of G. If $b_n \neq 1$ we say g has $-B$ form. We define $-A$, $B-$, and $A-$ form for elements of G in the same way.

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2. Preliminaries. We need the following group theoretic results. For any group G, we define $\Delta^+(G)$ by

$$\Delta^+(G) = \{x \in G : x \text{ has only a finite number of conjugates in } G \text{ and } x \text{ has finite order}\}.$$

LEMMA 1. *If $G = A *_H B$ then $\Delta^+(G) \leq \Delta^+(H)$.*

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Proof. This is straightforward.

THEOREM 1. *Let $G = A *_H B$, where H is a group with minimum condition. If H is not normal in G, and if H has no non-trivial subgroups which are normal in G, then there exists $g \in G$ such that $g^{-1}Hg \cap H = 1$.*

Proof. See [**3**, proof of Theorem 1].

THEOREM 2. *Let P be a group having subgroups A_i $(i \in I)$ which intersect pairwise in a common subgroup B. That is, for $i, j \in I$ with $i \neq j$, we have $A_i \cap A_j = B$. If every element $p \in P$ has a normal form as defined in the introduction and if normal forms of different lengths represent different elements of P, then P is the free product of the A_i amalgamating B.*

Proof. See [**8**, p. 511].

3. The main result.

THEOREM 3. *Let R be a ring with 1 and let $G = A *_H B$, where H is finite. If G is not finitely generated and if I is a non-zero ideal of $R(G)$, then there exist subgroups C and D of G, strictly containing the finite normal subgroup $\triangle^+(G)$, such that $I \cap R(M) \neq 0$, where $M = C *_{\triangle^+(G)} D$.*

Proof. By Lemma 1, $\triangle^+(G) \leq H$ and is hence a finite normal subgroup of G. Now $\triangle^+(G/\triangle^+(G)) = 1$ (see [**9**, 19.3, p. 81]) and $G/\triangle^+(G) = A/\triangle^+(G) *_{H/\triangle^+(G)} B/\triangle^+(G)$. Since $\triangle^+(G/\triangle^+(G)) = 1$, no non-trivial subgroup of $H/\triangle^+(G)$ is normal in $G/\triangle^+(G)$. Hence we know from theorem 1 that there exists $\bar{g} \in G/\triangle^+(G)$ such that $\bar{g}^{-1}(H/\triangle^+(G))\bar{g} \cap H/\triangle^+(G) = 1$. Let g be an inverse image of $\bar{g}$ in G. Then $g^{-1}Hg \cap H \leq \triangle^+(G)$. Since $\triangle^+(G)$ is normal in G and a subgroup of H, $g^{-1}Hg \cap H = \triangle^+(G)$. As G is not finitely generated, either A is not finitely generated or B is not finitely generated. We suppose the former. If g has $A-$ form, choose $b \in B$, $b \notin H$. Then if $h \in g^{-1}b^{-1}Hbg \cap H$, $h = g^{-1}b^{-1}h_1bg$ for some $h_1 \in H$. Since g is $A-$, $b^{-1}h_1b \in H$ and so $h \in g^{-1}Hg \cap H = \triangle^+(G)$. Thus $g^{-1}b^{-1}Hbg \cap H = \triangle^+(G)$ and we may assume that g has $B-$ form. Similarly we may suppose without loss of generality that g has BB form, if H is not normal in A, and that g has BA form otherwise. Let $0 \neq \theta \in I$ and let $L = \langle \text{supp } \theta, H \rangle$. Since A is not finitely generated and L is finitely generated we can choose $a \in A$ such that for all $c \in L$, $a^{-1}ca$ has AA form or $a^{-1}ca \in H$. Let $C = g^{-1}a^{-1}Lag$. If H is not normal in A, g has BB form and so for $c \in C$, c has BB form or $c \in \triangle^+(G)$. If H is normal in A, either H is not normal in B or H is normal in G. In the first case, the argument is analogous to what follows with elements of C having AA form or belonging to $\triangle^+(G)$. In the second case, $H = \triangle^+(G)$ and the result is trivial. Thus we may assume that H is not normal in A. Hence we can choose $a_1 \in A$ such that $a_1 \notin H$ and $a_1^2 \notin H$. Let $b \in B$ with $b \notin H$ and let $D = \langle a_1ba_1, \Delta^+(G) \rangle$. Elements of D will have the form $d(a_1ba_1)^n$, where $d \in \Delta^+(G)$. Consider the group $M = \langle C, D \rangle$. Any element of M can be written

$$d(a_1ba_1)^{n_1}m_1(a_1ba_1)^{n_2}m_2 \dots m_n, \tag{2}$$

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where m_i has BB form for $i=1,\ldots,n-1$, n_i is an integer for $i=1,\ldots,n$, $n_i\neq 0$ for $i=2,\ldots,n$ and m_n has BB form or $m_n=1$. Thus every element of M has a normal form and normal forms of different lengths represent different elements in M. Hence by Theorem 2, $M=C*_{\Delta^+(G)}D$. Since $\Delta^+(G)$ is normal in G it is normal in M and $0\neq g^{-1}a^{-1}\theta ag\in R(M)\cap I$, giving the required result.

NOTE. It is not known to the author whether the condition in Theorem 3, that G be not finitely generated, is necessary.

4. Applications. When H is a normal subgroup of $G=A*_H B$ we have the following results for $JR(G)$.

THEOREM 4. *Let R be a ring and let $G=A*_H B$ with H normal in G and $|A:H|\neq 2$ or $|B:H|\neq 2$. Suppose that $R(H)$ is a right (left) noetherian ring. Then $JR(G)=0$ if and only if $R(H)$ is semiprime.*

THEOREM 5. *Let K be a field of characteristic $p\neq 0$. Let $G=A*_H B$ with H normal in G. Suppose that H is a polycyclic-by-finite group. Then $JK(G)=NK(H)\,K(G)=NK(G)$.*

(Note that if the characteristic of K is 0, then $JK(G)=NK(G)=0$ by Theorem 4 and [**9**, 3.3, p. 9].)

These results can be obtained by modifying the proof of [**4**, Theorem 2], and considering the case $|A:H|=|B:H|=2$ separately. Details may be found in [**5**].

We use our main theorem to prove

THEOREM 6. *Let $G=A*_H B$, where H is a finite group, and let R be a right noetherian ring. If G is not finitely generated then*

(i) *$R(G)$ is semiprimitive if and only if $R(G)$ is semiprime,*

(ii) *if R is a field, $JR(G)=NR(\Delta^+(G))\,R(G)$.*

Proof. If H is normal in G, the result follows from Theorem 4 and Theorem 5. Thus we may assume that H is not normal in G. Let $0\neq\theta\in JR(G)$; then, by the proof of Theorem 3, there is $g\in G$ and $a\in A$ with $g^{-1}a^{-1}\theta ag\in R(M)\cap JR(G)$, where $M=C*_{\Delta^+(G)}D$. But $R(M)\cap JR(G)\subseteq JR(M)$ (see [**9**, 16.9, p. 68]). Thus $JR(M)\neq 0$. Since $\Delta^+(G)$ is finite, $R(\Delta^+(G))$ is right noetherian and so Theorem 4 shows that $R(\Delta^+(G))$ is not semiprime. Now $NR(\Delta^+(G))$ is nilpotent and so $NR(\Delta^+(G))R(G)$ is a nilpotent ideal in $R(G)$ and $R(G)$ is not semiprime. Clearly if $R(G)$ is not semiprime $R(G)$ is not semiprimitive and we have proved (i). For (ii) we apply Theorem 5 to obtain $JR(M)=NR(\Delta^+(G))R(G)=NR(G)$. Thus $g^{-1}a^{-1}\theta ag\in NR(\Delta^+(G))R(G)$. Since $NR(\Delta^+(G))$ is a nilpotent ideal of $R(\Delta^+(G))$ and invariant under automorphisms, $NR(\Delta^+(G))\,R(G)$ is a nilpotent ideal of $R(G)$. Thus $\theta\in NR(\Delta^+(G))\,R(G)$ and we have shown that $JR(G)\subseteq NR(\Delta^+(G))\,R(G)$. $NR(\Delta^+(G))\,R(G)\subseteq JR(G)$ since it is nilpotent, and we have the required equality.

The following result is a special case of Theorem 3.4 in [**1**].

THEOREM 7. *Let K be a field and $G = A *_H B$, where H is finite and normal in G. Then $ZK(G) = NK(G)$.*

We use this to obtain

THEOREM 8. *Let K be a field and $G = A *_H B$ with H finite and G not finitely generated. Then $ZK(G) = NK(G)$.*

Proof. If $H \rhd G$, the result follows by Theorem 7. Thus we may assume that H is not normal in G. Let $0 \neq \theta \in ZK(G)$. Then, by the proof of Theorem 3, $g^{-1}a^{-1}\theta ag \in K(M) \cap ZK(G)$, where $M = C *_{\Delta^+(G)} D$. Thus $g^{-1}a^{-1}\theta ag \in K(M) = ZK(M) = NK(M)$ by Theorem 7 and [**2**, Lemma 4.7]. Now since $\Delta^+(G)$ is finite and normal in M, $NK(M) = NK(\Delta^+(G))K(M)$, which is a nilpotent ideal invariant under automorphisms. Thus $\theta \in NK(\Delta^+(G))K(M)$ and hence $\theta \in NK(\Delta^+(G))K(G) \subseteq NK(G)$. Thus $ZK(G) \subseteq NK(G)$ and hence $ZK(G) = NK(G)$.

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