Digit Maps

NIPHAWAN PHOOPHA, PRAPANPONG PONGSRIIAM and
PHAKHINKON NAPP PHUNPHAYAP

1. *Happy Numbers*

The happy function S of each positive integer x is defined to be the sum of the squares of the decimal digits of x. For example, $S(2) = 4$ and $S(123) = 1^2 + 2^2 + 3^2 = 14$. It is well known that for any $x \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $S^{(n)}(x) \in \{1, 4\}$, where $S^{(n)}$ is the n-fold composition of S. In addition, if $x \in \mathbb{N}$ and $S^{(n)}(x) = 1$ for some $n \in \mathbb{N}$, then x is called a happy number. Moreover, we can generalise this concept to an (e, b)-happy function $S_{e,b}$ for $e, b \in \mathbb{N}$ and $e, b \geqslant 2$ by defining

$$S_{e,b}(x) = a_k^e + a_{k-1}^e + \dots + a_0^e,$$

if $x = (a_k a_{k-1} \dots a_0)_b = a_k b^k + a_{k-1} b^{k-1} + \dots + a_0$ is the b-adic expansion of x with $a_k \neq 0$ and $a_i \in \{0, 1, 2, \dots, b - 1\}$ for all $i = 0, 1, \dots, k$. Then a similar result still holds: there exists a finite set $A \subseteq \mathbb{N}$ such that for any $x \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $S_{e,b}^{(n)}(x) \in A$. For example, if $(e, b) = (2, 10)$, then $A = \{1, 4\}$; and if $(e, b) = (3, 10)$, then

$$A = \{1, 55, 136, 153, 160, 370, 371, 407, 919\}.$$

For more details about this, see for instance in the articles by El-Sedy and Siksek [1], Grundman and Teeple [2], and the book by Guy [3].

On one hand, we may focus on the study of long strings of consecutive integers which are happy or (e, b)-happy as given by El-Sedy and Siksek [1], Pan [4], Zhou and Cai [5], Gilmer [6], Styer [7], and Chase [8]. On the other hand, we may consider generalisations of the concept of (e, b)-happy functions as in the work of Chase [8], Grundman [9], Swart et al. [10], Noppakaew, Phoopha, and Pongsriiam [11], and Subwattanachai and Pongsriiam [12]. In this article, we focus on the latter and continue the study from those articles [11, 12]. Let us consider the following functions.

Definition 1: (The sum of factorials of digits)

Let $b \geqslant 2$ and let $f_b : \mathbb{N} \to \mathbb{N}$ be defined by

$$f_b(x) = a_k! + a_{k-1}! + \dots + a_0!$$

if $x = (a_k a_{k-1} \dots a_0)_b$ is the b-adic representation of x with $a_k \neq 0$.

Definition 2: (A power of sums of digits)

Let $e, b \geqslant 2$ and let $g_{e,b} : \mathbb{N} \to \mathbb{N}$ be defined by

$$g_{e,b}(x) = (a_k + a_{k-1} + \dots + a_0)^e$$

if $x = (a_k a_{k-1} \dots a_0)_b$ is the b-adic representation of x with $a_k \neq 0$.

The functions f_b, $g_{e,b}$, and similar variations are natural examples of new digit maps falling outside the scope of Chase's definition and other

https://doi.org/10.1017/mag.2023.5 Published online by Cambridge University Press

articles on digit maps, yet similar results still hold. That is, if f is such a function, then we can explicitly determine a finite set $A \subseteq \mathbb{N}$ such that for every $x \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $f^{(n)}(x) \in A$. In addition, our results can be interpreted as solutions to certain Diophantine equations which explain some popular mathematical curiosities.

2. *Main results*

Before proceeding further, we emphasise that throughout this Article, if we write a representation of a number without specifying a base, then it is always written in base 10. In this section, we first show the calculation related to f_b and $g_{e,b}$. After that we consider a similar function and give some calculations in less details. Our results are as follows.

Lemma 1: Let b be an integer greater than 1. Then there exists an integer $M = M_b \geqslant 1$ such that

$$(k + 1)(b - 1)! < b^k \text{ for all } k \geqslant M.$$

In particular, if $b = 10$, then we can choose $M_b = 7$.

Proof: By using a usual method in calculus, one can show that $\dfrac{b^k}{k + 1} \to \infty$ as $k \to \infty$. So there is an integer $M \geqslant 1$ such that if $k \geqslant M$, then $\dfrac{b^k}{k + 1}$ is greater than $(b - 1)!$. This proves the first part. For the second part, we prove by induction that

$$(k + 1)9! < 10^k \text{ for all } k \geqslant 7. \tag{1}$$

It is easy to see that (1) holds when $k = 7$. Suppose that $k \geqslant 7$ and (1) holds for k. Then

$$(k + 2)9! < (10k + 10)9! = 10(k + 1)9! < 10^{k+1}.$$

Therefore (1) is verified and the proof is complete.

Remark 1: By a similar method as in the proof of Lemma 1 for $2 \leqslant b \leqslant 9$, we can take M_b as follows: $M_2 = 2$, $M_3 = 2$, $M_4 = 3$, $M_5 = 3$, $M_6 = 4$, $M_7 = 5$, $M_8 = 5$ and $M_9 = 6$.

Theorem 1: Let b and M be integers as in Lemma 1. Then

$$f_b(x) < x \text{ for all } x \geqslant b^M. \tag{2}$$

In particular, $f_{10}(x) < x$ for all $x \geqslant 10^7$.

Proof: Let $x \geqslant b^M$. Then $x = (a_k a_{k-1} \dots a_0)_b$ where $k \geqslant M$, $a_k \neq 0$ and $0 \leqslant a_i \leqslant b - 1$ for all $i = 0, 1, \dots, k$. By Lemma 1, we obtain

$$f_b(x) = a_k! + a_{k-1}! + \dots + a_0! \leqslant (k + 1)(b - 1)! < b^k \leqslant a_k b^k \leqslant x.$$

This proves (2). The second part follows from (2) and Lemma 1.

https://doi.org/10.1017/mag.2023.5 Published online by Cambridge University Press

Remark 2: By Remark 1 and Theorem 1, we see that $f_2(x) < x$ for all $x \geqslant 2^2$, $f_3(x) < x$ for all $x \geqslant 3^2$, $f_4(x) < x$ for all $x \geqslant 4^3$, $f_5(x) < x$ for all $x \geqslant 5^3$, $f_6(x) < x$ for all $x \geqslant 6^4$, $f_7(x) < x$ for all $x \geqslant 7^5$, $f_8(x) < x$ for all $x \geqslant 8^5$ and $f_9(x) < x$ for all $x \geqslant 9^6$.

To obtain a finite set $A \subseteq \mathbb{N}$ satisfying $f_b^{(n)}(x) \in A$, we now only need to recall Theorem 1.2 of Noppakaew, Phoopha and Pongsriiam [11]. Consider the following two conditions for a function $f : \mathbb{N} \to \mathbb{N}$:

(A) There exists $N_f \in \mathbb{N}$ such that $f(x) < x$ for all $x \geqslant N_f$.

(B) For each $x \in \mathbb{N}$, the sequence $\left(f^{(n)}(x)\right)_{n \geqslant 1}$ converges to a fixed point or eventually enters into a cycle. In addition, the number of all such fixed points and cycles is finite.

Then we have the following results.

Theorem 2: (Noppakaew, Phoopha, and Pongsriiam [11]) If $f : \mathbb{N} \to \mathbb{N}$ satisfies the condition (A), then f satisfies the condition (B).

Theorem 3: Let $b \geqslant 2$ be an integer. Then there exists a finite set $A = A_b \subseteq \mathbb{N}$ such that for every $x \in \mathbb{N}$, there is an integer $n \geqslant 1$ such that $f_b^{(n)}(x) \in A$. In particular, if $b = 10$, then we can take

$$A = \{1,\ 2,\ 145,\ 40585,\ 169,\ 871,\ 872\}.$$

In fact 1, 2, 145, 40585 are the fixed points of f_b, and 169, 871, 872 are the elements of distinct cycles arising from the iteration $f_b^{(n)}(x)$ for any $n, x \in \mathbb{N}$.

Proof: By Theorems 1 and 2, we see that f_b satisfies the condition (B). Then we choose A_b to be the set of all elements in the cycles and fixed points of f_b, so that A_b is a finite subset of $\mathbb{N}$. Let $x \in \mathbb{N}$ be given. We know that $f_b : \mathbb{N} \to \mathbb{N}$, so if $f_b^{(n)}(x)$ converges to a fixed point $y \in \mathbb{N}$ as $n \to \infty$, then it means that there is $N \in \mathbb{N}$ such that $f_b^{(n)}(x) = y$ for all $n \geqslant N$. So in particular, $f_b^{(N)}(x) \in A_b$. Moreover, if $f_b^{(n)}(x)$ eventually enters into a cycle as $n \to \infty$, then $f_b^{(n)}(x) \in A_b$ for some n. This proves the first part. For the second part, let $b = 10$, and let F_{10} be the set of fixed points of f_{10} and C_{10} the set of all cycles (which are not fixed points) occurring in the iteration $f_{10}^{(n)}(x)$ for any $n, x \in \mathbb{N}$. We assert that

$$F_{10} = \{1,\ 2,\ 145,\ 40585\} \text{ and}$$

$$C_{10} = \{(169,\ 363601,\ 1454),\ (871,\ 45361),\ (872,\ 45362)\}.$$

It is easy to check that if $x \in \{1,\ 2,\ 145,\ 40585\}$, then $f_{10}(x) = x$. Suppose $x \in \mathbb{N}$ and $f_{10}(x) = x$. By Theorem 1, we obtain $x < 10^7$. So we only need to check the integers x in $[1,\ 10^7)$ whether or not they satisfy $f_{10}(x) = x$. After a computation in a computer, we find that $f_{10}(x) = x$ if, and only if, $x \in \{1,\ 2,\ 145,\ 40585\}$. This gives the set F_{10}. Similarly, to determine the set C_{10}, it is enough to look for the cycles occurring in the sequence $\left(f^{(n)}(x)\right)$ where x runs over the integers in $[1,\ 10^7)$. After a straightforward verification, we obtain C_{10} as asserted.

https://doi.org/10.1017/mag.2023.5 Published online by Cambridge University Press

Therefore we can take A to be the set consisting of 1, 2, 145, 40585, 169, 363601, 1454, 871, 45361, 872, 45362. But 169, 363601, 1454 are in the same cycle, so we need only one of them. For instance, if $f_{10}^{(n)}(x) = 169$, then $f_{10}^{(n+1)}(x) = 363601$, $f_{10}^{(n+2)}(x) = 1454$ and $f_{10}^{(n+3)} = 169$. Similarly, we can choose just one of 871, 45361 and one of 872, 45362. Therefore we can take A to be the set consisting of 1, 2, 145, 40585, 169, 871, 872 as required. This completes the proof.

Remark 3: By a similar method as in Theorem 3, we obtain for $2 \leqslant b \leqslant 9$ the set F_b of fixed points of f_b and the set C_b of cycles in the iteration $f_b^{(n)}(x)$ for any $n, x \in \mathbb{N}$ as follows. For $b = 2$, we only need to run a computation in a computer for x in $[1, 2^2)$ to obtain that $F_2 = \{1, 2\}$ and $C_2 = \varnothing$. Similarly, for $b = 3, 4, 5, 6, 7, 8, 9$, we run computations, for $x \in [1, 3^2)$, $x \in [1, 4^3)$, $x \in [1, 5^3)$, $x \in [1, 6^4)$, $x \in [1, 7^5)$, $x \in [1, 8^5)$, $x \in [1, 9^6)$ respectively, to obtain

$$F_3 = \{1, 2\}, C_3 = \varnothing,$$
$$F_4 = \{1, 2, 7\}, C_4 = \{(3, 6)\},$$
$$F_5 = \{1, 2, 49\}, C_5 = \varnothing,$$
$$F_6 = \{1, 2, 25, 26\}, C_6 = \varnothing,$$
$$F_7 = \{1, 2\}, C_7 = \{(38, 126, 27, 726, 243, 864)\},$$
$$F_8 = \{1, 2\}, C_8 = \{(3, 6, 720, 10), (125, 5161)\},$$
$$F_9 = \{1, 2, 41282\},$$

and C_9 consists of exactly one cycle, namely,

(1450, 80642, 251, 40327, 10803, 5173, 15121, 1445, 45481, 41094, 735, 723, 80646, 969, 41043).

The calculation for $g_{e,b}$ is similar to that for f_b, but the well-known Euler constant will appear in the proof. So to avoid confusion, we will write $E = 2.718\ldots$ to denote the base of the natural logarithm, while e is reserved for the integers appearing in the definition of $g_{e,b}$.

Lemma 2: We have $81(k+1)^2 < 10^k$ for all $k \geqslant 4$, $729(k+1)^3 < 10^k$ for all $k \geqslant 6$, $6561(k+1)^4 < 10^k$ for $k \geqslant 8$ and $59049(k+1)^5 < 10^k$ for $k \geqslant 10$. In general, if $e \geqslant 2$ is an integer, then

$$9^e(k+1)^e < 10^k \text{ for all } k \geqslant e^2. \tag{3}$$

Proof: The first four inequalities can be proved straightforwardly by induction, so we leave the details to the reader. For (3), let $e \geqslant 2$ be an integer. Observe that it can be proved by induction that $9(n^2+1) < 10^n$ for all $n \geqslant 2$, so in particular $9(e^2+1) < 10^e$. This implies that (3) holds when $k = e^2$. Next, suppose that $k \geqslant e^2$ and (3) holds for k. Recall that the sequence $(a_n) = \left((1 + \frac{1}{n})^n\right)$ is strictly increasing and converges to $E = 2.718\ldots$, the base of the natural logarithm. From this and the fact that

https://doi.org/10.1017/mag.2023.5 Published online by Cambridge University Press

$k \geqslant e^2$, we obtain

$$\frac{(k+2)^e}{(k+1)^e} = \left(1 + \frac{1}{k+1}\right)^e \leqslant \left(1 + \frac{1}{e^2+1}\right)^e < \left(1 + \frac{1}{e^2+1}\right)^{e^2+1}$$
$$= a_{e^2+1} \leqslant \sup\{a_n \mid n \in \mathbb{N}\} = \lim_{n\to\infty} a_n = E < 10.$$

Then $9^e(k+2)^e < 9^e(10)(k+1)^e < 10^{k+1}$, by the induction hypothesis. So the proof is complete.

Lemma 2 will be used in the calculation in some examples. For a general result, we have the following theorem.

Theorem 4: Let e, $b \geqslant 2$ be integers. Then the following statements hold.

(i) There exists an integer $M = M_{e,b} \geqslant 1$ such that $(k+1)^e(b-1)^e < b^k$ for all $k \geqslant M$.

(ii) $g_{e,b}(x) < x$ for all $x \geqslant b^M$.

(iii) $g_{e,b}$ satisfies the condition (B) and there exists a finite set $A = A_{e,b} \subseteq \mathbb{N}$ such that for every $x \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that $g_{e,b}^{(n)}(x) \in A$.

(iv) Let $F_{e,b}$ and $C_{e,b}$ be the sets of fixed points of $g_{e,b}$ and the cycles arising in the sequence $\left(g_{e,b}^{(n)}(x)\right)_{n\geqslant 1}$ for any $x \in \mathbb{N}$. Then we have

$$F_{2,10} = \{1,\ 81\}, C_{2,10} = \{(169,\ 256)\},$$

$$F_{3,10} = \{1,\ 512,\ 4913,\ 5832,\ 17576,\ 19683\}, C_{3,10} = \{(6859,\ 21952)\},$$

$$F_{4,10} = \{1,\ 2401,\ 234256,\ 390625,\ 614656,\ 1679616\},$$

$$C_{4,10} = \{(104976,\ 531441)\},$$

$$F_{5,10} = \{1,\ 17210368,\ 52521875,\ 60466176,\ 205962976\},$$

and $C_{5,10}$ consists of the following cycles:

(16807, 5153632, 9765625, 102400000), (6436343, 20511149),
(28629151, 45435424).

Proof: Since e, b are already given, we obtain $\dfrac{b^k}{(k+1)^e} \to \infty$ as $k \to \infty$ and so there exists $M \geqslant 1$ such that $\dfrac{b^k}{(k+1)^e} > (b-1)^e$ for all $k \geqslant M$. This proves (i). Suppose $x \geqslant b^M$. Then $x = (a_k a_{k-1} \dots a_0)_b$ where $k \geqslant M$, $a_k \neq 0$ and $0 \leqslant a_i \leqslant b-1$ for all $i = 0, 1, \dots, k$. Then by (i), we obtain

$$g_{e,b} = (a_k + a_{k-1} + \dots + a_0)^e \leqslant ((k+1)(b-1))^e < b^k \leqslant a_k b^k \leqslant x.$$

This proves (ii). Then (iii) follows from (ii), Theorem 2, and exactly the same argument as in Theorem 3. For (iv), to determine the set $F_{e,b}$ and $C_{e,b}$ for a particular pair of (e, b), we only need to apply Lemma 2 and run a

https://doi.org/10.1017/mag.2023.5 Published online by Cambridge University Press

computation on the integers in $[1, b^M)$ as in the proof of Theorem 3. If $e = 2$ and $b = 10$, we can take $M_{e,b} = 4$. After checking $\left(g_{e,b}^{(n)}(x)\right)$ for x in the interval $[1, 10^4)$, we obtain $F_{2,10} = \{1, 81\}$, $C_{2,10} = \{(169, 256)\}$. If $e = 3$ and $b = 10$, we can take $M_{e,b} = 6$. Then running a computation for $g_{e,b}^{(n)}(x)$ where $n \in \mathbb{N}$ and $x \in [1, 10^6)$, we obtain

$$F_{3,10} = \{1, 512, 4913, 5832, 17576, 19683\} \text{ and } C_{3,10} = \{(6859, 21952)\}.$$

Similarly, if $(e, b) = (4, 10)$, then we take $M_{e,b} = 8$; if $(e, b) = (5, 10)$, then we take $M_{e,b} = 10$, After running a computation in a computer, we obtain $F_{4,10}$, $C_{4,10}$, $F_{5,10}$ and $C_{5,10}$ as given above. So the proof is complete.

Observing that $3435 = 3^3 + 4^4 + 3^3 + 5^5$, we are interested in determining all numbers with this property. So we should consider $h(x) = a_k^{a_k} + a_{k-1}^{a_{k-1}} + \dots + a_0^{a_0}$ if $x = (a_k a_{k-1} \dots a_0)_{10}$ but there is a problem with this definition since 0^0 is not defined. One way to avoid this is to skip the zero digit and define $h(x) = b_m^{b_m} + b_{m-1}^{b_{m-1}} + \dots + b_0^{b_0}$ if $x = (a_k a_{k-1} \dots a_0)_{10}$ and $b_m, b_{m-1}, \dots, b_0$ are taken from $a_k, a_{k-1}, \dots, a_0$ but without zero. Equivalently, we can temporarily assign the value $0^0 = 0$ and study the following function.

Definition 3: Let $h = \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be defined by $h(0) = 0$, $h(a) = a^a$ if $a \in \{1, 2, \dots, 9\}$ and

$$h(x) = h(a_k) + h(a_{k-1}) + \dots + h(a_0)$$

if $x \geqslant 10$ and $x = (a_k a_{k-1} \dots a_0)_{10}$ is the decimal representation of x with $a_k \neq 0$. Equivalently, we can assign $0^0 = 0$ and define h by

$$h(x) = a_k^{a_k} + a_{k-1}^{a_{k-1}} + \dots a_0^{a_0}$$

for each $x = (a_k a_{k-1} \dots a_0)_{10}$.

The calculation for h can be done in the same way as that for f_b and $g_{e,b}$, so we skip the details and leave them to the reader. We have the following result.

Theorem 5: The following statements hold.

(i) $(k + 1)9^9 < 10^k$ for all $k \geqslant 10$.

(ii) $h(x) < x$ for all $x \geqslant 10^{10}$.

(iii) h satisfies the condition (B) and there exists a finite set $A \subseteq \mathbb{N}$ such that for every $x \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that $h^{(n)}(x) \in A$.

(iv) The set of fixed points of h is $\{1, 3435, 438579088\}$.

Proof: The statement (i) can be proved by induction. If $x \geqslant 10^{10}$, then we write $x = (a_k a_{k-1} \dots a_0)_{10}$ with $k \geqslant 10$ and $a_k \neq 0$, and so

$$h(x) \leqslant 9^9(k + 1) < 10^k \leqslant a_k 10^k \leqslant x.$$

Then (iii) follows from (ii), Theorem 2, and exactly the same argument as before. Then running a computation in a computer, we obtain (iv).

https://doi.org/10.1017/mag.2023.5 Published online by Cambridge University Press

3. *Diophantine equations and proofs of some mathematical curiosities*

Many people have enjoyed numerical curiosities which are distributed via social media worldwide. They can be discovered by anyone and can definitely be appreciated without proofs or explanations. Nevertheless, we show that our results can be interpreted as solutions to certain Diophantine equations and use them to explain or create some curiosities. For example, the only fixed points of f_{10} are 1, 2, 145 and 40585, and so the solutions in non-negative integers $a_k, a_{k-1}, \dots, a_0$ with $a_k \neq 0$ to the Diophantine equation

$$a_k! + a_{k-1}! + \dots + a_0! = (a_k a_{k-1} \dots a_0)_{10}$$

are given by the numbers 1, 2, 145 and 40585.

Corollary: $1 = 1!$, $2 = 2!$, $145 = 1! + 4! + 5!$, $40585 = 4! + 0! + 5! + 8! + 5!$ and these are the only positive integers with this property. That is, a positive integer x is the sum of the factorials of all its decimal digits (except the leading zeros) if, and only if, $x = 1, 2, 145$ or 40585.

Proof: This follows immediately from Theorem 3.

Corollary:

$$1 = (1)_9 = 1!, \ 2 = (2)_9 = 2!, \ 41282 = (62558)_9 = 6! + 2! + 5! + 5! + 8!$$

and these are the only positive integers with this property. That is, if $x \in \mathbb{N}$, then x is the sum of factorials of its digits (in base 9) if, and only if,

$$x = (1)_9, (2)_9, (62558)_9 .$$

Proof: This follows immediately from Remark 3.

Corollary: We have

$$1 = 1^3, \ 512 = (5 + 1 + 2)^3, \ 4913 = (4 + 9 + 1 + 3)^3,$$

$$5832 = (5 + 8 + 3 + 2)^3, \ 17576 = (1 + 7 + 5 + 7 + 6)^3,$$

$$19683 = (1 + 9 + 6 + 8 + 3)^3,$$

and these are the only positive integers with this property. That is, if $x \in \mathbb{N}$, then x is the cubes of the sum of its decimal digits if, and only if,

$$x = 1, \ 512, \ 4913, \ 5832, \ 17576 \text{ or } 19683.$$

Similarly,

$$1 = 1^4, \qquad 2401 = (2 + 4 + 0 + 1)^4$$

$$234256 = (2 + 3 + 4 + 2 + 5 + 6)^4, \qquad 390625 = (3 + 9 + 0 + 6 + 2 + 5)^4$$

$$614656 = (6 + 1 + 4 + 6 + 5 + 6)^4, \qquad 1679616 = (1 + 6 + 7 + 9 + 6 + 1 + 6)^4$$

https://doi.org/10.1017/mag.2023.5 Published online by Cambridge University Press

are the only positive integers that are equal to the 4th power of the sum of their decimal digits;

$$1 = 1^5,$$
$$17210368 = (1 + 7 + 2 + 1 + 0 + 3 + 6 + 8)^5,$$
$$52521875 = (5 + 2 + 5 + 2 + 1 + 8 + 7 + 5)^5,$$
$$60466176 = (6 + 0 + 4 + 6 + 6 + 1 + 7 + 6)^5,$$
$$205962976 = (2 + 0 + 5 + 9 + 6 + 2 + 9 + 7 + 6)^5,$$

are the only positive integers that are equal to the 5th power of the sum of their decimal digits.

Proof: This follows immediately from Theorem 4.

Corollary:

$$1 = 1^1, \; 3435 = 3^3 + 4^4 + 3^3 + 5^5,$$
$$438579088 = 4^4 + 3^3 + 8^8 + 5^5 + 7^7 + 9^9 + 8^8 + 8^8,$$

and these are the only positive integers with this property.

Proof: This follows immediately from Theorem 5.

Other known results in the literature can be used to produce similar relationships too. Here we rewrite the results of Grundman and Teeple [2], and Hargreaves and Siksek [13].

Corollary: (Grundman and Teeple [2], and Hargreaves and Siksek [13]) We have

$$1 = 1^3, \; 153 = 1^3 + 5^3 + 3^3, \; 370 = 3^3 + 7^3 + 0^3, \; 371 = 3^3 + 7^3 + 1^3, \; 407 = 4^3 + 0^3 + 7^3,$$

and these are the only positive integers with this property. That is, if $x \in \mathbb{N}$, then x is the sum of the cubes of its decimal digits if, and only if, $x = 1$, 153, 370, 371, 407. Similarly,

$$1 = 1^4, \; 1634 = 1^4 + 6^4 + 3^4 + 4^4, \; 8208 = 8^4 + 2^4 + 0^4 + 8^4, \; 9474 = 9^4 + 4^4 + 7^4 + 4^4,$$

are the only positive integers that are equal to the sum of the 4th powers of their decimal digits. In addition,

$$1 = 1^5, \; 4150 = 4^5 + 1^5 + 5^5 + 0^5, \; 4151 = 4^5 + 1^5 + 5^5 + 1^5,$$
$$54748 = 5^5 + 4^5 + 7^5 + 4^5 + 8^5, \; 92727 = 9^5 + 2^5 + 7^5 + 2^5 + 7^5,$$
$$93084 = 9^5 + 3^5 + 0^5 + 8^5 + 4^5, \; 194979 = 1^5 + 9^5 + 4^5 + 9^5 + 7^5 + 9^5$$

are the only positive integers that are equal to the sum of the 5th powers of their decimal digits.

Acknowledgement: We would like to thank the referee for his or her comments which improve the presentation of this article. Niphawan Phoopha receives a scholarship from DPST, Thailand. Prapanpong Pongsriiam's research project is funded by the National Research Council of

Thailand (NRCT), Grant Number NRCT5–RSA63021–02. He is also supported by the Tosio Kato Fellowship during his visit to Nagoya University from July 2022 to July 2023. Prapanpong Pongsriiam is the corresponding author of this manuscript.

References

1. E. El-Sedy and S. Siksek, On happy numbers, *Rocky Mountain J. Math.* **30** (2000) pp. 565-570.
2. H. G. Grundman and E. A. Teeple, Generalized happy numbers, *Fibonacci Quart.* **39**(5) (2001) pp. 462-466.
3. R. K. Guy, *Unsolved problems in number theory* (3rd edn.), Springer-Verlag (1981).
4. H. Pan, On consecutive happy numbers, *J. Number Theory* **128** (2008) pp. 1646-1654.
5. X. Zhou and T. Cai, On *e*-power *b*-happy numbers, *Rocky Mountain J. Math.* **39** (2009) pp. 2073-2081.
6. J. Gilmer, On the density of happy numbers, *Integers* **13** (2013), pp. 1-25.
7. R. Styer, Smallest examples of strings of consecutive happy numbers, *J. Integer Seq.* **13** (2010), Article 10.6.3.
8. Z. Chase, On the iterates of digit maps, *Integers* **18** (2018) pp. 1-7.
9. H. G. Grundman, Semihappy numbers, *J. Integer Seq.* **13** (2010), Article 10.4.8.
10. B. B. Swart, K. A. Beck, S. Crook, C. E. Turner, H. G. Grundman, M. Mei and L. Zack, Augmented generalized happy functions, *Rocky Mountain J. Math.* **47** (2017) pp. 403-417.
11. P. Noppakeaw, N. Phoopha, and P. Pongsriiam, Composition of happy functions, Notes *Number Theory Discrete Math.* **25** (2019) pp. 13-20.
12. K. Subwattanachai and P. Pongsriiam, Composition of happy functions and digit maps, *Int. J. Math. Comp. Sci.* **16**(1) (2021) pp. 169-178.
13. K. Hargreaves and S. Siksek, Cycles and fixed points of happy functions, *J. Comb. Number Theory* **2**(3) (2010) pp. 65-77.

10.1017/mag.2023.5
Published by Cambridge University Press
on behalf of The Mathematical Association

NIPHAWAN PHOOPHA,
PRAPANPONG PONGSRIIAM
Department of Mathematics,
Faculty of Science, Silpakorn University,
Nakhon Pathom, 73000, Thailand
e-mails: *phoopha.miw@gmail.com, prapanpong@gmail.com*
PHAKHINKON NAPP PHUNPHAYAP
Department of Mathematics,
Faculty of Science, Burapha University,
Chonburi, 20131 Thailand
e-mails: *phakhinkon.ph@go.buu.ac.th*, *phakhinkon@gmail.com*