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THE AUGMENTED BASE LOCUS IN POSITIVE CHARACTERISTIC

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Abstract Let L be a nef line bundle on a projective scheme X in positive characteristic. We prove that the augmented base locus of L is equal to the union of the irreducible closed subsets V of X such that $L|_V$ is not big. For a smooth variety in characteristic 0, this was proved by Nakamaye using vanishing theorems.

Keywords: stable base locus; augmented base locus; big and nef line bundle

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1. Introduction

Let X be a projective scheme over an algebraically closed field k, and let L be a line bundle on X. The base locus $\mathsf{Bs}(L)$ of L is the closed subset of X consisting of those $x \in X$ such that every section of L vanishes at x. It is easy to see that if m_1 and m_2 are positive integers such that m_1 divides m_2, then $\mathsf{Bs}(L^{m_2}) \subseteq \mathsf{Bs}(L^{m_1})$. It follows from the noetherian property that $\mathsf{Bs}(L^m)$ is independent of m if m is divisible enough; this is the *stable base locus* $\mathsf{SB}(L)$ of L.

The stable base locus is a very interesting geometric invariant of L, but it is quite subtle; for example, there exist numerically equivalent Cartier divisors whose stable base loci are different. Nakamaye introduced in [**5**] the following upper approximation of $\mathsf{SB}(L)$, the *augmented base locus* $\mathsf{B}_+(L)$. If $L \in \operatorname{Pic}(X)$ and $A \in \operatorname{Pic}(X)$ is ample, then

$$\mathsf{B}_+(L) := \mathsf{SB}(L^m \otimes A^{-1})$$

for $m \gg 0$. It is easy to check that this is well defined, is independent of A, and only depends on the numerical equivalence class of L. The following is our main result.

Theorem 1.1. *Let X be a projective scheme over an algebraically closed field of positive characteristic. If L is a nef line bundle on X, then $\mathsf{B}_+(L)$ is equal to $L^{\perp}$, the union of all irreducible closed subsets V of X such that $L|_V$ is not big.*

We note that since L is nef, for an irreducible closed subset V of X, the restriction $L|_V$ is not big if and only if V has positive dimension and $(L|_V^{\dim(V)}) = 0$. When X is a smooth projective variety in characteristic 0, the above theorem was proved in [**5**], making use of the Kawamata–Viehweg vanishing theorem. It is an interesting question as to whether the result holds in characteristic 0 when the variety is singular.

The proof of Theorem 1.1 makes use, in an essential way, of the Frobenius morphism. The following is a key ingredient in the proof.

Theorem 1.2. *Let X be a projective scheme over an algebraically closed field of positive characteristic. If L is a nef line bundle on X and D is an effective Cartier divisor such that $L(-D)$ is ample, then $\mathsf{B}_+(L) = \mathsf{B}_+(L|_D)$.*

In the proofs of the above results we make use of techniques introduced by Keel in [**2**]. In fact, if we replace in Theorem 1.2 the two augmented base loci by the corresponding stable base loci, we recover one of the main results in [**2**]. We give a somewhat simplified proof of this result (see Corollary 3.6), and this proof extends to also give Theorem 1.2.

In the next section we recall some basic facts about augmented base loci. The proofs of Theorems 1.2 and 1.1 are then given in § 3.

2. Augmented base loci and big line bundles

In this section we review some basic facts about the augmented base locus. This notion is usually defined for integral schemes. However, even if one is only interested in this restrictive setting, for the proof of Theorem 1.1 we need to also consider possibly reducible, or even non-reduced, schemes. We, therefore, define the augmented base locus in the more general setting that we need. Its general properties follow as in the case of integral schemes, for which we refer the reader to [**1**].

Let X be a projective scheme over an algebraically closed field k. If L is a line bundle on X and $s \in H^0(X, L)$, then we denote by $Z(s)$ the zero locus of s (with the obvious scheme structure). Note that $Z(s)$ is defined by a locally principal ideal, but in general it is not an effective Cartier divisor (if X is reduced, then $Z(s)$ is an effective Cartier divisor if and only if no irreducible component of X is contained in $Z(s)$). The base locus of L is by definition the closed subset of X given by

$$\mathsf{Bs}(L) := \bigcap_{s \in H^0(X,L)} Z(s)_{\mathrm{red}}.$$

If m is a positive integer and $s \in H^0(X, L)$, then it is clear that $Z(s)_{\mathrm{red}} = Z(s^{\otimes m})_{\mathrm{red}}$; hence, $\mathsf{Bs}(L^m) \subseteq \mathsf{Bs}(L)$. More generally, we have $\mathsf{Bs}(L^{mr}) \subseteq \mathsf{Bs}(L^r)$ for every $m, r \geqslant 1$; hence, by the noetherian property there exists $m_0 \geqslant 1$ such that

$$\mathsf{SB}(L) := \bigcap_{r \geqslant 1} \mathsf{Bs}(L^r)$$

is equal to $\mathsf{Bs}(L^m)$ whenever m is divisible by m_0. The closed subset $\mathsf{SB}(L)$ of X is the *stable base locus* of L. It follows by definition that $\mathsf{SB}(L) = \mathsf{SB}(L^r)$ for every $r \geqslant 1$.

Since X is projective, every line bundle is of the form $\mathcal{O}_X(D)$, for some Cartier divisor D (see [**4**]). We sometimes find it convenient to work with Cartier divisors, rather than line bundles. Let $\mathrm{Cart}(X)_{\mathbb{Q}} := \mathrm{Cart}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the group of Cartier $\mathbb{Q}$-divisors and let $\mathrm{Pic}(X)_{\mathbb{Q}} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. For a Cartier divisor D, we set $\mathsf{SB}(D) = \mathsf{SB}(\mathcal{O}_X(D))$. Since $\mathsf{SB}(D) = \mathsf{SB}(rD)$ for every $r \geqslant 1$, the definition extends in the obvious way to $\mathrm{Cart}(X)_{\mathbb{Q}}$.

Given a Cartier $\mathbb{Q}$-divisor D, the augmented base locus of D is

$$\mathsf{B}_+(D) := \bigcap_A \mathsf{SB}(D - A),$$

where the intersection is over all ample Cartier $\mathbb{Q}$-divisors on X. It is easy to see that if A_1 and A_2 are ample Cartier $\mathbb{Q}$-divisors such that $A_1 - A_2$ is ample, then $\mathsf{SB}(D - A_2) \subseteq \mathsf{SB}(D - A_1)$. It follows from the noetherian property that there exists an ample Cartier $\mathbb{Q}$-divisor A such that $\mathsf{B}_+(D) = \mathsf{SB}(D - A)$. Furthermore, in this case, if A' is ample and $A - A'$ is also ample, then $\mathsf{B}_+(D) = \mathsf{SB}(D - A')$. It is then clear that if H is any ample Cartier divisor on X, then for $m \gg 0$ we have that

$$\mathsf{B}_+(D) = \mathsf{SB}\left(D - \frac{1}{m}H\right) = \mathsf{SB}(mD - H).$$

The following properties of the augmented base locus are direct consequences of the definition (see [**1**, § 1]).

(1) For every Cartier $\mathbb{Q}$-divisor D, we have $\mathsf{SB}(D) \subseteq \mathsf{B}_+(D)$.

(2) If D_1 and D_2 are numerically equivalent Cartier $\mathbb{Q}$-divisors, then $\mathsf{B}_+(D_1) = \mathsf{B}_+(D_2)$.

If D is a Cartier divisor and $L = \mathcal{O}_X(D)$, we also write $\mathsf{B}_+(L)$ for $\mathsf{B}_+(D)$.

Lemma 2.1. *If L is a line bundle on the projective scheme X, and Y is a closed subscheme of X, then*

(i) $\mathsf{SB}(L|_Y) \subseteq \mathsf{SB}(L)$,

(ii) $\mathsf{B}_+(L|_Y) \subseteq \mathsf{B}_+(L)$.

Proof. The first assertion follows from the fact that if $s \in H^0(X, L)$, then $Z(s|_Y) \subseteq Z(s)$; hence, $\mathsf{Bs}(L^m|_Y) \subseteq \mathsf{Bs}(L^m)$ for every $m \geqslant 1$. For the second assertion, fix an ample line bundle A on X, and let $m \gg 0$ be such that $\mathsf{B}_+(L) = \mathsf{SB}(L^m \otimes A^{-1})$. Since $A|_Y$ is ample on Y, using (i) and the definition of the augmented base locus of $L|_Y$, we obtain that

$$\mathsf{B}_+(L|_Y) \subseteq \mathsf{SB}(L^m|_Y \otimes A^{-1}|_Y) \subseteq \mathsf{SB}(L^m \otimes A^{-1}) = \mathsf{B}_+(L).$$

$\square$

Recall that a line bundle L on an integral n-dimensional scheme X is *big* if there exists $C > 0$ such that $h^0(X, L^m) \geqslant Cm^n$ for $m \gg 0$. Equivalently, this is the case if and only if there exist Cartier divisors A and E, with A ample and E effective, such that $L^m \simeq \mathcal{O}_X(A + E)$ for some $m \geqslant 1$. We refer the reader to [**3**, § 2.2] for basic facts about big line bundles on integral schemes. The following lemma is well known, but we include a proof for completeness.

Lemma 2.2. *Let X be an n-dimensional projective scheme and let L be a line bundle on X. For every coherent sheaf $\mathcal{F}$ on X, there exists $C > 0$ such that $h^0(X, \mathcal{F} \otimes L^m) \leqslant Cm^n$ for every $m \geqslant 1$.*

Proof. We write $L \simeq A \otimes B^{-1}$ for suitable very ample line bundles A and B. For every $m \geqslant 1$, the line bundle B^m is very ample. By choosing a section $s_m \in H^0(X, B^m)$ such that $Z(s_m)$ does not contain any of the associated subvarieties of $\mathcal{F}$, we obtain an inclusion $H^0(X, \mathcal{F} \otimes L^m) \hookrightarrow H^0(X, \mathcal{F} \otimes A^m)$. Since $h^0(X, \mathcal{F} \otimes A^m) = P(m)$ for $m \gg 0$, where P is a polynomial of degree less than or equal to n, we obtain the assertion in the lemma. □

If X is reduced, and A, D are Cartier divisors on X, with A ample and D effective, then the restriction of $\mathcal{O}_X(A+D)$ to every irreducible component Y of X is big (note that the restriction $D|_Y$ is well defined and gives an effective divisor on Y). As a consequence of the next lemma, we obtain a converse to this statement.

Lemma 2.3. *Let X be a reduced projective scheme. Given the line bundles L and A on X, with A ample, if $m \gg 0$ and $s \in H^0(X, L^m \otimes A^{-1})$ is general, then, for every irreducible component Y of X such that $L|_Y$ is big, we have $Y \not\subseteq Z(s)$.*

Note that since A is ample, if $s \in H^0(X, L^m \otimes A^{-1})$ and Y' is an irreducible component of X (considered with the reduced scheme structure) such that $L|_{Y'}$ is not big, then $Y' \subseteq Z(s)$.

Proof of Lemma 2.3. Suppose that Y is an irreducible component of X (considered with the reduced structure) such that $L|_Y$ is big, but such that, for infinitely many m, we have $Y \subseteq Z(s)$ for every $s \in H^0(X, L^m \otimes A^{-1})$. If W is the union of the other irreducible components of X, also considered with the reduced scheme structure, then we have an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_W \to \mathcal{O}_{Y \cap W} \to 0,$$

where $Y \cap W$ denotes the (possibly non-reduced) scheme-theoretic intersection of Y and W. After tensoring with $L^m \otimes A^{-1}$ and taking global sections, this induces the exact sequence

$$\begin{aligned} 0 \to H^0(X, L^m \otimes A^{-1}) \to H^0(Y, L^m \otimes A^{-1}|_Y) \oplus H^0(W, L^m \otimes A^{-1}|_W) \\ \to H^0(Y \cap W, L^m \otimes A^{-1}|_{Y \cap W}). \end{aligned}$$

By assumption, the map $H^0(X, L^m \otimes A^{-1}) \to H^0(Y, L^m \otimes A^{-1}|_Y)$ is 0 for infinitely many m, in which case the above exact sequence implies that

$$h^0(Y, L^m \otimes A^{-1}|_Y) \leqslant h^0(T, L^m \otimes A^{-1}|_T). \tag{2.1}$$

Let $n = \dim(Y)$. Since $\dim(T) \leqslant n-1$, it follows from Lemma 2.2 that we can find $C > 0$ such that

$$h^0(T, L^m \otimes A^{-1}|_T) \leqslant Cm^{n-1}$$

for all m. On the other hand, since $L|_Y$ is big, it is easy to see that there exists $C' > 0$ such that $h^0(Y, L^m \otimes A^{-1}|_Y) \geqslant C'm^n$ for all $m \gg 0$. These two estimates contradict (2.1) when $m \gg 0$. □

Corollary 2.4. *Let L be a line bundle on the reduced projective scheme X. If the restriction of L to every irreducible component of X is big, then, for every ample line bundle A and every $m \gg 0$, the zero locus of a general section in $H^0(X, L^m \otimes A^{-1})$ defines an effective Cartier divisor on X.*

3. Main results

In this section we assume that all our schemes are of finite type over an algebraically closed field k of characteristic $p > 0$. For such a scheme X we denote by $F = F_X$ the absolute Frobenius morphism of X. This is the identity on the topological space, and it takes a section f of $\mathcal{O}_X$ to f^p. Note that F is a finite morphism of schemes (not preserving the structure of schemes over k). We also consider the iterates F^e of F, for $e \geqslant 1$.

We recall some basic facts concerning pullback of line bundles, sections and subschemes. Suppose that L is a line bundle on X and Z is a closed subscheme of X.

(1) There exists a canonical isomorphism of line bundles $(F^e)^*(L) \simeq L^{p^e}$.

(2) The scheme-theoretic inverse image $Z^{[e]} := (F^e)^{-1}(Z)$ is a closed subscheme of X defined by the ideal $I_Z^{[p^e]}$ such that if I_Z is locally generated by $(f_i)_i$, then $I_Z^{[p^e]}$ is generated by $(f_i^{p^e})_i$. In particular, if Y is another closed subscheme of X, having the same support as Z, there exists some e such that Y is a subscheme of $Z^{[e]}$.

(3) If $s \in H^0(Z, L|_Z)$, then $(F^e)^*(s)$ is a section in $H^0(Z^{[e]}, (F^e)^*(L)|_{Z^{[e]}})$, whose restriction to Z gets identified with $s^{\otimes p^e} \in H^0(Z, L^{p^e}|_Z)$.

Lemma 3.1. *If X is a projective scheme over k and L is a line bundle on X, then*

(i) $\mathsf{SB}(L) = \mathsf{SB}(L|_{X_{\mathrm{red}}})$,

(ii) $\mathsf{B}_+(L) = \mathsf{B}_+(L|_{X_{\mathrm{red}}})$.

Proof. The inclusions '$\supseteq$' in both (i) and (ii) follow from Lemma 2.1. We prove the reverse implication in (i). Let m be such that $\mathsf{SB}(L|_{X_{\mathrm{red}}}) = \mathsf{Bs}(L^m|_{X_{\mathrm{red}}})$. Given $x \in X$, suppose that $x \notin \mathsf{Bs}(L^m|_{X_{\mathrm{red}}})$. Consider $s \in H^0(X_{\mathrm{red}}, L^m|_{X_{\mathrm{red}}})$ such that $x \notin Z(s)$. Let

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J denote the ideal defining $X_{\rm red}$, and let $e \gg 0$ be such that $J^{[p^e]} = 0$. In this case $(F^e)^*(s)$ gives a section in $H^0(X, L^{mp^e})$ whose restriction to $X_{\rm red}$ is equal to $s^{\otimes p^e}$. In particular, $x \notin Z((F^e)^*(s))$. We conclude that $x \notin \mathsf{Bs}(L^{mp^e})$; hence, $x \notin \mathsf{SB}(L)$. This completes the proof of (i).

Let A be an ample line bundle on X, and let $m \gg 0$ be such that $\mathsf{B}_+(L|_{X_{\rm red}}) = \mathsf{SB}(L^m \otimes A^{-1}|_{X_{\rm red}})$ and $\mathsf{B}_+(L) = \mathsf{SB}(L^m \otimes A^{-1})$. The assertion in (ii) now follows by applying (i) to $L^m \otimes A^{-1}$. □

The following is a key result from [**2**]. We give a different proof, which has the advantage that it also applies when replacing the stable base loci by the augmented base loci.

Theorem 3.2. *If L is a nef line bundle on a projective scheme X, and D is an effective Cartier divisor on X such that $L(-D)$ is ample, then*

$$\mathsf{SB}(L) = \mathsf{SB}(L|_D).$$

We isolate the key point in the argument in a lemma that we will use several times.

Lemma 3.3. *Let A be an ample line bundle on a projective scheme X, and let D be an effective Cartier divisor on X. If $L := A \otimes \mathcal{O}_X(D)$ is nef, then, for every $m \geqslant 1$ and every section $s \in H^0(D, L^m|_D)$, there exists $e \geqslant 1$ such that $s^{\otimes p^e} \in H^0(D, L^{mp^e}|_D)$ is the restriction of a section in $H^0(X, L^{mp^e})$.*

Proof. Consider the short exact sequence

$$0 \to L^m(-D) \to L^m \to L^m|_D \to 0.$$

Pulling back by F^e gives the exact sequence

$$0 \to L^{mp^e}(-p^eD) \to L^{mp^e} \to L^{mp^e}|_{p^eD} \to 0.$$

Note that $L^m(-D) = L^{m-1} \otimes L(-D)$ is ample, since L is nef and $L(-D)$ is ample. By asymptotic Serre vanishing, we conclude that for $e \gg 0$ we have that

$$H^1(X, L^{mp^e}(-p^eD)) = 0,$$

and, therefore, the restriction map

$$H^0(X, L^{mp^e}) \to H^0(X, L^{mp^e}|_{p^eD})$$

is surjective. Therefore, there exists $t \in H^0(X, L^{mp^e})$ such that $t|_{p^eD} = (F^e)^*(s)$. In this case the restriction of t to D is equal to $s^{\otimes p^e}$. □

Proof of Theorem 3.2. It follows from Lemma 2.1 that it is enough to show that if P is a point on X that does not lie in $\mathsf{SB}(L|_D)$, then P does not lie in $\mathsf{SB}(L)$. If P does not lie on D, then it is clear that $P \notin \mathsf{SB}(L)$, since $A := L \otimes \mathcal{O}_X(-D)$ is ample. On the other hand, if $P \in D$, let $m \geqslant 1$ be such that there exists a section $s \in H^0(D, L^m|_D)$, with $P \notin Z(s)$. Since $Z(s^{\otimes p^e})_{\rm red} = Z(s)_{\rm red}$, in order to show that $P \notin \mathsf{SB}(L)$ it is enough to show that, for some e, the section $s^{\otimes p^e}$ lifts to a section in $H^0(X, L^{mp^e})$. This is a consequence of Lemma 3.3. □

Corollary 3.4. *Let X be a reduced projective scheme. If L and A are line bundles on X, with A ample and L nef, and $Z = Z(s)$ for some $s \in H^0(X, L \otimes A^{-1})$, then $\mathsf{SB}(L) = \mathsf{SB}(L|_Z)$.*

Proof. Let X' be the union of the irreducible components of X that are contained in Z, and let X'' be the union of the other components (we consider the reduced scheme structures on both X' and X''). If $X' = X$, then $Z = X$ and there is nothing to prove, while if $X' = \emptyset$, then Z is an effective Cartier divisor and the assertion follows from Theorem 3.2. Therefore, we may and will assume that both X' and X'' are non-empty.

Using the fact that A is ample and the definition of the stable base locus, we obtain $\mathsf{SB}(L) \subseteq \mathsf{SB}(L \otimes A^{-1}) \subseteq Z$. As in the proof of Theorem 3.2, we see that it is enough to show that if $t \in H^0(Z, L^m|_Z)$ for some m, then there exists $e \geqslant 1$ such that $t^{\otimes p^e}$ can be lifted to a section in $H^0(X, L^{mp^e})$. By applying Lemma 3.3 to X'', $D = Z \cap X''$ and the ample line bundle $L|_{X''} \otimes \mathcal{O}_{X''}(-D)$, we see that for some e we can lift $t^{\otimes p^e}|_{X'' \cap Z}$ to a section $t'' \in H^0(X'', L^{mp^e}|_{X''})$. Since $X' \subseteq Z$, the restriction of t'' to $X'' \cap X'$ is equal to $t^{\otimes p^e}|_{X' \cap X''}$. Therefore, we can glue $t^{\otimes p^e}|_{X'}$ with t'' to get a section in $H^0(X, L^{mp^e})$ lifting $t^{\otimes p^e}$. □

Recall that if L is a nef line bundle on the projective scheme X, then the *exceptional locus* $L^{\perp}$ is the union of all closed irreducible subsets $V \subseteq X$ such that $L|_V$ is not big. Since L is nef, this condition is equivalent to the fact that $\dim(V) > 0$ and $(L|_V^{\dim(V)}) = 0$.

Remark 3.5. It is easy to see by induction on $\dim(X)$ that $L^{\perp}$ is a closed subset of X. Note first that if $X_1, \ldots, X_r$ are the irreducible components of X (with the reduced scheme structures), then clearly $L^{\perp} = (L|_{X_1})^{\perp} \cup \cdots \cup (L|_{X_r})^{\perp}$. Therefore, we may assume that X is integral. In this case, if L is not big, then $L^{\perp} = X$. Otherwise, we can find an effective Cartier divisor D and a positive integer m such that $L^m(-D)$ is ample. It is clear that if $L|_V$ is not big, then $V \subseteq D$. Therefore, $L^{\perp} = (L|_D)^{\perp}$; hence, it is closed by induction.

The following result is one of the main results from [**2**]. As we see, this is an easy consequence of Corollary 3.4.

Corollary 3.6. *If L is a nef line bundle on the projective scheme X, then $\mathsf{SB}(L) = \mathsf{SB}(L|_{L^{\perp}})$.*

Proof. Arguing by noetherian induction, we may assume that the result holds for every proper closed subscheme of X. Since $L^{\perp} = (L|_{X_{\mathrm{red}}})^{\perp}$, it follows from Lemma 3.1 that we may assume that X is reduced. If the restriction of L to every irreducible component of X is not big, then $L^{\perp} = X$, and there is nothing to prove. From now on we assume that this is not the case, and let X'' (respectively, X') be the union of those irreducible components of X on which the restriction of L is big (respectively, is not big). On both X' and X'' we consider the reduced scheme structures. Note that by assumption X'' is non-empty.

Consider an ample line bundle A on X. It follows from Lemma 2.3 that if $m \gg 0$, there exists a section $s \in H^0(X, L^m \otimes A^{-1})$ such that no irreducible component of X''

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is contained in $Z = Z(s)$ (but such that $X' \subseteq Z$). It is clear that if V is an irreducible closed subset of X such that $L|_V$ is not big, then $V \subseteq Z$. Therefore, $L^{\perp} = (L|_Z)^{\perp}$. Since X'' is non-empty, it follows that $Z \neq X$; hence, the inductive assumption gives that $\mathsf{SB}(L|_Z) = \mathsf{SB}(L|_{L^{\perp}})$. On the other hand, Corollary 3.4 gives that

$$\mathsf{SB}(L) = \mathsf{SB}(L^m) = \mathsf{SB}(L^m|_Z) = \mathsf{SB}(L|_Z),$$

which completes the proof. $\square$

We can now prove the second theorem stated in § 1.

Proof of Theorem 1.2. We suitably modify the argument in the proof of Theorem 3.2. By Lemma 2.1, it is enough to prove the inclusion $\mathsf{B}_+(L) \subseteq \mathsf{B}_+(L|_D)$. Furthermore, Lemma 3.1 implies that $\mathsf{B}_+(L|_D) = \mathsf{B}_+(L|_{2D}) = \mathsf{B}_+(L^2|_{2D})$, and we have that $\mathsf{B}_+(L) = \mathsf{B}_+(L^2)$; hence, we may replace L by L^2 and D by $2D$ to assume that $L(-D) \simeq A^2$ for some ample line bundle A.

Suppose that P is a point that does not lie on $\mathsf{B}_+(L|_D)$. If $P \notin D$, since $L(-D)$ is ample, it follows that $P \notin \mathsf{B}_+(L)$. Hence, from now on we may assume that $P \in D$. By assumption, for $m \gg 0$ we have that $P \notin \mathsf{SB}(L^m \otimes A^{-1}|_D)$. We choose $r \geqslant 1$ such that there exists $t \in H^0(D, L^{rm} \otimes A^{-r}|_D)$ with $P \notin Z(t)$. Furthermore, since we may take r large enough, we may assume that $A^{r-1}|_D$ is globally generated. Let $t' \in H^0(D, A^{r-1}|_D)$ be such that $P \notin Z(t')$. Therefore, $t \otimes t' \in H^0(D, L^{rm} \otimes A^{-1})$ is such that $P \notin Z(t \otimes t')$. Note that $L^{rm} \otimes A^{-1}(-D) \simeq L^{rm-1} \otimes A$ is ample, since L is nef and A is ample. Therefore, Lemma 3.3 implies that, for some $e \geqslant 1$, the section $t^{\otimes p^e} \otimes t'^{\otimes p^e}$ can be lifted to a section in $H^0(X, L^{rmp^e} \otimes A^{-p^e})$, and this section clearly does not vanish at P. This shows that $P \notin \mathsf{B}_+(L)$, and completes the proof of the theorem. $\square$

Corollary 3.7. *Let X be a reduced projective scheme. If L and A are line bundles on X, with L nef and A ample, and $Z = Z(s)$ for some $s \in H^0(X, L \otimes A^{-1})$, then $\mathsf{B}_+(L) = \mathsf{B}_+(L|_Z)$.*

Proof. We slightly modify the argument in the proof of Theorem 1.2, along the lines of the proof of Corollary 3.4. By Lemma 2.1, it is enough to show that if $P \notin \mathsf{B}_+(L|_Z)$, then $P \notin \mathsf{B}_+(L)$. Let X' be the union of the irreducible components of X that are contained in Z, and let X'' be the union of the other components, both considered with the reduced scheme structures. If $X' = X$, then $Z = X$ and there is nothing to prove, while if $X' = \emptyset$, then Z is an effective Cartier divisor, and the assertion follows from Theorem 1.2. From now on, we assume that both X' and X'' are non-empty.

After replacing L and A by L^2 and A^2, respectively, and s by $s^{\otimes 2}$, we may assume that $A = B^2$ for some ample line bundle B (note that $\mathsf{B}_+(L|_{Z(s)}) = \mathsf{B}_+(L|_{Z(s^{\otimes 2})})$ by Lemma 3.1). Suppose that $P \notin \mathsf{B}_+(L|_Z)$. If $P \notin Z$, then $P \notin \mathsf{SB}(L \otimes A^{-1})$. Since A is ample, we have that $\mathsf{B}_+(L) \subseteq \mathsf{SB}(L \otimes A^{-1})$; hence, $P \notin \mathsf{B}_+(L)$. From now on we assume that P lies in Z.

Arguing as in the proof of Theorem 1.2, we find a section

$$t \otimes t' \in H^0(Z, L^{rm} \otimes A^{-1}|_Z)$$

such that $P \notin Z(t \otimes t')$, and we use Lemma 3.3 to deduce that, for some $e \geqslant 1$, we can lift $t^{\otimes p^e} \otimes t'^{p^e}|_{Z \cap X''}$ to a section $t'' \in H^0(X'', L^{rmp^e} \otimes A^{-p^e}|_{X''})$. Recall that $X' \subseteq Z$; hence, $X' \cap X'' \subseteq Z \cap X''$, and, therefore, $t''|_{X' \cap X''} = t^{\otimes p^e} \otimes t'^{\otimes p^e}|_{X' \cap X''}$. Since X is reduced, it follows that we can glue t'' and $t^{\otimes p^e} \otimes t'^{\otimes p^e}|_{X'}$ to a section in $H^0(X, L^{rmp^e} \otimes A^{-p^e})$ that does not vanish at P. Therefore, $P \notin \mathsf{B}_+(L)$, which concludes the proof. $\square$

We now give the proof of the characteristic p version of Nakamaye's theorem.

Proof of Theorem 1.1. We argue as in the proof of Corollary 3.6. By noetherian induction, we may assume that the theorem holds for every proper closed subscheme of X. Lemma 3.1 implies that $\mathsf{B}_+(L) = \mathsf{B}_+(L|_{X_{\mathrm{red}}})$, and, since $L^{\perp} = (L|_{X_{\mathrm{red}}})^{\perp}$, we may assume that X is reduced.

Note that the inclusion $L^{\perp} \subseteq \mathsf{B}_+(L)$ is clear: if V is a closed irreducible subset of X that is not contained in $\mathsf{B}_+(L)$, then we can find an ample line bundle A, a positive integer m, and $s \in H^0(X, L^m \otimes A^{-1})$ such that $V \nsubseteq Z(s)$. Therefore, $s|_V$ gives a non-zero section of $L^m \otimes A^{-1}|_V$; hence, $L|_V$ is big. This shows that it is enough to prove the inclusion $\mathsf{B}_+(L) \subseteq L^{\perp}$.

If the restriction of L to all the irreducible components of X is not big, then $L^{\perp} = X$, and the assertion is clear. Otherwise, let X' denote the union of the irreducible components of X on which the restriction of L is not big, and let X'' denote the union of the other components, both with the reduced scheme structures. It follows from Lemma 2.3 that, given any ample line bundle A, we can find $m \geqslant 1$ and a section $s \in H^0(X, L^m \otimes A^{-1})$ whose restriction to every component of X'' is non-zero (and whose restriction to X' is zero). Let $Z = Z(s)$. By assumption X'' is non-empty, and, therefore, Z is a proper closed subscheme of X; hence, by the inductive assumption we have that $\mathsf{B}_+(L|_Z) = (L|_Z)^{\perp}$. If $V \subseteq X$ is an irreducible closed subset such that $L|_V$ is not big, then $V \subseteq Z$; hence, $L^{\perp} = (L|_Z)^{\perp}$. On the other hand, Corollary 3.7 gives that $\mathsf{B}_+(L) = \mathsf{B}_+(L|_Z)$, and we conclude that $\mathsf{B}_+(L) = L^{\perp}$. $\square$

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