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Nagoya Math. J.
Vol. 121 (1991), 195–199

ON THE STRONG UNIMODALITY OF LÉVY PROCESSES

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§ 1. Introduction and results

A measure $\mu(dx)$ on R is said to be unimodal with mode a if $\mu(dx) = c\delta_a(dx) + f(x)dx$, where $c \geqq 0$, $\delta_a(dx)$ is the delta measure at a and $f(x)$ is non-decreasing for $x < a$ and non-increasing for $x > a$. A measure $\mu(dx) = \sum_{n=-\infty}^{\infty} p_n\delta_n(dx)$ on $Z = \{0, \pm 1, \pm 2, \cdots\}$ is said to be unimodal with mode a if p_n is non-decreasing for $n \leqq a$ and non-increasing for $n \geqq a$. A probability measure $\mu(dx)$ on R (resp. on Z) is said to be strongly unimodal on R (resp. on Z) if, for every unimodal probability measure $\eta(dx)$ on R (resp. on Z), the convolution $\mu * \eta(dx)$ is unimodal on R (resp. on Z). Let X_t, $t \in [0, \infty)$, be a Lévy process (that is, a process with stationary independent increments starting at the origin) on R (resp. on Z) with the Lévy measure $\nu(dx)$. The process X_t is said to be unimodal on R (resp. on Z) if, for every $t > 0$, the distribution of X_t is unimodal on R (resp. on Z). It is said to be strongly unimodal on R (resp. on Z) if, for every $t > 0$, the distribution of X_t is strongly unimodal on R (resp. on Z). In this paper we shall characterize strongly unimodal Lévy processes on R and Z.

THEOREM 1. *Let X_t be a Lévy process on R. Then X_t is strongly unimodal on R if and only if*

$$X_t = \sigma B(t) + \gamma t\,,$$

where $B(t)$ is a Brownian motion and σ and γ are constants, $\sigma \geqq 0$.

THEOREM 2. *Let X_t be a Lévy process on Z. Then X_t is strongly unimodal on Z if and only if*

$$X_t = X_{at}^{(1)} - X_{bt}^{(2)}\,,$$

Received June 25, 1990.

https://doi.org/10.1017/S0027763000003470 Published online by Cambridge University Press

where $X_t^{(1)}$ and $X_t^{(2)}$ are independent Poisson processes and a and b are non-negative constants.

Ibragimov [1] proves that a probability measure on R is strongly unimodal if and only if it is a delta measure or absolutely continuous with support being an interval and the density being log-concave. As a counterpart on Z, Keilson-Gerber [2] proves that a probability measure $\mu(dx) = \sum_{n=-\infty}^{\infty} p_n \delta_n(dx)$ on Z is strongly unimodal if and only if $p_n^2 \geqq p_{n+1} p_{n-1}$ for every $n \in Z$. These results play an essential role in our proof.

The following are main related results. Yamazato [9] shows that if the density of $|x|\nu(dx)$ is log-concave on R-{0}, then the distribution of X_t is strongly unimodal on R for sufficiently large $t > 0$. It is an open problem to characterize unimodal Lévy processes on R or Z in terms of their Lévy measures. Wolfe [7] proves that, if X_t is unimodal on R (resp. on Z), then $\nu(dx)$ (resp. $\nu(dx) + c\delta_0(dx)$ for some $c > 0$) is unimodal on R (resp. on Z) with mode 0, and that the converse does not hold. Medgyessy [3] shows that if $\nu(dx)$ is symmetric and unimodal on R, then X_t is unimodal on R. The analogous result on Z is observed by Wolfe [7]. As a big advancement, Yamazato [8] shows that Lévy processes of class L are unimodal on R. Steutel-van Harn [4] proves the unimodality of Lévy processes on the non-negative integers analogous to class L. Watanabe [5] constructs non-symmetric unimodal Lévy processes on R that are not of class L. Watanabe [6] gives a similar result for Lévy processes on the non-negative integers.

§ 2. Proof of Theorem 1

Let $\mu_t(dx)$ be the distribution of X_t. Then we have

$$\int_{-\infty}^{\infty} e^{izx}\mu_t(dx) = e^{t\psi(z)}\,, \tag{2.1}$$

$$\psi(z) = i\gamma z - 2^{-1}\sigma^2 z^2 + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx(1+x^2)^{-1})\nu(dx)\,,$$

where $\gamma \in R$, $\sigma^2 \geqq 0$, and

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x^2(1+x^2)^{-1}\nu(dx) < \infty\,.$$

The measure $\nu(dx)$ is called the Lévy measure of X_t.

https://doi.org/10.1017/S0027763000003470 Published online by Cambridge University Press

Proof of "if" part. Since normal distributions are strongly unimodal on R by Ibragimov's result [1], $X_t = \sigma B(t) + \gamma t$ is strongly unimodal on R.

Proof of "only if" part. Suppose that X_t is strongly unimodal on R and not deterministic. Then, for each $t > 0$,

$$\mu_t(dx) = f_t(x)dx\,, \tag{2.2}$$

the set $\{x: f_t(x) > 0\}$ is an interval, and $\log f_t(x)$ is concave on this set. This is by Ibragimov's result [1]. By Wolfe's theorem [7],

$$\nu(dx) = \phi(x)dx\,, \tag{2.3}$$

with $\phi(x)$ non-decreasing for $x < 0$ and non-increasing for $x > 0$. It is well-known that, for any bounded continuous function $g(x)$ with support in $R - \{0\}$, it holds that

$$\lim_{t\to 0} t^{-1}\int_{-\infty}^{\infty} g(x)\mu_t(dx) = \int_{-\infty}^{\infty} g(x)\nu(dx)\,. \tag{2.4}$$

Hence, by Lemma 3 of Ibragimov [1], we can choose a sequence $t(n)$ such that, as $n\to\infty$, $t(n)\to 0$ and

$$t(n)^{-1}f_{t(n)}(x) \longrightarrow \phi(x) \tag{2.5}$$

for a.e. $x \in R$. It follows that $\log\phi(x)$ is concave on the support of $\phi(x)$ by (2.5). Therefore, $\phi(x)$ is bounded on R and

$$c = \nu(R) = \int_{-\infty}^{\infty}\phi(x)dx < \infty\,. \tag{2.6}$$

Suppose that $c > 0$. We shall show that this leads to a contradiction. Let

$$\gamma_0 = \gamma - \int_{-\infty}^{\infty} x(1+x^2)^{-1}\nu(dx)\,.$$

We can assume $\gamma_0 = 0$, because we can consider $X_t - \gamma_0 t$ instead of X_t. There are two possible cases.

Case 1. $\sigma = 0$. The process X_t is a compound Poisson process and hence $\mu_t(\{0\}) > 0$. This is a contradiction because non-trivial strongly unimodal probability measure on R has no point mass.

Case 2. $\sigma^2 > 0$. We get, for any $t > 0$,

$$\mu_t(dx) = \mu_t^{(1)} * \mu_t^{(2)}(dx)\,, \tag{2.7}$$

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where $\mu_t^{(1)}(dx) = g_t(x)dx$ is the normal distribution with mean 0 and variance $\sigma^2 t$, and $\mu_t^{(2)}(dx)$ is a compound Poisson distribution. Since $\mu_t^{(2)}(\{0\}) \to 1$ as $t \to 0$, we obtain from (2.7) that

$$\lim_{t\to 0} \{g_t(0)\}^{-1} f_t(0) = \lim_{t\to 0} (2\pi t)^{1/2} \sigma f_t(0) = 1 . \tag{2.8}$$

We have, by Ibragimov's theorem [1],

$$\{f_t(x)\}^2 \geqq f_t(0) f_t(2x) \tag{2.9}$$

for any $t > 0$ and $x \in R$. Hence we obtain from (2.5), (2.8), and (2.9) that

$$\begin{aligned} 0 &= \lim_{n\to\infty} (2\pi)^{1/2} \sigma \{t(n)\}^{3/2} \{(t(n))^{-1} f_{t(n)}(x)\}^2 \\ &\geqq \lim_{n\to\infty} (2\pi t(n))^{1/2} \sigma f_{t(n)}(0) \{(t(n))^{-1} f_{t(n)}(2x)\} = \phi(2x) \end{aligned} \tag{2.10}$$

for a.e. $x \in R$. It follows that $\phi(x) = 0$ for a.e. $x \in R$. This contradicts the assumption $c > 0$.

Therefore, if X_t is strongly unimodal on R, then $\nu(dx) = 0$. Thus we have proved Theorem 1.

§ 3. Proof of Theorem 2

Let X_t be a Lévy process on Z. Then we can write (2.1) as

$$\psi(z) = \int_Z (e^{izx} - 1) \nu(dx) \tag{3.1}$$

with $\nu(\{0\}) = 0$ and $\nu(Z) < \infty$.

Proof of "if" part. Since Poisson distributions are strongly unimodal on Z by Keilson-Gerber [2], $X_t = X_{at}^{(1)} - X_{bt}^{(2)}$ is strongly unimodal on Z.

Proof of "only if" part. Suppose that X_t is strongly unimodal on Z. Let $\mu_t(dx) = \sum_{n=-\infty}^{\infty} p_n(t) \delta_n(dx)$ be the distribution of X_t. By Keilson-Gerber's theorem [2], we have

$$\{p_1(t)\}^2 \geqq p_0(t) p_2(t) \tag{3.2}$$

for any $t > 0$. Since $\mu_t(dx)$ converges weakly to $\delta_0(dx)$ as $t \to 0$, we get

$$\lim_{t\to 0} p_0(t) = 1 . \tag{3.3}$$

Since (2.4) holds, we have

$$\lim_{t\to 0} t^{-1} p_n(t) = \nu(\{n\}) \tag{3.4}$$

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for $n \neq 0$. Hence we obtain from (3.2), (3.3), and (3.4) that

$$0 = \lim_{t \to 0} t(t^{-1}p_1(t))^2 \geqq \lim_{t \to 0} p_0(t)t^{-1}p_2(t) = \nu(\{2\}) . \tag{3.5}$$

Therefore we get $\nu(\{2\}) = 0$. Since $\nu(\{n\})$ is non-increasing for $n \geqq 1$ by Wolfe's theorem [7], this implies that $\nu(\{n\}) = 0$ for $n \geqq 2$. Similarly we have $\nu(\{n\}) = 0$ for $n \leqq -2$. The proof of Theorem 2 is complete.

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https://doi.org/10.1017/S0027763000003470 Published online by Cambridge University Press