Glasgow Math. J. **41** (1999) 43–44. © Glasgow Mathematical Journal Trust 1999. Printed in the United Kingdom

A NOTE ON TIGHTNESS

MOHAMMAD SALEH†

Mathematics Department, Birzeit University, P.O. Box 14, Birzeit, West Bank, Palestine

(Received 14 February, 1997)

Abstract. The purpose of this note is to prove a results of Jain and López-Permouth under a weaker conditions replacing R-weak injectivity by R-tightness and even getting a simpler proof.

1. Introduction. Throughout this paper all rings are associative with identity and all modules are unitary right modules. We denote the category of all right R-modules by Mod-R. Given a module M_R the injective hull of M in Mod-R is denoted by $E(M)$. The purpose of this paper is to further the study of the concept of tightness [**1**], [**4**]. Following Jain and López-Permouth, given two modules M and N $\in$Mod-R, a module M is N-*tight* if every quotient of N which is embeddable in $E(M)$ is embeddable in M. A module is *tight* if it is tight relative to all finitely generated submodules of $E(M)$.

A ring R is called right *CEP*-ring if every cyclic right R-module is essentially embeddable in a projective module. In this paper we assume all modules are unital right R-modules unless otherwise indicated.

We start first with some basic results that will be needed in this note.

LEMMA 1. *Let* R *be an artinian ring, and let* N, M *be finitely generated modules. If* M *is* N-*tight and* N *is* M-*tight and* Soc(M) $\simeq$ Soc(N) *then* M $\simeq$ N.

Proof. Let σ: $N \to E(M)$ be the monomorphism induced by the isomorphism between Soc(M) and Soc(N). Since M is N-tight, N is embeddable in M. Similarly, M is embeddable in N. Since M and N are finitely generated over artinian ring, $M \simeq N$.

LEMMA 2. [**2**, **3**, **4**] *A right* CEP-*ring is right artinian. All projective indecomposable right modules over a right* CEP-*ring are uniform.*

LEMMA 3. *Let* R *be a right artinian ring such that all indecomposable projective right* R-*modules are uniform and* R-*tight. Then the following holds:*

(i) *every simple right* R-*module is isomorphic to the socle of an indecomposable projective module,*
(ii) *every simple right* R-*module is embeddable in* Soc(R),
(iii) *if* P *and* Q *are projective right modules with* Soc(P) $\simeq$ Soc(Q) *then* P $\simeq$ Q.

Proof. The proof follows from Lemma 1 and [**3**, Lemma 5. 1].

†The author was supported by Birzeit University under grant 235-17-98-9.

The next Theorem was proved by Jain and López-Permouth in [**3**] with the assumption that every indecomposable projective right module is weakly R-injective. We show that the theorem is true under a weaker condition. In fact all we need is to have every indecomposable projective right module is R-tight.

THEOREM. *A ring* R *is a right* CEP-*ring if and only if the following holds:*

(i) R *is right artinian,*
(ii) *every indecomposable projective right module is uniform and R-tight.*

Proof. Let R be a CEP-ring. By Lemma 2, R is right artinian. Let P be an indecomposable projective right module. Once more Lemma 2 implies that $Soc(P)$ is simple. Let $\sigma : R/I \to E(P)$ be a monomorphism. Then $Soc(R/I) \simeq Soc(E(P)) = Soc(P)$. Since R is a CEP-ring, R/I embeds essentially in some projective module, say, Q. Thus $Soc(P) \simeq Soc(Q)$. Hence by Lemma 3, $P \simeq Q$, and thus R/I embeds in P, proving that P is R-tight. Conversely, assume R satisfies the two conditions. Write $R = \oplus \sum_{i=1}^{n} e_i R$ as a direct sum of indecomposable right ideals. Let I be a right ideal of R. By Lemma 3, $Soc(R/I) \simeq \oplus \sum_{i=1}^{k} Soc(e_i R)^{n_i}$. Let $P = \oplus \sum_{i=1}^{k} (e_i R)^{n_i}$. Since $Soc(P) = Soc(E(P))$, the above isomorphism between $Soc(R/I)$ and $\oplus \sum_{i=1}^{k} Soc(e_i R)^{n_i} = Soc(P)$ may be looked upon as an essential embedding φ : $Soc(R/I) \to E(P)$. This extends into an essential embedding $\hat{\varphi}$: $R/I \to E(P)$. By tightness, there exists an embedding $\sigma : R/I \to P$ which is essential in P, proving that R is a CEP-ring.

ACKNOWLEDGEMENT. The author would like to thank the referee for his suggestions and bringing to my attention the paper by Gomez Pardo and Guil Asensio.

REFERENCES

1. J. S. Golan and S. R. López-Permouth, QI-filters and tight modules, *Comm. Algebra* **19**(8) (1991), 2217–2229.

2. J. L. Gomez Pardo and P. A. Guil Asensio, Essentials embedding of cyclic modules in projectives, *Trans. Amer. Math. Soc.*, to appear.

3. S. K. Jain and S. R. López-Permouth, Rings whose cyclics are essentially embeddable in projectives, *J. Algebra* **128** (1990), 257–269.

4. S. K. Jain, S. R. López-Permouth and S. Singh, On a class of QI-rings, *Glasgow J. Math.* **34** (1992), 75–81.