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Maximal absolutely continuous invariant measures for piecewise linear Markov transformations

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Abstract. Let $\mathbb{A}$ be an irreducible 0–1 matrix such that the non-zero entries in each row are consecutive. Let $\mathfrak{C}_{\max}$ be the class of piecewise linear Markov transformations τ on [0, 1] into [0, 1] induced by $\mathbb{A}$ for which the absolutely continuous invariant measure has maximal entropy. The main result presents necessary and sufficient slope conditions on τ which guarantee that $\tau \in \mathfrak{C}_{\max}$.

1. *Introduction*

There are two measures which appear prominently in the dynamical systems theory literature: measures which are absolutely continuous with respect to Lebesgue measure [**10, 11**] and the maximal measures [**6–9**], i.e. those which maximize the measure theoretic entropy. The maximal measure reflects the maximum randomness that can be generated by a dynamical system while the absolutely continuous invariant measure (a.c.i.m.) is the one which arises naturally in physical situations, as for example in computer simulations. When an a.c.i.m. is also maximal it says that the most chaotic situation possible can be realized by the physical system. Transformations, which have this property are, therefore, of interest.

In [**8**] the maximal measures for a restrictive class of piecewise monotonic transformations is characterized and under a mild restriction uniqueness of the maximal measure is established. In this paper we shall be concerned with piecewise linear Markov transformations associated with an irreducible 0–1 matrix $\mathbb{A}$. Using the structure of $\mathbb{A}$, we shall derive a system of equations which provide necessary and sufficient conditions for the unique maximal measure to be absolutely continuous. In [**9**] it is stated that the 'absolutely continuous invariant measure in general is not the measure with maximal entropy'. For the class of piecewise linear Markov transformations associated with $\mathbb{A}$, we will be able to make this statement more precise. For example, we shall be able to specify the dimension of the family

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of piecewise linear Markov transformations with the property that the a c i m is maximal

We shall also study the case when $\mathbb{A}$ is not irreducible Then it is possible that there is no a c i m which is maximal The study of transformations compatible with such a matrix clarifies the distinction between a c i m and maximal measures

Let $\mathbb{A}=(a_{ij})$ be a fixed $n\times n$ 0–1 irreducible matrix such that the non-zero entries in each row are consecutive Let $\mathscr{A}$ denote this class of matrices Let $\mathbf{I}=[0,1]$ and let $0=a_0<a_1<\quad<a_n=1$ be a partition of $\mathbf{I}$ denoted by $\mathfrak{P}$ We say that a transformation τ $\mathbf{I}\to\mathbf{I}$ is Markov if $\tau(\{a_0,a_1,\quad,a_n\})\subset\{a_0,a_1,\quad,a_n\}$ The Markov transformation τ is compatible with the matrix $\mathbb{A}$ when $\tau(\mathbf{I}_i)\supseteq\mathbf{I}_j$ if and only if $a_{ij}=1$ for all $\mathbf{I}_i,\mathbf{I}_j\in\mathfrak{P}$ Let $\mathfrak{C}_{\mathbb{A}}$ be the class of piecewise linear Markov transformations compatible with $\mathbb{A}\in\mathscr{A}$ It is easy to show that $\tau\in\mathfrak{C}_{\mathbb{A}}$ admits a unique absolutely continuous invariant measure $\mu=fm$, where m is Lebesgue measure on $\mathbf{I}$ The function f is a τ-invariant density and is constant on elements of the defining partition of τ, $\mathfrak{P}$

The measure-theoretic entropy $h_\mu(\tau)$ can be computed by means of the formula **[2]**

$$h_\mu(\tau)=\int_{\mathbf{I}}\ln|\tau'|\,d\mu$$

The topological entropy of τ is denoted by $h_{top}(\tau)$ If τ is not continuous, we define $h_{top}(\tau)=\ln\lambda$, where λ is the maximal eigenvalue of $\mathbb{A}$ (We note that $\tau\in\mathfrak{C}_{\mathbb{A}}$ is isomorphic to the subshift of finite type associated with $\mathbb{A}$ in the sense of Definition 2 4 in **[13]**)

In this paper we completely characterize the piecewise linear Markov transformations whose a c i m is maximal That is, we find necessary and sufficient conditions for $\tau\in\mathfrak{C}_{\mathbb{A}}$ to belong to the subclass

$$\mathfrak{C}_{\max}=\{\tau\in\mathfrak{C}_{\mathbb{A}}\ \ h_\mu(\tau)=h_{top}(\tau)=\ln\lambda\}$$

It is obvious that if τ has constant slope, then it is equal to λ and $\tau\in\mathfrak{C}_{\max}$ Moreover, if some iterate of τ, say τ^k, has constant slope, then $\tau\in\mathfrak{C}_{\max}$, since

$$h_\mu(\tau)=(1/k)h_\mu(\tau^k)=(1/k)h_{top}(\tau^k)=h_{top}(\tau)$$

In the sequel we shall show that $\mathfrak{C}_{\max}$ is much richer than those examples might suggest In particular, we will give an example of a $\tau\in\mathfrak{C}_{\max}$ such that no iterate τ^k of τ has constant slope

In **[4]** we treated the foregoing problem for a very restricted class of matrices $\mathscr{A}_0$, where $\mathbb{A}\in\mathscr{A}_0\subset\mathscr{A}$ if there are integers p and q, $1\le p\leqq q\leqq n$ such that every row of $\mathbb{A}$ either consists of a block of 1's $a_{ij}=1$ if and only if $j=p,\quad,q$, or else the row contains a unique nonzero element The main result of **[4]** is

Theorem 0 *Let* $\mathbb{A}\in\mathscr{A}_0$ *and* $1\le p,p+1,\quad,q\le n$ *denote the indices of the block of* 1's *Let* τ *be a piecewise linear Markov transformation on* $\mathbf{I}$ *compatible with* $\mathbb{A}$ *and having the defining partition* $\mathfrak{P}=\{\mathbf{I}_1,\quad,\mathbf{I}_n\}$ *Let* μ *be the* a c i m *associated with* τ *If* μ *is maximal, then for those* i*'s,* $p\le i\le q$, *for which* $\tau(\mathbf{I}_i)\supseteq\mathbf{I}_j$, *with* $p\le j\le q$, *we have* $|\tau'|_{\mathbf{I}_i}|=\lambda$, *where* λ *is the maximal eigenvalue of* $\mathbb{A}$

In Theorem 3 of [4] this result was stated incorrectly to apply to all i's, $p \le i \le q$ The mathematical method of the proof of Theorem 3 was correct, but too general a conclusion was drawn

2 *Background*

Let $\mathbb{A}$ be an irreducible 0–1 matrix, and let $(\Sigma_{\mathbb{A}}^{+}, \sigma)$ be the one-sided subshift of finite type associated with $\mathbb{A}$ $\Sigma_{\mathbb{A}}^{+}$ is a metric space with metric $d(\underline{x}, \underline{y}) = 2^{-N}$, where $N = \inf\{n \;\; x_n \ne y_n\}$ for $\underline{x} = (x_0, x_1, \;\;)$, $\underline{y} = (y_0, y_1, \;\;)$ in $\Sigma_{\mathbb{A}}^{+}$ Let $\mathcal{F}$ be a set of Holder continuous functions on $\Sigma_{\mathbb{A}}^{+}$ For any $\varphi \in \mathcal{F}$, we define the operator

$$\mathcal{L}_{\varphi} \;\; \mathcal{C}(\Sigma_{\mathbb{A}}^{+}) \to \mathcal{C}(\Sigma_{\mathbb{A}}^{+})$$

by the formula

$$\mathcal{L}_{\varphi}(f)(\underline{x}) = \sum_{\underline{y} \;\; \sigma \underline{y} = \underline{x}} \exp(\varphi(\underline{y})) f(\underline{y})$$

THEOREM 1

(1) *There exist a unique* $\lambda_{\varphi} \in \mathbb{R}$, *a function* h_{φ} (*unique up to constant multiples*), *such that* $\mathcal{L}_{\varphi} h_{\varphi} = \lambda_{\varphi} h_{\varphi}$, *and a unique probability measure* ν_{φ} *such that* $\mathcal{L}_{\varphi}^{*} \nu_{\varphi} = \lambda_{\varphi} \nu_{\varphi}$

(2) *The measure* $\mu_{\varphi} = h_{\varphi} \nu_{\varphi}$ *is* σ*-invariant, ergodic, positive on non-empty open sets, and it is the unique measure which maximizes the expression* $h_{\mu}(\sigma) + \mu(\varphi)$ *The measure* μ_{φ} *is called the equilibrium state for* φ

(3) *For* $\varphi, \psi \in \mathcal{F}$, *we have* $\mu_{\varphi} = \mu_{\psi}$ *if and only if there exists a function* $t \in \mathcal{C}(\Sigma_{\mathbb{A}}^{+})$ *and a number* $c \in \mathbb{R}$ *such that*

$$\varphi - \psi = c + t - t \circ \sigma$$

(4) *If* $\varphi = \ln g$ *where*

$$\sum_{\underline{y} \Sigma \sigma \underline{y} = \underline{x}} g(\underline{y}) = 1$$

for any $\underline{x} \in \Sigma_{\mathbb{A}}^{+}$, *then* $\lambda_{\varphi} = 1$, $h_{\varphi} = 1$, *and* $h_{\mu_{\varphi}}(\sigma) + \mu_{\varphi}(\varphi) = 0$

(5) *If* $\varphi, \psi \in \mathcal{F}$, $\varphi = \ln g_1$, $\psi = \ln g_2$ *with*

$$\sum_{\underline{y} \;\; \sigma \underline{y} = \underline{x}} g_i(\underline{y}) = 1 \quad i = 1, 2,$$

then $\mu_{\varphi} = \mu_{\psi}$ *implies* $g_1 = g_2$

Proof The proof of this theorem is an extension of the proof of an analogous theorem for primitive matrices $\mathbb{A}$ [1] □

PROPOSITION 1 *Let* $\tau \;\; \mathbf{I} \to \mathbf{I}$ *be a piecewise linear Markov map with irreducible transition matrix* $\mathbf{A}$ *Let* $\mu = fm$ *be the unique absolutely continuous measure invariant under* τ *Then the dynamical system* $(\mathbf{I}, \tau, \mu)$ *is isomorphic to* $(\Sigma_{\mathbf{A}}^{+}, \sigma, \mu_{\varphi})$, *and*

(1) *the isomorphism* $\pi \;\; \Sigma_{\mathbf{A}}^{+} \to \mathbf{I}$ *is Holder continuous and* 1–1 *on the set of full* μ_{φ}*-measure where* $\varphi = -\ln|\tau' \circ \pi|$,

(2) *for* $\varphi = -\ln|\tau' \circ \pi|$, *the measure* $\nu_{\varphi} = m \circ \pi^{-1}$, $h_{\varphi} = f \circ \pi$, *and* $\lambda_{\varphi} = 1$

Proof Let $\mathfrak{P} = \{\mathbf{I}_1, \mathbf{I}_2, \;\; , \mathbf{I}_n\}$ be the defining partition of τ The standard isomorphism π is defined by

$$\pi((x_0, x_1, x_2, \;\;)) = \mathbf{I}_{x_0} \cap \tau^{-1}(\mathbf{I}_{x_1}) \cap \;\; \cap \tau^{-n}(\mathbf{I}_{x_n}) \cap$$

It can be proved that π is Holder continuous It follows that $\varphi = -\ln|\tau' \circ \pi|$ belongs to $\mathscr{F}$, and by Theorem 1, there exists the unique measure μ_φ (the equilibrium state for φ) The fact that π is 1-1 on the set of full μ_φ-measure is well known

We now prove that $h_\varphi = f \circ \pi$ It is enough to show that

$$\mathscr{L}_\varphi(f \circ \pi) = f \circ \pi$$

We have

$$\mathscr{L}_\varphi(f \circ \pi)(\underline{x}) = \sum_{\underline{y}\ \sigma\underline{y}=\underline{x}} |\tau'(y)|^{-1} f(y) = \sum_{y\ \tau y = x} |\tau'(y)|^{-1} f(y) = f(x) = (f \circ \pi)(\underline{x}),$$

where $x = \pi(\underline{x})$ and $y = \pi(\underline{y})$

Analogously, we can prove that $\mathscr{L}_\varphi^*(m \circ \pi^{-1}) = m \circ \pi^{-1}$ From the above it follows that $\mu_\varphi = \mu \circ \pi^{-1}$ □

PROPOSITION 2 *In the notation of Proposition* 1, *the function*

$$g = \frac{f \circ \pi}{|\tau' \circ \pi|(f \circ \pi \circ \sigma)}$$

satisfies

$$\sum_{\underline{y}\ \sigma\underline{y}=\underline{x}} g(\underline{y}) = 1$$

for any $\underline{x} \in \Sigma_{\mathbb{A}}^+$

Proof Follows by the τ-invariance of f □

3 *Blocks and paths*

Let $\mathbb{A}$ be an $n \times n$ irreducible 0–1 matrix with consecutive non-zero entries in each row A digraph having n vertices can be associated with such a matrix We now partition the n vertices into blocks

Definition 1 Vertices j and j^* belong to the same block B if and only if there exist integers $i_1, \quad , i_k$ and $j_1, \quad , j_k$ such that

$$a_{i_p j_p} = 1 \quad 1 \le p \le k,$$

where $j_1 = j$, $i_2 = i_1, j_3 = j_2, i_4 = i_3, \quad , i_k = i_{k-1}, j_k = j^*$

Example 1

$$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

There are two blocks $B_1 = \{1\}$ and $B_2 = \{2, 3, 4\}$

A convenient equivalent way of determining the blocks can be described as follows let C_j denote the positions of the non-zero entries in the jth row and let c_j, d_j denote the two extreme integers in C_j We say C_j and C_j^* combine if either

$\{c_j, d_j\} \cap C_j^* \neq \varnothing$ or $\{c_j^*, d_j^*\} \cap C_j \neq \varnothing$ Then we define $C_{jj}^* = C_j \cup C_j^*$ We continue this process until we obtain a maximal set This set is a block

In this way we can find the blocks by examining the matrix $\mathbb{A}$ In example 1, we have $C_1 = \{3, 4\}$ and $C_2 = \{2, 3\}$ Since C_1 and C_2 combine, $\{2, 3, 4\}$ is a block

Let us assume that we have $m \leq n$ blocks: $B_1, \ldots, B_m$

Definition 2 We say there is a path P of length p from B_l to B_k if and only if for some vertex $i \in B_l$ and for some vertex $j \in B_k$ there exist integers $i_1, i_2, \ldots, i_{p-1}, j_1, \ldots, j_{p-1}$, where i_s and j_s are in the same block $(1 \leq s \leq p-1)$, such that

$$a_{ij_1} = 1,$$
$$a_{i_1 j_2} = 1,$$
$$\vdots$$
$$a_{i_{p-1} j} = 1$$

This path will be denoted by (P, p)

Let us now associate with any vertex i a number $s_i > 0$, $i = 1, 2, \ldots, n$ We associate with a path (P, p) a number

$$\gamma(P) = s_i s_{i_1} s_{i_2} \cdots s_{i_{p-1}},$$

which we shall refer to as a path product associated with a path (P, p) With a path of length 0, we associate the path product 1

We shall now present a system of equations associated with the matrix $\mathbb{A}$ We consider all paths between blocks $B_1, \ldots, B_m$ of $\mathbb{A}$, including paths of length 0 Whenever there are two paths (P_1, p_1), (P_2, p_2) with the same starting and ending blocks, we write the equation

$$\gamma(P_1)/\lambda^{p_1} = \gamma(P_2)/\lambda^{p_2} \tag{1}$$

The system of all such equations will be referred to as the system of structural equations of $\mathbb{A}$

4 *Main theorem*

Let us consider $\tau \in \mathfrak{C}_{\mathbf{A}}$ Let s_i be the slope of τ on the interval $I_i \in \mathfrak{P}$ The main result of this note is

THEOREM 2 *$\tau \in \mathfrak{C}_{\max}$ if and only if the slopes $(s_1, s_2, \ldots, s_n)$ of τ satisfy the system of structural equations of $\mathbb{A}$*

Proof First we shall prove that this is a necessary condition Let $\tau \in \mathscr{C}_{\max}$, and let τ_λ denote the transformation with constant slope λ Clearly, $\tau_\lambda \in \mathfrak{C}_{\max}$

As in Proposition 1, we construct isomorphisms

$$\pi_1 \ (\Sigma_{\mathbf{A}}^+, \sigma, \mu_1) \to (\mathbf{I}, \tau, \mu)$$
$$\pi_2 \ (\Sigma_{\mathbf{A}}^+, \sigma, \mu_2) \to (\mathbf{I}, \tau_\lambda, \mu_\lambda)$$

Since the measures μ and μ_λ maximize measure theoretic entropy, the measures μ_1, μ_2 also maximize entropy and by uniqueness of the equilibrium state, $\mu_1 = \mu_2$

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By Propositions 1, 2 and property (5) of Theorem 1, we obtain

$$\frac{f_\lambda \circ \pi_2}{\lambda (f_\lambda \circ \pi_2 \circ \sigma)} = \frac{f \circ \pi_1}{|\tau' \circ \pi_1|(f \circ \pi_1 \circ \sigma)},$$

which can be written more conveniently in the form

$$\frac{|\tau' \circ \pi_1|}{\lambda} = \left(\frac{f \circ \pi_1}{f_\lambda \circ \pi_2}\right)\left(\frac{f_\lambda \circ \pi_2 \circ \sigma}{f \circ \pi_1 \circ \sigma}\right) \tag{2}$$

Equation (2) will be used extensively in the sequel

Let w_i be the value of $(f \circ \pi_1 / f_\lambda \circ \pi_2)$ on the cylinder sets $(x_0 = i)$, $i = 1, \quad , n$ We recall that s_i is the value of $|\tau' \circ \pi_1|$ on $(x_0 = i)$, $i = 1, \quad , n$ First we shall prove that w_i is the same for all i's in the same block

Let j and j^* belong to the same block By definition, there exist integers $i_1, \quad , i_k$ and $j_1, \quad , j_k$ such that

$$a_{i_p j_p} = 1, \quad 1 \le p \le k,$$

and $j_1 = j$, $i_2 = i_1, j_3 = j_2, i_4 = i_3, \quad , i_k = i_{k-1}, j_k = j^*$ Using (2), $i_2 = i_1$, $a_{i_1 j_1} = 1$ and $a_{i_2 j_2} = 1$, we get

$$s_{i_1}/\lambda = w_{i_1}/w_{j_1}$$
$$s_{i_1}/\lambda = w_{i_1}/w_{j_2}$$

Therefore, $w_{j_1} = w_{j_2}$ Since $j_3 = j_2$, we get $w_{j_1} = w_{j_2} = w_{j_3}$ We now proceed by this argument

Let $(\boldsymbol{P}_1, p_1)$ and $(\boldsymbol{P}_2, p_2)$ be two paths from B_l to B_k The existence of $(\boldsymbol{P}_1, p_1)$ and (2) imply

$$s_i/\lambda = w_i/w_{j_1}$$
$$s_{i_1}/\lambda = w_{i_1}/w_{j_2}$$

$$s_{i_{p_1-1}}/\lambda = w_{i_{p_1-1}}/w_j$$

The existence of $(\boldsymbol{P}_2, p_2)$ and (2) imply

$$s_i/\lambda = w_i/w_{j_1^*}$$
$$s_{i_1^*}/\lambda = w_{i_1^*}/w_{j_2^*}$$

$$s_{i_{p_1-1}^*}/\lambda = w_{i_{p_1-1}^*}/w_j$$

Since the pairs $i_1, j_1, \quad , i_{p_1-1}, j_{p_1-1}, i_1^*, j_1^*, \quad , i_{p_2-1}^*, j_{p_2-1}^*$ belong to the same blocks,

$$w_{i_1} = w_{j_1}, \quad , w_{i_{p_1-1}} = w_{j_{p_1-1}} \quad w_{i_1^*} = w_{j_1^*}, \quad , w_{i_{p_2-1}^*} = w_{j_{p_2-1}^*},$$

and the above set of equations yields

$$\gamma(\boldsymbol{P}_1)/\lambda^{p_1} = \gamma(\boldsymbol{P}_2)/\lambda^{p_2}$$

This completes the necessity part of the proof

We shall now prove that the system of structural equations of $\mathbb{A}$ provides a sufficient condition for τ to belong to $\mathfrak{C}_{\max}$ By property (3) of Theorem 1, it is

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enough to construct a continuous function $t\ \Sigma_{\mathbb{A}}^{+}\to\mathbb{R}$ such that

$$|\tau'\circ\pi_1|/\lambda = t/t\circ\sigma$$

We will construct this function using the structural equations of $\mathbb{A}$ This will be done by defining t as constants t_i on the cylinder sets $(x_0=i)$, $i=1,2,\quad,n$ The function t will be made constant on blocks of $\mathbb{A}$

Let $B_1=\{i_1,\quad,i_k\}$ be a block We put $t_i=a$ for all $i\in B_1$, where a is a fixed real number Now we consider those j's, $1\le j\le n$, such that $a_{i_rj}=1$, $i_r\in B_1$ For any such j, we define

$$t_j=t_{i_r}\lambda/s_{i_r}$$

Moreover, we define $t_j^*=t_j$ for all j^* for which j^* and j are in the same block We now prove that this assignment of values is consistent The only situation in which a contradiction can occur is if there are two different ways of reaching the same block $a_{i_{r_1}j_1}=1$ and $a_{i_{r_2}j_2}=1$, where j_1 and j_2 are in the same block But then from (2) we obtain

$$s_{i_{r_1}}/\lambda=s_{i_{r_2}}/\lambda$$

and there is no contradiction

Proceeding in this way, we can define values for t on all cylinder sets At every step we check that there is no contradiction in defining the t_i's Such a contradiction can occur only if we use two different paths between the same starting and ending blocks,

$$B_{l_1}\to B_{l_2}\to\quad\to B_{l_v},$$

$$B_{k_1}\to B_{k_2}\to\quad\to B_{k_w},$$

$(l_1=k_1,\ l_v=k_w)$ to define $t_{j(l_v)}$, where $j(l_v)\in B_{l_v}$, and $t_{j^*(k_w)}$, where $j^*(k_w)\in B_{k_w}=B_{l_v}$ We have

$$i(l_1)\in B_{l_1},\quad j(l_2),\ i(l_2)\in B_{l_2},\quad ,j(l_{v-1}),\quad i(l_{v-1})\in B_{l_{v-1}},$$

$$j(l_v)\in B_{l_v},\quad i^*(k_1)\in B_{k_1},\quad j^*(k_2),\ i^*(k_2)\in B_{k_2},\quad ,j^*(k_{w-1}),$$

$$i^*(k_{w-1})\in B_{k_{w-1}},\quad j^*(k_w)\in B_{k_w},$$

and

$$t_j(l_2)=t_{i(l_1)}\lambda/s_{i(l_1)}\quad t_{j(l_3)}=t_{i(l_2)}\lambda/s_{i(l_2)},\quad ,$$

$$t_{j(l_v)}=t_{i(l_{v-1})}\lambda/s_{i(l_{v-1})},$$

$$t_{j^*(k_2)}=t_{i^*(k_1)}\lambda/s_{i^*(k_1)},\quad t_{j^*(k_3)}=t_{i^*(k_2)}\lambda/s_{i^*(k_2)},\quad ,$$

$$t_{j^*(k_w)}=t_{i^*(k_{w-1})}\lambda/s_{i^*(k_{w-1})}$$

On the other hand we have the structural equation

$$s_{i(l_1)}s_{i(l_2)}\quad s_{i(l_{v-1})}/\lambda^{v-1}=s_{i^*(k_1)}s_{i^*(k_2)}\quad s_{i^*(k_{w-1})}/\lambda^{w-1}$$

Hence $t_{j(l_v)}=t_{j^*(k_w)}$ This proves that the function t is well defined and thus proves the sufficiency part of the theorem □

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COROLLARY 1 *If the matrix* $\mathbb{A}$ *admits exactly one block, then the piecewise linear Markov transformations compatible with* $\mathbb{A}$ *for which the* a c i m *is maximal are precisely those of constant slope* λ

5 *Consequences of the Main Theorem*

Let $\tau \in \mathfrak{C}_{\mathbb{A}}$ be a transitive piecewise linear Markov transformation (the transitivity of τ is equivalent to the irreducibility of $\mathbb{A}$) It is known that τ is conjugate via a homeomorphism Φ to a transformation τ_λ of constant slope [**12**] Using our main theorem we now prove some properties of Φ

THEOREM 3 *If* Φ $(\mathbf{I}, \tau) \to (\mathbf{I}, \tau_\lambda)$ *is a topological conjugation, then* Φ *is absolutely continuous if and only if the slopes of* τ *satisfy the system of structural equations*

Proof If the slopes of τ satisfy the system of structural equations, then the absolutely continuous τ-invariant measure is maximal As a maximal measure, it is equal to $\mu_\lambda \circ \Phi^{-1}$, where μ_λ is an a c i m for τ_λ Hence Φ is absolutely continuous

If the slopes of τ do not satisfy the structural equations, then μ is not maximal Hence the maximal measure $\mu_\lambda \circ \Phi^{-1}$ is different from μ and so it is singular (since μ is ergodic) Hence Φ is singular □

Example 2 For the matrix

$$\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

the structural equations are $s_1 = \lambda$, $s_2 = \lambda$, $\lambda = 2$ Let, for $0 < \alpha < 1$

$$\tau^{(\alpha)}(x) = \begin{cases} x/\alpha & 0 \le x \le \alpha \\ (1-x)/(1-\alpha) & \alpha < x \le 1 \end{cases}$$

By Theorem 3 we obtain that for any $\alpha \neq \frac{1}{2}$ the conjugacy Φ_α between $\tau^{(\alpha)}$ and $\tau^{(1/2)}$ is singular

This result is proved by a completely different method in [**3**]

Remark The method of § 4 has, in effect, solved an optimization problem which is difficult to handle by the standard method of Lagrange multipliers To write the metric entropy in a manner that would be amenable to Lagrange multipliers would require defining $3n$ variables The natural constraints yield $2n+2$ equations The large number of variables and the nonlinear appearance of these variables renders this approach intractable

6 *Examples*

In the following examples we shall illustrate a number of points in regard to the foregoing theory

Example 3 There exists a transformation τ $\mathbf{I} \to \mathbf{I}$ such that no iterate τ^n of τ has constant slope, yet $\tau \in \mathfrak{C}_{\max}$ Let

$$\mathbb{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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and let $\mathcal{P}=\{\mathbf{I}_1,\mathbf{I}_2,\mathbf{I}_3\}$ be the partition of $\mathbf{I}$ It is easy to see that the sets $D=\mathbf{I}_3\cap(\bigcap_{k=1}^{n}\tau^{-k}(\mathbf{I}_3))$ and $E=\mathbf{I}_1\cap(\bigcap_{k=1}^{n}\tau^{-k}(\mathbf{I}_3))$ are non-empty and $|(\tau^n)'|_{|D}=s_3^n$, while $|(\tau^n)'|_{|E}=s_1s_3^{n-1}$, where $s_i=|\tau'|_{|\mathbf{I}_i}$, $i=1,2,3$

Now the system of structural equations for $\mathbb{A}$ is

$$s_3=\lambda \quad\text{and}\quad s_1s_2=\lambda^2$$

If we choose $s_1\neq s_2$, then

$$|(\tau^n)'|_{|D}\neq|(\tau^n)'|_{|E}$$

Hence τ^n does not have constant slope, yet $\tau\in\mathfrak{C}_{\max}$

By Corollary 1 we see that if $\mathbb{A}$ has one block only, the system of structural equations of $\mathbb{A}$ only yields the constant slope solutions In general, the number of degrees of freedom among the variables $\{s_1,s_2,\quad,s_n\}$ can vary from 0 (as in the case of a single block) to $n-1$ We shall show this by means of the following three examples

Example 4 One degree of freedom Let

$$\mathbb{A}=\begin{bmatrix}0&0&1&1\\0&0&0&1\\0&0&0&1\\1&1&0&0\end{bmatrix}$$

There are two blocks $B_1=\{1,2\}$ and $B_2=\{3,4\}$, and the structural equations are

$$s_1/\lambda=s_2/\lambda \quad s_3/\lambda=1 \quad s_1s_4/\lambda^2=1,$$

which yield $s_1=s_2$, $s_3=\lambda$, $s_1s_4=\lambda^2$ There are 4 unknowns and three independent equations Hence there is one degree of freedom

Example 5 Three degrees of freedom Let

$$\mathbb{A}=\begin{bmatrix}0&0&0&1&1\\0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\1&1&0&0&0\end{bmatrix}$$

There are two independent structural equations

$$s_1s_5=\lambda^2 \quad\text{and}\quad s_2s_3s_4=\lambda^3$$

Hence there are 3 degrees of freedom

Example 6 $n-1$ degrees of freedom Let $\mathbb{A}$ be an $n\times n$ irreducible matrix with only one non-zero entry in each row Then there is only one structural equation Since there are n unknown slopes, there are $n-1$ degrees of freedom

Remarks

(1) The matrices in the foregoing examples are all primitive, indicating that primitivity and the degree of freedom are not dependent on each other

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(2) If $\mathbb{A}$ has one block only, we obtain a unique solution Conversely, if $\tau \in \mathfrak{C}_{\max}$ only when τ has constant slope λ, it is easy to see that $\mathbb{A}$ must consist of a unique block

(3) In order to determine $\mathfrak{C}_{max}$, we need to find the lengths of the intervals of the partition $\mathfrak{P}$ associated with a given set of slopes This is accomplished by means of the Frobenius–Perron operator Consider Example 4 The Frobenius–Perron operator is given by the matrix

$$\mathbb{M} = \begin{bmatrix} 0 & 0 & 1/s_1 & 1/s_1 \\ 0 & 0 & 0 & 1/s_1 \\ 0 & 0 & 0 & 1/s_3 \\ 1/s_4 & 1/s_4 & 0 & 0 \end{bmatrix}$$

The lengths of the partition of $\tau \in \mathfrak{C}_{\max}$ are given by the normalized right eigenvector $\hat{l}$ of $\mathbb{M}$, i e $\mathbb{M}\hat{l} = \hat{l}$, where $\hat{l} = (l_1, l_2, l_3, l_4)$ Thus,

$$(l_3 + l_4)/s_1 = l_1$$
$$l_4/s_1 = l_2$$
$$l_4/s_3 = l_3$$
$$(l_1 + l_2)/s_4 = l_4$$

subject to $l_1 + l_2 + l_3 + l_4 = 1$ Solving this system, we obtain

$$l_1 = (s_4 - 1/s_1)l_4 \quad l_2 = l_4/s_1 \quad l_3 = l_4/s_3$$

On normalizing, we get

$$l_4 = (1 + 1/\lambda + s_4)^{-1}$$

Thus, there is also one degree of freedom in the partition

7 *Reducible matrices*

To motivate the material of this section, consider the 4×4 reducible matrix

$$\mathbb{A} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbb{A}_{11}} & 0 \\ \hline 1 & 1 & \\ 0 & 0 & \mathbb{A}_{22} \end{array}\right],$$

where

$$\mathbb{A}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbb{A}_{22} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The maximal eigenvalue of $\mathbb{A}$ is the same as the maximal eigenvalue of $\mathbb{A}_{22}$ which is equal to 2 Therefore, $h_{\text{top}}(\tau) = \ln 2$ On the other hand, the support of any τ-invariant absolutely continuous measure is on the intervals corresponding to the block $\mathbb{A}_{11}$ [**5**] Hence, for any τ-invariant absolutely continuous measure μ, we have

$$h_\mu(\tau) \le \ln \lambda_1 < h_{\text{top}}(\tau),$$

where $\lambda_1 < 2$ is the maximal eigenvalue of $\mathbb{A}_{11}$

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The above example illustrates the difference between topological and metric entropy Whereas $h_{\text{top}}(\tau)$ is determined by the submatrix $\mathbb{A}_{22}$, the metric entropy $h_\mu(\tau)$ for any a c i m μ is determined by $\mathbb{A}_{11}$

Let $\mathbb{A}$ be an $n \times n$ reductible 0-1 matrix such that all the non-zero entries in each row are consecutive By relabelling, $\mathbb{A}$ can be written in the form

$$\mathbb{A}' = \begin{bmatrix} \mathbb{A}_{11} & & & & \\ & \mathbb{A}_{22} & & 0 & \\ & & & & \\ \mathbb{A}_{i1} & \mathbb{A}_{i2} & \mathbb{A}_{ii} & & \\ & & & & \\ \mathbb{A}_{k1} & \mathbb{A}_{k2} & & & \mathbb{A}_{kk} \end{bmatrix}$$

where $\mathbb{A}_{ii}$ are irreducible submatrices Let λ be the maximal eigenvalue of $\mathbb{A}$ Since $\mathbb{A}'$ is derived from $\mathbb{A}$ by a similarity transformation, λ is also the maximal eigenvalue of $\mathbb{A}'$ From this it follows that at least one of the $\mathbb{A}_{ii}$'s has λ as its maximal eigenvalue

Definition 3 We say the irreducible submatrix $\mathbb{A}_{ii}$ is final if $\mathbb{A}_{ij} \equiv 0$ for $j \neq i$

THEOREM 4 *Suppose $\mathbb{A}$ is reducible Then one of the following must occur*

(1) *there exists no final irreducible submatrix $\mathbb{A}_{ii}$ having maximal eigenvalue λ In this case, $\mathfrak{C}_{\max} = \varnothing$*

(2) *There exist final irreducible submatrices $\mathbb{A}_{i_1 i_1}, \ldots, \mathbb{A}_{i_m i_m}$ such of which has maximal eigenvalue λ Then $\tau \in \mathfrak{C}_{\max}$ if and only if there exists some i, $i_1 \le i \le i_m$, such that τ satisfies the structural equations associated with $\mathbb{A}_{ii}$ on the intervals corresponding to $\mathbb{A}_{ii}$*

Proof The topological entropy of τ is equal to $\ln \lambda$ By graph theoretic methods [**6**], it can be shown that any ergodic a c i m must have support on intervals corresponding to a final irreducible submatrix Then the proof of (1) is obvious since $h_\mu(\tau) < \ln \lambda$ for all a c i m μ To prove (2) we choose any final irreducible submatrix, say $\mathbb{A}_{ii}$, which has maximal eigenvalue λ On the intervals corresponding to $\mathbb{A}_{ii}$ we proceed as in section 4 to construct τ using the structural equations derived from $\mathbb{A}_{ii}$ □

REFERENCES

[1] P Walters Ruelle s operator theorem and g-measures *Trans Amer Math Soc* **214** (1975), 375-387

[2] F Ledrappier Some properties of absolutely continuous invariant measures on the interval *Ergod Th & Dynam Sys* **1** (1981), 77-93

[3] H Proppe, W Byers & A Boyarsky Singularity of topological conjugacies between certain unimodal maps of the interval *Israel J Math* **44** (1983), 277-288

[4] W Byers & A Boyarsky Absolutely continuous invariant measures that are maximal *Trans Amer Math Soc* **290** (1985), 303-314

[5] W Byers & A Boyarsky A graph theoretic bound on the number of independent absolutely continuous invariant measures *J Math Anal Appl* **139** (1989), 139-151

[6] F Hofbauer β-shifts have unique maximal measure *Monatsh Math* **85** (1978), 189-198

[7] F Hofbauer Maximal measures for piecewise monotonically increasing functions on [0, 1] In *Lecture Notes in Math* **729**, Springer, Berlin and New York (1979), 66-77

[8] F Hofbauer On intrinsic ergodicity of piecewise monotonic transformations with positive entropy *Israel J Math* **34** (1979), 213-237

https://doi.org/10.1017/S0143385700005824 Published online by Cambridge University Press

[9] M Misiurewicz Absolutely continuous measures for certain maps of an interval *Publ Math IHES* **53** (1981), 17-52
[10] A Lasota & J A Yorke On the existence of invariant measures for piecewise monotonic transformations *Trans Amer Math Soc* **186** (1974), 481-488
[11] R Bowen Invariant measures for Markov maps of the interval *Comm Math Phys* **69** (1979), 1-17
[12] W Parry Intrinsic Markov chains *Trans Amer Math Soc* **112** (1964), 55-65
[13] P Walters *An Introduction to Ergodic Theory* Springer-Verlag, New York (1982)