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# CLOSED IDEALS IN THE BANACH ALGEBRA $\ell^1(\omega)$ WHEN $\omega$ IS $\epsilon$-STAR SHAPED

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Provided that the weight function $\omega$ satisfies certain submultiplicative and decay conditions, the discrete convolution algebra $\ell^1(\omega)$ becomes a commutative radical Banach algebra with identity adjoined. There are obvious closed ideals in $\ell^1(\omega)$ and these are denoted *standard* ideals. Earlier results of Thomas, strengthened by Yakubovich and Domar, showed that if the weight $\omega$ is star-shaped then all closed ideals are standard. Consequently, the closed ideal generated by any element $f$ in $\ell^1(\omega)$ must be standard.

The requirement that $\omega$ be star-shaped (essentially that $\omega(n)^{1/n}$ must decrease to zero) is somewhat restrictive in that no local maxima of $\omega(n)^{1/n}$ are allowed. We generalize this previous result to apply to the larger class of $\epsilon$-star shaped weights ($0 < \epsilon \leq 1$) which allow such local maxima. If $f$ is a non-zero element on $\ell^1(\omega)$ we let the integer $\alpha(f) = k_0$ denote the index of its first non-zero term. We introduce the concept of an $\epsilon$-peak point for $k_0$. If $\epsilon = 1$ then $\omega$ is star-shaped in the usual sense and there are an infinite number of 1-peak points for any $k_0$. Although this latter fact may fail if $0 < \epsilon < 1$, if $\omega(n)^{1/n}$ tends to zero sufficiently quickly (dependent on $k_0$ and $\epsilon$) there will always be an infinite number of $\epsilon$-peak points for $k_0$.

Our main result is that if $\omega$ is an $\epsilon$-star shaped weight, if $f$ is an non-zero element of $\ell^1(\omega)$, if $\alpha(f) = k_0$, and if the number of $\epsilon$-peak points for $k_0$ is infinite, then the closed ideal generated by $f$ is standard.

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## 1. Introduction

We are concerned here with the closed ideal structure of discrete convolution algebras. However, we require some preliminary definitions and remarks about radical algebra weight functions. We will use the term "decreasing" in the usual sense of non-increasing.

**Definition 1.1.** Let $\omega$ be a function from the non-negative integers, $\mathbf{Z}^+$, to the positive reals. We say that $\omega$ is a *weight* or a *weight function* if the following condition is satisfied

(i) $\omega(0) = 1$ and $\omega(n)$ is decreasing in $n$.

If, in addition to condition (i), the inequality

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(ii) $\omega(n+m) \leq \omega(n)\omega(m)$ for all $n, m$ in $\mathbf{Z}^+$

holds, we say that $\omega$ is an *algebra weight*. Finally, if in addition to conditions (i) and (ii), the following equation

(iii) $\lim_{n\to\infty} \omega(n)^{1/n} = 0$,

holds, we say that $\omega$ is a *radical algebra weight*.

**Definition 1.2.** Let $\omega$ be a weight function. We say that $\omega$ is *star-shaped* provided that $\omega(n)^{1/n}$ decreases to zero.

If $\omega$ is a star-shaped weight it is routine to show that

$$\omega(n+m) \leq \omega(n)^{(n+m)/n} \leq \omega(n)\omega(n)^{m/n} \leq \omega(n)\omega(m),$$

if $m \leq n$, so that condition (ii) in Definition 1.1 is automatically satisfied. Therefore, a star-shaped weight is a radical algebra weight. When considering such a weight $\omega$ it is often illustrative to consider the graph of $\log \omega$. The name "star-shaped" is used because it is essentially equivalent to saying that the region below the graph of $\log \omega$ is "illuminated" by the origin.

One can generalize this geometrical way of looking at (the log of) a radical algebra weight and compare (for $m < n$) the slope of the line segment between $(m, \log \omega(m))$ and $(n, \log \omega(n))$ with the slope of the star segment from $(0,0)$ to $(n, \log \omega(n))$. If there exists $\epsilon$ in the range $0 < \epsilon \leq 1$ such that

$$\inf_{m<n,\omega(n)<1} \left[ \frac{(1/(n-m))(\log \omega(n) - \log \omega(m))}{(1/n)(\log \omega(n))} \right] \geq \epsilon, \tag{1.3}$$

we say that $\omega$ is $\epsilon$-star-shaped. Although an initial segment of the weight $\omega$ could be level (i.e. $\omega(0) = \omega(1) = \ldots \omega(k)$ for $k$ in $\mathbf{N}$), the inequality above will eventually preclude the radical algebra weight function $\omega$ from having level or even "near-level" segments (a short sketch of $\log \omega$ should make this concept clearer).

Simplifying the above inequality in order to remove the logarithms, we obtain an equivalent formal definition in terms of the weight $\omega$ itself.

**Definition 1.4.** Let $\omega$ be a radical algebra weight. If there exists $\epsilon$ in the range $0 < \epsilon \leq 1$ such that

$$\omega(n)^{\left(1 - \frac{\epsilon(n-m)}{n}\right)} \leq \omega(m),$$

for all $m < n$ in $\mathbf{Z}^+$ we say that $\omega$ is *$\epsilon$-star shaped.*

It is routine to check that the above definition reduces to the star-shaped definition if $\epsilon = 1$. However, if $\epsilon$ is in the range $0 < \epsilon < 1$, the condition that $\omega$ be a radical algebra weight is necessary; condition (ii) in Definition 1.1 is not automatic from the inequality (in Definition 1.4) alone if $0 < \epsilon < 1$.

We will often use the inequality in Definition 1.4 in the slightly different (but equivalent) forms

$$\left[\frac{\omega(n)}{\omega(m)}\right] \leq \omega(n)^{\left(\frac{\epsilon(n-m)}{n}\right)}, \tag{1.5}$$

for all $m < n$ in $\mathbf{Z}^+$, or

$$\omega(n) \leq \omega(m)^{\left(\frac{n}{(1-\epsilon)n+\epsilon m}\right)} \tag{1.6}$$

for all $m < n$ in $\mathbf{Z}^+$.

Let $\mathbb{C}[[\mathbf{X}]]$ denote the Fréchet algebra of formal power series (in the indeterminate $X$) over the complex field $\mathbb{C}$. The discrete convolution algebras that we will consider will be subalgebras of $\mathbb{C}[[\mathbf{X}]]$.

**Definition 1.7.** Let $\omega$ be a radical algebra weight. We define

$$\ell^1(\omega) \equiv \left\{ f = \sum_{k=0}^{\infty} \lambda_k X^k \in \mathbb{C}[[\mathbf{X}]] \mid \|f\| = \sum_{k=0}^{\infty} |\lambda_k| \omega(k) < \infty \right\}.$$

Multiplication is, of course, inherited from $\mathbb{C}[[\mathbf{X}]]$. If we regard $\ell^1(\omega)$ as a space of sequences then multiplication is simply convolution. It is routine to check that $\ell^1(\omega)$ is a normed subalgebra (as a consequence of condition (ii) in Definition 1.1), that the norm is complete, and that $\ell^1(\omega)$ is a local ring whose unique maximal ideal is the (Jacobson) radical of $\ell^1(\omega)$ (as a consequence of condition (iii) in Definition 1.1). Hence, $\ell^1(\omega)$ is a commutative radical Banach algebra with identity adjoined. The reader may wish to refer to the literature (**[1, 3, 4, 5** and **6]**) for additional details.

There are always non-trivial closed ideals in $\ell^1(\omega)$.

**Definition 1.8.** Let $\omega$ be a radical algebra weight. If $f$ is a non-zero element of $\ell^1(\omega)$ we say that $\alpha(f) = k_0$ if $f = \sum_{k=k_0}^{\infty} \lambda_k X^k$ and $\lambda_{k_0} \neq 0$. Define the following closed ideals

$$M_k \equiv \{f \in \ell^1(\omega) \mid \alpha(f) \geq k\},$$

for $k \in \mathbf{N}$, and let $M_0 = \ell^1(\omega)$, $M_\infty = \{0\}$. These closed ideals will be denoted *standard* ideals. Other closed ideals, if any exist, will be denoted *non-standard* ideals (no connection with non-standard analysis).

The reader will note that the (Jacobson) radical of $\ell^1(\omega)$ is $M_1$. It is also straightforward to show that a non-zero closed ideal $J$ is standard if and only if $X^n \in J$

for some $n \in \mathbf{N}$. Hence, a non-zero closed ideal $J$ is standard if and only if its codimension, $[\ell^1(\omega) : J]$, is finite.

There are two central questions which recur throughout the study of the discrete convolution algebras $\ell^1(\omega)$:

Q1. Let $f \in \ell^1(\omega)$ where $\omega$ is a radical algebra weight. If $f = 0$ then $f$ can only generate the zero ideal. If $\alpha(f) = 0$ then $f$ is invertible and $f\ell^1(\omega) = \ell^1(\omega)$. However, suppose that $\alpha(f) \in \mathbf{N}$; when does $f$ generate a *standard* closed ideal, i.e. when is

$$\overline{f\ell^1(\omega)} = M_{\alpha(f)}\,?$$

Q2. Let $\omega$ be a radical algebra weight. What additional conditions on $\omega$ will ensure that $\ell^1(\omega)$ has only *standard* closed ideals?

These questions are clearly related since, if all elements $f \in \ell^1(\omega)$ with $\alpha(f) \in \mathbf{N}$ generate standard closed ideals then all closed ideals in $\ell^1(\omega)$ must be standard. However, complete answers to these questions are not currently known. It has been shown that there are radical algebra weights $\omega$ such that $\ell^1(\omega)$ contains a non-standard closed ideal (see [**8**, Theorem 3.20]), so (Q2) is not vacuous.

The strongest previous result concerning sufficiency is that all closed ideals of $\ell^1(\omega)$ are standard if $\omega$ is star-shaped (see [**2**] which strengthened [**7**, Corollary 3.6]). In order to further strengthen this result we will allow $\omega$ to belong to the more general class of $\epsilon$-star shaped weights (for $0 < \epsilon \le 1$). This is desirable since the definition of star-shaped ($\epsilon = 1$) is rather restrictive in that $\omega(n)^{1/n}$ must decrease to zero. We want to allow some local maxima of $\omega(n)^{1/n}$ (provided that $\lim_{n\to\infty} \omega(n)^{1/n} = 0$). But "near-level" segments in a radical algebra weight are definitely harmful to the goal of generating only standard closed ideals. We wish to stress that the only known examples of radical algebra weights $\omega$ for which $\ell^1(\omega)$ has *non-standard* closed ideals satisfy

$$\inf_{m<n,\,\omega(n)<1}\left[\frac{(1/(n-m))(\log\omega(n) - \log\omega(m))}{(1/n)(\log\omega(n))}\right] = 0; \tag{1.9}$$

(cf. inequality (1.3)) references are [**8**] and [**9**]. The calculation is routine in the case of the former (non-regulated) example; it is more tedious in the case of the latter (regulated) example but still straightforward. Consequently, it is natural to try to extend the star-shaped result to $\epsilon$-star shaped weights.

Unfortunately, we cannot yet prove that $\ell^1(\omega)$ has only standard closed ideals for *all* $\epsilon$-star shaped weights. We need an additional requirement concerning the existence of points where $\omega$ makes sufficiently large drops relative to the placement of the local maxima for $\omega(n)^{1/n}$.

**Definition 1.10.** Let $k_0 \in \mathbf{N}$ and let $\omega$ be an $\epsilon$-star shaped weight for $0 < \epsilon \le 1$. We say that $n \in \mathbf{N}$ is an *$\epsilon$-peak point for $k_0$* provided that

$$\frac{\omega(k_0+m_0)^{\epsilon j/m_0}}{\omega(k_0+j)} \le 1 \text{ for } j \in \{1,2,\ldots,n-1\},$$

where the supremum, $\sup_{m\ge n}\omega(k_0+m)^{1/(k_0+m)}$, is *first* attained at $m_0$.

We remark that the supremum in the above definition is attained at only a *finite* number of points $m \ge n$ due to condition (iii)

$$\lim_{k\to\infty}\omega(k)^{1/k}=0,$$

in Definition 1.1.

If $\omega$ is actually star-shaped ($\epsilon = 1$) then it can be shown that the set of 1-peak points for $k_0$ is infinite for each $k_0 \in \mathbf{N}$ (see argument in [7, Theorem 2.7]).

In this case, it is clear that

$$\sup_{m\ge n}\omega(k_0+m)^{1/(k_0+m)}=\omega(k_0+n)^{1/(k_0+n)},$$

since $\omega(k)^{1/k}$ is decreasing, so that $m_0 = n$. If $\omega$ is only $\epsilon$-star shaped with $0<\epsilon<1$ there may *not* be an infinite number of $\epsilon$-peak points (for some, or, indeed, all values of $k_0$). However, if $\omega(k)^{1/k}$ tends to zero sufficiently quickly we will show that there will always be an infinite number of such points (see Lemma 2.7).

Our main result will be proved in Section 2: if $\omega$ is an $\epsilon$-star shaped weight for $0<\epsilon\le 1$, if $f$ is a non-zero element of $\ell^1(\omega)$ with $\alpha(f)=k_0$ in $\mathbf{N}$, and if the set of $\epsilon$-peak points for $k_0$ is infinite then the closed ideal generated by $f$ is standard (see Theorem 2.9).

This theorem not only covers all previously known cases, it also adds a large number of weights to the class of weights $\omega$ for which it is known that $\ell^1(\omega)$ has only standard closed ideals. For example, one consequence of our theorem (together with Lemma 2.6) is the following

**Application 1.11** *There exists a one-parameter family $\{\omega_\epsilon\}_{0\le\omega<1}$ of radical algebra weights with the following properties:*

(i) *If $0<\epsilon<1$ then $\omega_\epsilon(n)=\omega_0(n)^{(1-\epsilon)^{-n}}$ for all $n$ in $\mathbf{Z}^+$, $\omega_\epsilon$ is $\epsilon$-star shaped and has an infinite number of $\epsilon$-peak points for any $k_0\in\mathbf{N}$. Consequently,* ***all*** closed ideals of $\ell^1(\omega_\epsilon)$ *are* ***standard****.*

(ii) *the radical algebra weight $\omega_0$ is a semi-multiplicative weight (in the sense of* [8]*), $\omega_0$ is* ***not*** *regulated, and there exists $f\in\ell^1(\omega_0)$ with $\alpha(f)=1$ such that the closed ideal generated by $f$ is* ***not*** *standard.*

(iii) $\lim_{\epsilon\to 0^+}\omega_\epsilon(n)=\omega_0(n)$ *for each $n$ in $\mathbf{Z}^+$.*

We sketch the construction of such a one-parameter family as follows. First choose

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a decreasing sequence of $\epsilon_k$'s tending to zero. In the construction of the semi-multiplicative weight in [**8**, p. 205] *add* (after (B4)) *the additional inductive requirement*

$$\text{(B5)}\quad \omega(n(k))^{\left(\frac{\epsilon_k}{2n(k)}\right)} \le \min\{\omega(m) \mid m = 1, 2, \ldots, n(k) - 1\}.$$

This is allowable since the right hand side above is determined as soon as $\omega(n(k-1))$ is chosen (for a semi-multiplicative weight $\omega(n(k-1)+s) = \omega(n(k-1))\omega(s)$ if $n(k-1)+s < n(k)$). Let $\omega_0$ be the resulting semi-multiplicative weight $\omega$ and define

$$\omega_\epsilon(n) = \omega(n)^{(1-\epsilon)^{-n}},$$

for all $n$ in $\mathbf{Z}^+$ and $0 < \epsilon < 1$. Assertion (ii) follows by exactly the same arguments as those used to prove [**8**, Theorem 3.20] and assertion (iii) is immediate. However, we have several things to check regarding assertion (i). The first condition in Definition 1.1 is clear and note that each $\omega_\epsilon$ is an algebra weight because

$$\begin{aligned}\omega_\epsilon(n+m) &= \omega(n+m)^{(1-\epsilon)^{-(n+m)}} \\ &\le \omega(n)^{(1-\epsilon)^{-(n+m)}}\omega(m)^{(1-\epsilon)^{-(n+m)}} \\ &\le \omega(n)^{(1-\epsilon)^{-n}}\omega(m)^{(1-\epsilon)^{-m}} \\ &= \omega_\epsilon(n)\omega_\epsilon(m),\end{aligned}$$

for all $n, m$ in $\mathbf{Z}^+$. Also, because

$$\lim_{n\to\infty} \omega_\epsilon(n)^{1/n} = \lim_{n\to\infty}(\omega(n)^{1/n})^{(1-\epsilon)^{-n}} = 0,$$

we see that $\omega_\epsilon$ is a radical algebra weight for each $0 < \epsilon < 1$.

Next, let $m < n$ and note that

$$\begin{aligned}\omega_\epsilon(n)^{((1-\epsilon)n+\epsilon m)/n} &= \omega(n)^{(1-\epsilon)^{-n}(((1-\epsilon)n+\epsilon m)/n)} \\ &\le \omega(n)^{(1-\epsilon)^{-n}(((1-\epsilon)n)/n)} \\ &= \omega(n)^{(1-\epsilon)^{-(n-1)}} \\ &\le \omega(m)^{(1-\epsilon)^{-m}} = \omega_\epsilon(m),\end{aligned}$$

and, by inequality (1.6), it follows that $\omega_\epsilon$ is, indeed, an $\epsilon$-star shaped weight for each $0 < \epsilon < 1$.

Finally, let $\epsilon > 0$ and $k_0 \in \mathbf{N}$ be given. We claim that the (infinite) set

$$\{n(k) - k_0 \mid k \in \mathbf{N} \text{ with } k \ge k_0 \text{ and } \epsilon_k \le \epsilon\}$$

contains only $\epsilon$-peak points for $k_0$ (where $\{n(k)$ is the subsequence used in the construction of the semi-multiplicative weight $\omega$ and note that $n(k) \gg k$ from [**8**, condition (A), p. 204]). Hence, let $k \in \mathbf{N}$ with $k \geq k_0$ and $\epsilon_k \leq \epsilon$ be fixed for the following. As in Definition 1.10, let $m_0$ be the integer where the supremum, $\sup_{m \geq n(k)-k_0} \omega(k_0+m)^{1/(k_0+m)}$, is *first* attained. We will prove later (see Lemma 2.6) that because $n(k) \leq k_0 + m_0$ it follows that $\omega_\epsilon(k_0+m_0)^{1/(k_0+m_0)} \leq (\omega_\epsilon(n(k))^{1/n(k)})^{1/2}$. Consequently, we have the following train of inequalities

$$\begin{aligned}\omega_\epsilon(k_0+m_0)^{\left(\frac{\epsilon}{m_0}\right)} &\leq \omega_\epsilon(k_0+m_0)^{\left(\frac{\epsilon}{k_0+m_0}\right)}\\ &\leq \omega_\epsilon(n(k))^{\left(\frac{\epsilon}{2n(k)}\right)}\\ &\leq \omega(n(k))^{\left(\frac{\epsilon_k}{2n(k)}\right)\left(\frac{1}{1-\epsilon}\right)^{n(k)}}.\end{aligned}$$

Now let $t \in \{1, 2, 3, \ldots, (n(k)-k_0)-1\}$ and use the above inequality and the (B5) inductive hypothesis to obtain

$$\begin{aligned}\omega_\epsilon(k_0+m_0)^{\left(\frac{\epsilon t}{m_0}\right)} &\leq \omega(n(k))^{\left(\frac{\epsilon_k}{2n(k)}\right)\left(\frac{1}{1-\epsilon}\right)^{n(k)} t}\\ &\leq \omega(k_0+t)^{\left(\frac{1}{1-\epsilon}\right)^{n(k)} t}\\ &\leq \omega(k_0+t)^{\left(\frac{1}{1-\epsilon}\right)^{(k_0+t)} t},\end{aligned}$$

since $(1/(1-\epsilon)) > 1$ and $k_0 + t < n(k)$, and the above is then

$$\begin{aligned}&= \omega_\epsilon(k_0+t)^t\\ &\leq w_\epsilon(k_0+t).\end{aligned}$$

This shows that $n(k)-k_0$ is an $\epsilon$-peak point for $k_0$, provided that $k$ is chosen to satisfy $k \geq k_0$ and $\epsilon_k \leq \epsilon$. Therefore we conclude that the set of $\epsilon$-peak points for $k_0$ is infinite. Since $k_0$ was arbitrary we have shown, by Theorem 2.9 that $\ell^1(\omega_\epsilon)$ contains only standard closed ideals for each $0 < \epsilon < 1$, thus establishing assertion (i) of Application 1.11.

We will prove our theorem in the next section, but the above application raises the following question (which we cannot at this time answer):

Q3. If $\omega$ is a radical algebra weight such that $\ell^1(\omega)$ has non-standard closed ideals, is it *necessary* that

$$\inf_{m<n,\omega(n)<1}\left[\frac{(1/(n-m))(\log\omega(n)-\log\omega(m))}{(1/n)(\log\omega(n))}\right] = 0\,?$$

Equivalently, are all closed ideals standard in $\ell^1(\omega)$ when $\omega$ is $\epsilon$-star shaped with $0 < \epsilon \leq 1$ even if there are values of $k_0$ in $\mathbf{N}$ for which the set of $\epsilon$-peak points for $k_0$ is a *finite* set?

**2. $\epsilon$-star shaped weights: proof of main result**

In the following we will *fix* an $\epsilon$-star shaped weight $\omega$, where $0 < \epsilon \leq 1$, and we will *fix* a non-zero element $f \in \ell^1(\omega)$ with $\alpha(f) = k_0$ in $\mathbf{N}$, so that

$$f = \sum_{k=k_0}^{\infty} \lambda_k X^k,$$

and $\lambda_{k_0} \neq 0$. We want to establish sufficient conditions on $\omega$ and $k_0$ so that the closed ideal generated by $f$ is standard, i.e.

$$\overline{f\ell^1(\omega)} = M_{k_0}.$$

As discussed in the Introduction, it will suffice to show that $X^m \in \overline{f\ell^1(\omega)}$ for some $m \in \mathbf{N}$.

It is also clear that multiplying $f$ by any non-zero scalar does not change the closed ideal which $f$ generates. Therefore, in order to simplify the following calculations *we will assume without loss of generality that* $\lambda_{k_0} = 1$.

We will temporarily work in the Fréchet algebra of formal power series. Divide $f$ by $X^{k_0}$ in $\mathbb{C}[[\mathbf{X}]]$ to obtain an element $(1-b)$, i.e. $b = \sum_{j=1}^{\infty} b_j X^j$, where

$$f = (1-b)X^{k_0},$$

and $b_j = -\lambda_{k_0+j}$ for all $j \in \mathbf{N}$. Of course, $b$ is only a formal series and is *not* in general an element of $\ell^1(\omega)$. Since $(1-b)$ is invertible in $\mathbb{C}[[\mathbf{X}]]$ we can find an element $c = \sum_{j=1}^{\infty} c_j X^j \in \mathbb{C}[[\mathbf{X}]]$ satisfying

$$(1+c)(1-b) = 1,$$

in $\mathbb{C}[[\mathbf{X}]]$. Generally, we will rearrange this to

$$c = \sum_{s=1}^{\infty} b^s = b(1-b)^{-1},$$

the convergence of the geometric series being in the Fréchet topology of coefficient-wise convergence in $\mathbb{C}[[\mathbf{X}]]$ (*not* in the topology of $\ell^1(\omega)$). Our immediate goal is to obtain an upper bound for certain combinations of the $c_j$'s. The following lemma is due to Y. Domar but we repeat the proof for the sake of completeness of the exposition.

**Lemma 2.1.** *Let $b = \sum_{j=1}^{\infty} b_j X^j$ and $c = \sum_{j=1}^{\infty} c_j X^j$ be formal series in $\mathbb{C}[[\mathbf{X}]]$ which satisfy the equation $c = b(1-b)^{-1}$. Suppose that there exists $n \in \mathbf{N}$ and $\zeta$ in the non-negative reals $\mathbf{R}^+$ such that*

$$\sum_{j=1}^{n-1} |b_j|\zeta^j < 1.$$

*Then*

$$\sum_{j=1}^{n-1} |c_j|\zeta^j \le \left(\sum_{j=1}^{n-1} |b_j|\zeta^j\right)\left(1 - \sum_{j=1}^{n-1} |b_j|\zeta^j\right)^{-1}.$$

**Proof.** By the absolute value of a formal series we mean the new series obtained by taking the absolute value of each coefficient of the given series. We will use "$\le$" in this same *coefficient-wise* sense. It is then clear that

$$|c| \le |\sum_{s=1}^{\infty} b^s| \le \sum_{s=1}^{\infty} |b^s| \le \sum_{s=1}^{\infty} |b|^s.$$

Let $n \in \mathbf{N}$ be *fixed*. Note that the series $b$ has no constant term so that $|b|^s$ contains no power of the indeterminate $X$ lower than $X^s$. Also note that all powers occurring in the expansion of $(\sum_{j=n}^{\infty} |b_j|X^j)^s$ are at least of the degree of $X^n$, for all $s \in \mathbf{N}$. We therefore conclude that

$$\sum_{j=1}^{n-1} |c_j|X^j \le \sum_{s=1}^{\infty}\left(\sum_{j=1}^{n-1} |b_j|X^j\right)^s$$

$$= \left(\sum_{j=1}^{n-1} |b_j|X^j\right)\left(1 - \sum_{j=1}^{n-1} |b_j|X^j\right)^{-1},$$

again using the expression for the sum of a geometric series in the Fréchet algebra $\mathbb{C}[[X]]$. Now observe that given the above coefficient-wise inequality, we can substitute the real number $\zeta$ for $X$, preserving the inequality (as real numbers) because $\sum_{j=1}^{n-1} |b_j|\zeta^j < 1$. This ends the proof of the lemma. □

We note in passing that if we make $c_0 = 1$ then the sequence $\{c_j\}_{j=0}^{\infty}$ is also called the *associated sequence* for $f$ in the literature (see [7, Definition 2.1]). This sequence could also have been defined inductively via

$$c_0 = (\lambda_{k_0})^{-1} \text{ and } c_n = -(\lambda_{k_0})^{-1} \sum_{j=0}^{n-1} c_j \lambda_{k_0+n-j},$$

for $n = 1, 2, 3, \ldots$.

In the proof that all closed ideals are standard if $\omega$ is star-shaped (i.e. $\epsilon = 1$) [2] the following linear combination of translates of $f$

$$\sum_{j=0}^{n-1} c_j X^{k_0+j} f$$

was used to approximate $X^{2k_0}$ with the values of $n$ being carefully chosen. If $0 < \epsilon < 1$ there are technical problems in doing this. Consequently, we must *choose and fix* an $r \in \mathbf{N}$ satisfying $\epsilon(k_0 + r) \geq k_0$. In the remainder of the paper we will develop sufficient conditions for

$$\liminf_{n\to\infty} \|X^{2k_0+r} - \sum_{j=0}^{n-1} c_j X^{k_0+r+j} f\|$$

to be zero. This will suffice to prove that $\overline{f\ell^1(\omega)}$ is standard, from which it must follow that $\overline{f\ell^1(\omega)} = M_{\alpha(f)}$. We require some definitions.

**Definition 2.2.** Let $m \in \mathbf{N}$. Define the natural projection $Q_m$ on $\ell^1(\omega)$ by

$$Q_m\left(\sum_{j=0}^{\infty} \gamma_j X^j\right) = \sum_{j=m}^{\infty} \gamma_j X^j,$$

for $\sum_{j=0}^{\infty} \gamma_j X^j \in \ell^1(\omega)$.

**Definition 2.3.** Let $\omega$ be an $\epsilon$-star shaped weight for $0 < \epsilon \leq 1$. If $k \in \mathbf{N}$ define $\mu(k)$ as follows. If $\omega(k)^{1/k} \geq \omega(n)^{1/n}$ for all $n \geq k$ let $\mu(k) = k$. Otherwise, let $\mu(k)$ be the first index after $k$ which satisfies

$$\sup_{n\geq k} \omega(n)^{1/n} = \omega(\mu(k))^{1/\mu(k)},$$

noting that this supremum is attained (at most a finite number of times) since $\lim_{n\to\infty} \omega(n)^{1/n} = 0$.

We remark that if $\epsilon = 1$, clearly $\mu(k) = k$ for all $k \in \mathbf{N}$: but in the general case a finite number of local maxima of $\omega(n)^{1/n}$ may follow $\omega(k)^{1/k}$ and exceed it. We now give an equivalent, but notationally simpler, formulation of Definition 1.10.

**Definition 2.4.** Let $\omega$ be an $\epsilon$-star shaped weight for $0 < \epsilon \leq 1$. Let $k_0 \in \mathbf{N}$. We say that $n \in \mathbf{N}$ is an *$\epsilon$-peak point for* $k_0$ provided that

$$\frac{\omega(\mu(k_0+n))^{(\epsilon j/(\mu(k_0+n)-k_0))}}{\omega(k_0+j)} \leq 1 \text{ for } j \in \{1, 2, \ldots, n-1\}.$$

We remark that a point may be an $\epsilon$-peak point for some values of $k_0$ and not for others. Candidates for $\epsilon$-peak points for $k_0$ are values $n$ where there is a large "drop" in the weight at $\omega(k_0+n)$ and where $\omega(k_0+n)^{1/(k_0+n)}$ is not too small in comparison with $\sup_{m \geq k_0+n} \omega(m)^{1/m}$. In general, the larger $k_0$ is, the harder it is to find $\epsilon$-peak points (if we allowed $k_0 = 0$ in the above inequality, *every* point would qualify in the star-shaped case of $\epsilon = 1$). The pertinence of the expression in Definition 2.4 is not apparent until one considers the mass of the tail of a translate of $f$.

**Lemma 2.5.** *Let $\omega$ be an $\epsilon$-star shaped weight for $0 < \epsilon \leq 1$. Let $r$ be as above and satisfy $\epsilon(k_0+r) \geq k_0$. Let $n \in \mathbf{N}$ with $n > k_0 + r$ and $j \in \{1, 2, \ldots, n-1\}$. Then*

$$\|Q_{k_0+n}(X^{k_0+r+j}f)\| \leq \omega(\mu(k_0+n))^{(\epsilon j/(\mu(k_0+n)-k_0))}\|f\|.$$

**Proof.** Recall that $f = \sum_{k=k_0}^{\infty} \lambda_k X^k$. Applying the projection we see that

$$\begin{aligned}
\|Q_{k_0+n}(X^{k_0+r+j}f)\| &= \|Q_{k_0+n}\left(\sum_{k=k_0}^{\infty} \lambda_k X^{k_0+r+j+k}\right)\| \\
&= \| \sum_{k=n-r-j \text{ and } k \geq k_0}^{\infty} \lambda_k X^{k_0+r+j+k} \| \\
&= \sum_{k=n-r-j \text{ and } k \geq k_0}^{\infty} |\lambda_k| \omega(k_0+r+j+k) \\
&= \sum_{k=n-r-j \text{ and } k \geq k_0}^{\infty} |\lambda_k| \omega(k) \left(\frac{\omega(k_0+r+j+k)}{\omega(k)}\right) \\
&\leq \sum_{k=n-r-j \text{ and } k \geq k_0}^{\infty} |\lambda_k| \omega(k) \omega(k_0+r+j+k)^{\left(\frac{\epsilon(k_0+r+j)}{k_0+r+j+k}\right)},
\end{aligned}$$

applying inequality (1.5). Noting that $k_0 + r + j + k \geq k_0 + n$ we see that the above is

$$\begin{aligned}
&\leq \sum_{k=n-r-j \text{ and } k \geq k_0}^{\infty} |\lambda_k| \omega(k) \omega(\mu(k_0+n))^{\left(\frac{\epsilon(k_0+r)+\epsilon j}{\mu(k_0+n)}\right)} \\
&\leq \omega(\mu(k_0+n))^{\left(\frac{k_0+\epsilon j}{\mu(k_0+n)}\right)} \|f\|,
\end{aligned}$$

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since $\epsilon(k_0+r) \geq k_0$. Since $(k_0+t)/(k_0+s) \geq t/s$ if $t \leq s$, we see that the above is

$$\leq \omega(\mu(k_0+n))^{(\epsilon j/(\mu(k_0+n)-k_0))}\|f\|.$$

This completes the proof of the lemma. □

In general, there may be only finitely many $\epsilon$-peak points for a given $k_0$. However, if the $\epsilon$-star shaped weight decreases sufficiently rapidly, we can show that there are always an infinite number of these points. We require a preliminary lemma.

**Lemma 2.6.** *Let $\omega$ be an $\epsilon$-star shaped weight for $0 < \epsilon \leq 1$ and let $k_0 \in \mathbf{N}$. Suppose for $n \in \mathbf{N}$ that*

$$C(k_0+n) \leq \mu(k_0+n) < (C+1)(k_0+n),$$

*for some $C \in \mathbf{N}$. Then*

$$\omega(\mu(k_0+n))^{1/\mu(k_0+n)} \leq \left(\omega(k_0+n)^{1/(k_0+n)}\right)^{C/(C+1)}.$$

**Proof.** We remark that we always have

$$\omega(\mu(k_0+n))^{1/\mu(k_0+n)} \geq \omega(k_0+n)^{1/(k_0+n)},$$

so that the usefulness of this lemma is the establishment of an *upper* bound. First, use the fact that the weight is $\epsilon$-star shaped by applying inequality (1.6) to obtain

$$\begin{aligned}\omega(\mu(k_0+n))^{1/\mu(k_0+n)} &\leq \omega(C(k_0+n))^{1/((1-\epsilon)\mu(k_0+n)+\epsilon C(k_0+n))}\\ &\leq \omega(C(k_0+n))^{1/((C+1)(k_0+n))},\end{aligned}$$

since $((1-\epsilon)\mu(k_0+n)+\epsilon C(k_0+n))$ lies in the interval $[C(k_0+n),(C+1)(k_0+n)]$. Use the submultiplicative property (ii) of Definition 1.1 to see that the above is

$$\begin{aligned}&\leq (\omega(k_0+n)^C)^{1/((C+1)(k_0+n))}\\ &= \left(\omega(k_0+n)^{1/(k_0+n)}\right)^{C/(C+1)},\end{aligned}$$

and the lemma is proved. □

Note in the above lemma that $C/(C+1) \geq 1/2$ and that $\lim_{C\to\infty} C/(C+1) = 1$ so that the further $\mu(k_0+n)$ is from $(k_0+n)$ the closer the trailing local maximum $\omega(\mu(k_0+n))^{1/\mu(k_0+n)}$ is to $\omega(k_0+n)^{1/(k_0+n)}$. We can now prove

**Lemma 2.7.** *Let $\omega$ be an $\epsilon$-star shaped weight for $0 < \epsilon \le 1$ and suppose for some $k_0 \in \mathbf{N}$ that the number of $\epsilon$-peak points for $k_0$ is finite. Then there exists a constant $0 < B < 1$ such that the following lower bound holds*

$$\omega(k_0 + n)^{1/n} \ge B^{(2(k_0+1)/\epsilon)^n},$$

*for all $n \in \mathbf{N}$.*

**Proof.** By hypothesis, there exists $N \in \mathbf{N}$ such that $n$ is *not* an $\epsilon$-peak point for $k_0$ if $n > N$. Without loss of generality we may also assume that $k_0 < N$ and $\omega(k_0 + N) < 1$. Pick any integer $n_1 > N$. Apply Lemma 2.6, noting that $(C+1)/C \le 2$, to obtain

$$\omega(k_0 + n_1)^{1/(k_0+n_1)} \ge \left(\omega(\mu(k_0 + n_1))^{1/\mu(k_0+n_1)}\right)^2.$$

Using the above inequality and the fact that $1/n_1 \le (k_0+1)/(k_0+n_1)$ we have that

$$\begin{aligned} \omega(k_0+n_1)^{1/n_1} &\ge \omega(k_0+n_1)^{(k_0+1)/(k_0+n_1)} \\ &\ge \left(\omega(\mu(k_0+n_1))^{1/\mu(k_0+n_1)}\right)^{2(k_0+1)} \\ &\ge \left(\omega(\mu(k_0+n_1))^{1/(\mu(k_0+n_1)-k_0)}\right)^{2(k_0+1)} \\ &\ge \left(\omega(k_0+n_2)^{1/(\epsilon n_2)}\right)^{2(k_0+1)}, \end{aligned}$$

for some $n_2 \in \{1, 2, \ldots, n_1 - 1\}$ since $n_1$ is not an $\epsilon$-peak point for $k_0$. We have thus shown that if $n_1 > N$ there exists $n_2 < n_1$ satisfying

$$\omega(k_0+n_1)^{1/n_1} \ge \left(\omega(k_0+n_2)^{1/n_2}\right)^{2(k_0+1)/\epsilon}.$$

If $n_2 > N$ we can continue this process and find an $n_3 < n_2$ such that

$$\omega(k_0+n_2)^{1/n_2} \ge \left(\omega(k_0+n_3)^{1/n_3}\right)^{2(k_0+1)/\epsilon}.$$

At some point we will obtain an $n_k \le N$. If we let

$$B \equiv \min\{\omega(k_0+j)^{1/j} \mid j = 1, 2, \ldots, N\},$$

we see that $0 < B < 1$ and that

$$\omega(k_0+n_1)^{1/n_1} \ge B^{(2(k_0+1)/\epsilon)^{(k-1)}}.$$

Since $(k-1) < n_1$, the assertion of the lemma now follows. □

We need one more lemma before we can prove our main theorem. The proof of this lemma uses an earlier technique of Y. Domar applied to our specific situation of $\epsilon$-peak points.

**Lemma 2.8.** *Let $\omega$ be an $\epsilon$-star shaped weight for $0<\epsilon\le 1$ and let $k_0\in\mathbf{N}$. Suppose that there are an **infinite** number of $\epsilon$-peak points for $k_0$ and that they have been formed into a subsequence $\{n_i\}_{i=1}^{\infty}$. Then*

$$\lim_{i\to\infty}\left(\sum_{j=1}^{n_i-1}|\lambda_{k_0+j}|\omega(\mu(k_0+n_i))^{(\epsilon j/(\mu(k_0+n_i)-k_0))}\right)=0,$$

*where the $\lambda_k$'s are, of course, the coefficients of $f$.*

**Proof.** First note that since $\mu(k_0+n)\ge(k_0+n)$ it follows that

$$\lim_{n\to\infty}\omega(\mu(k_0+n))^{1/\mu(k_0+n)}=\lim_{m\to\infty}\omega(m)^{1/m}=0.$$

Consequently, for *fixed* $j\in\mathbf{N}$, it is the case that

$$\lim_{n\to\infty}\omega(\mu(k_0+n))^{(\epsilon j/(\mu(k_0+n)-k_0))}=0,$$

also. Rewrite the above sum as

$$\sum_{j=1}^{n_i-1}|\lambda_{k_0+j}|\omega(\mu(k_0+n_i))^{(\epsilon j/(\mu(k_0+n_i)-k_0))}$$

$$=\sum_{j=1}^{n_i-1}\left[\frac{\omega(\mu(k_0+n_i))^{(\epsilon j/(\mu(k_0+n_i)-k_0))}}{\omega(k_0+j)}\right]|\lambda_{k_0+j}|\omega(k_0+j)$$

$$=\int_N\left[\left(\frac{\omega(\mu(k_0+n_i))^{(\epsilon j/(\mu(k_0+n_i)-k_0))}}{\omega(k_0+j)}\right)\chi_{[1,n_i-1]}(j)\right]d\lambda(k_0+j),$$

where $d\lambda$ is a *finite* measure on $\mathbf{N}$. Since each $n_i$ is an $\epsilon$-peak point for $k_0$ the integrand is bounded above by 1, and, for each $j\in\mathbf{N}$ converges pointwise to zero as $i$ tends to infinity. The Lebesgue Dominated Convergence Theorem now justifies the assertion of the lemma. □

We now have our main result.

**Theorem 2.9.** *Let $\omega$ be an $\epsilon$-star shaped weight for $0<\epsilon\le 1$. Let $f$ be a non-zero element of $\ell^1(\omega)$ with $\alpha(f)=k_0$ in $\mathbf{N}$. If the set of $\epsilon$-peak points for $k_0$ is **infinite** then*

$$\overline{f\ell^1(\omega)} = M_{\alpha(f)},$$

*and thus the closed ideal generated by $f$ is* ***standard.***

**Proof.** Recall our earlier definition of the associated sequence $\{c_j\}_{j=0}^{\infty}$. Let $S$ be the set of $\epsilon$-peak points for $k_0$. We will show that

$$\lim_{n\in S, n\to\infty} \|X^{2k_0+r} - \sum_{j=0}^{n-1} c_j X^{k_0+r+j} f\| = 0,$$

where $r$ was fixed above to satisfy $\epsilon(k_0+r) \geq k_0$. Note that the expression $(X^{2k_0+r} - \sum_{j=0}^{n-1} c_j X^{k_0+r+j} f)$ vanishes on the (discrete) interval $[0, k_0+n) \cap \mathbf{N}$ (see [7, Definition 2.1, assertion (2.1)]). We can clearly restrict our attention to those $n$ for which $n > (k_0+r)$ and for these cases

$$\|X^{2k_0+r} - \sum_{j=0}^{n-1} c_j X^{k_0+r+j} f\| = \|Q_{k_0+n}\left(\sum_{j=0}^{n-1} c_j X^{k_0+r+j} f\right)\|,$$

since $Q_{k_0+n}(X^{2k_0+r}) = 0$. The above expression is then

$$\leq \|Q_{k_0+n}(X^{k_0+r} f)\| + \sum_{j=1}^{n-1} |c_j| \|Q_{k_0+n}(X^{k_0+r+j} f)\|.$$

The first term above tends to zero as $n$ tends to infinity since $f \in \ell^1(\omega)$. It therefore suffices to prove that

$$\lim_{n\in S, n\to\infty} \sum_{j=1}^{n-1} |c_j| \|Q_{k_0+n}(X^{k_0+r+j} f)\| = 0.$$

However, if we apply Lemma 2.5, we see that it suffices to prove that

$$\lim_{n\in S, n\to\infty} \sum_{j=1}^{n-1} |c_j| \omega(\mu(k_0+n))^{(\epsilon j/(\mu(k_0+n)-k_0))} = 0.$$

Let $\delta$ be a positive real number in the range $0 < \delta < 1$. Apply Lemma 2.8 to obtain $N \in \mathbf{N}$ such that

$$\sum_{j=1}^{n-1} |\lambda_{k_0+j}| \omega(\mu(k_0+n))^{(\epsilon j/(\mu(k_0+n)-k_0))} < \delta,$$

for all $n \in S$ with $n \geq N$. Lemma 2.1, with

$$\zeta = \omega(\mu(k_0+n))^{(\epsilon/(\mu(k_0+n)-k_0))},$$

now implies that

$$\sum_{j=1}^{n-1} |c_j| \omega(\mu(k_0+n))^{(\epsilon j/(\mu(k_0+n)-k_0))} \le \delta(1-\delta)^{-1},$$

for all $n \in S$ with $n \ge N$ (since $b_j = -\lambda_{k_0+j}$). This establishes the result, since $\delta(1-\delta)^{-1}$ can be made arbitrarily small. □

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