ON WELL-BOUNDED OPERATORS

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1. Introduction

This note is concerned with a question arising from some work of D. R. Smart (**2**). In the introduction we indicate the nature of the problem. Notations will be explained only in as far as they differ from those of (**2**).

Let $\mathfrak{B}$ be a reflexive Banach space and T a bounded linear operator from $\mathfrak{B}$ into itself. Following Smart we shall say that T is *well-bounded* if it is possible to choose a constant K and a finite closed interval $J = [a, b]$ of the real line in such a way that

$$||p(T)|| \leqq K\{\sup_{\lambda \in J} |p(\lambda)| + \operatorname*{var}_J p(\lambda)\}$$

for every real [1] polynomial $p(\lambda)$. We may write the above inequality in the form

$$||p(T)|| \leqq K\mathbf{|}p\mathbf{|}_J, \tag{1}$$

where $\mathbf{|}p\mathbf{|}_J$ is defined to be the bracketed expression on the right hand side of the original inequality. We shall usually omit the suffix J and simply write $\mathbf{|}p\mathbf{|}$: but at one stage of § 3 the interval J enters essentially into the argument. Throughout this note the symbol K denotes the constant occurring in (1).

In his paper Smart develops for well-bounded operators a theory analogous to the spectral theory of bounded Hermitian operators in Hilbert space. He produces a functional calculus in which to every function $p(\lambda)$ which is absolutely continuous on J there corresponds an operator $p(T)$ for which (1) is valid. He also constructs a spectral family $\{E_\lambda\}$ of projections in $\mathfrak{B}$, and proves the existence of the "scalar operator"

$$S = \int \lambda dE_\lambda.$$

[1] When $\mathfrak{B}$ is a complex Banach space we may deduce that a similar condition, in which K is replaced by $2K$, is valid for *complex* polynomials. In this note, complex polynomials are not needed, and the term "polynomial" is therefore used for "real polynomial."

The operator $S-T$ is shown to be quasi-nilpotent, and it is conjectured that $S = T$. In the present note we shall prove this conjecture.

2. Preliminaries

Throughout the remainder of this note it will be assumed that T is a well-bounded operator acting in a reflexive Banach space $\mathfrak{B}$, that $\{E_\lambda\}$ is the associated spectral family of projections in $\mathfrak{B}$, and that S is the corresponding scalar operator. For the definitions and properties of these operators we refer to (**2**).

Let μ be an arbitrary real number. We define a function $f_\mu^{(n)}(\lambda)$ for each positive integer n as follows:

$$f_\mu^{(n)}(\lambda) = \begin{cases} 1 & (\lambda \leqq \mu - 1/n), \\ 0 & (\lambda \geqq \mu + 1/n), \end{cases}$$

and $f_\mu^{(n)}(\lambda)$ is linear on the interval $[\mu - 1/n, \mu + 1/n]$. We note that $f_\mu^{(n)}(\lambda)$ is an absolutely continuous function of λ which takes the value $\frac{1}{2}$ when $\lambda = \mu$.

LEMMA 1. *Let $x \in \mathfrak{B}$. Then*

$$f_\mu^{(n)}(T)x \to \tfrac{1}{2}(E_\mu + E_{\mu-})x \tag{2}$$

as $n \to \infty$. Convergence is in the norm topology of $\mathfrak{B}$, and $E_{\mu-}$ denotes the (strong) limit of E_λ as $\lambda \to \mu - 0$.

PROOF. We shall first consider the case in which x may be expressed in the form

$$\begin{aligned} x &= p(T)x_1 + q(T)x_2 + w \\ &= \quad u \quad + \quad v \quad + w, \end{aligned} \tag{3}$$

where $p(\lambda)$ is an absolutely continuous function vanishing on a neighbourhood of $[\mu, \infty)$, $q(\lambda)$ is an absolutely continuous function vanishing on a neighbourhood of $(-\infty, \mu]$, $x_1, x_2 \in \mathfrak{B}$ and $Tw = \mu w$. The set $\mathfrak{B}_d$ of all such elements x forms a dense subspace of $\mathfrak{B}$ (see (**2**) § 4, the proof of Theorem A (vi)).

Since $f_\mu^{(n)}(\lambda)$ takes the value unity on $(-\infty, \mu - 1/n)$ and zero on $(\mu + 1/n, \infty)$, it is clear that, for all sufficiently large n,

$$f_\mu^{(n)}(\lambda)p(\lambda) \equiv p(\lambda), \qquad f_\mu^{(n)}(\lambda)q(\lambda) \equiv 0.$$

We deduce that

$$\begin{aligned} f_\mu^{(n)}(T)u &= f_\mu^{(n)}(T)p(T)x_1 \\ &= p(T)x_1 \\ &= u, \end{aligned}$$

while

$$f_\mu^{(n)}(T)v = f_\mu^{(n)}(T)q(T)x_2 = 0,$$

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for all sufficiently large n. Finally we note that, since $Tw = \mu w$, we have

$$f(T)w = f(\mu)w$$

for every polynomial $f(\lambda)$ and hence, by continuity, for every absolutely continuous function $f(\lambda)$. Thus

$$f_\mu^{(n)}(T)w = \tfrac{1}{2}w.$$

From equation (3) we now obtain

$$\begin{aligned} f_\mu^{(n)}(T)x &= f_\mu^{(n)}(T)(u + v + w) \\ &= u + \tfrac{1}{2}w \end{aligned}$$

for all sufficiently large n. Since

$$\begin{aligned} E_\mu x &= u + w, \\ E_{\mu-} x &= u, \end{aligned}$$

(see (**2**) § 4), we have in fact shown that

$$f_\mu^{(n)}(T)\,x = \tfrac{1}{2}(E_\mu + E_{\mu-})\,x$$

for all sufficiently large n. Thus (2) is proved for the case in which x is an element of the dense subspace $\mathfrak{B}_d$.

We now remark that, since

$$\| f_\mu^{(n)} \| = \sup_{\lambda \in J} |f_\mu^{(n)}(\lambda)| + \operatorname*{var}_J f_\mu^{(n)}(\lambda) \leqq 2,$$

we have

$$\| f_\mu^{(n)}(T) \| \leqq 2K \quad (n = 1, 2, 3, \cdots).$$

From one of the Banach-Steinhaus theorems (see, for example, (**1**) 25, Theorem 2.12.1) it now follows that (2) holds for all $x \in \mathfrak{B}$. This proves the lemma.

Suppose now that μ_1, μ_2 are real numbers such that $\mu_1 < \mu_2$. For integers n such that $\mu_2 - \mu_1 > 2/n$, we define a function $\phi_{\mu_1, \mu_2}^{(n)}(\lambda)$ as follows:

$$\phi_{\mu_1, \mu_2}^{(n)}(\lambda) = \begin{cases} 0 & \text{unless } \mu_1 - 1/n < \lambda < \mu_2 + 1/n, \\ 1 & \text{when } \mu_1 + 1/n \leqq \lambda \leqq \mu_2 - 1/n, \end{cases}$$

and $\phi_{\mu_1, \mu_2}^{(n)}(\lambda)$ is linear on each of the intervals $[\mu_1 - 1/n, \mu_1 + 1/n]$ and $[\mu_2 - 1/n, \mu_2 + 1/n]$. We shall later use such functions in approximating to step functions by absolutely continuous functions. It is apparent that $\phi_{\mu_1, \mu_2}^{(n)}(\lambda)$ is absolutely continuous, and we have the following result.

LEMMA 2. *Let $x \in \mathfrak{B}$. Then*

$$\phi_{\mu_1, \mu_2}^{(n)}(T)x \to \tfrac{1}{2}(E_{\mu_2} + E_{\mu_2-} - E_{\mu_1} - E_{\mu_1-})\,x \tag{4}$$

as $n \to \infty$. Convergence is in the norm topology of $\mathfrak{B}$.

PROOF. Since
$$\phi^{(n)}_{\mu_1,\mu_2}(\lambda) \equiv f^{(n)}_{\mu_2}(\lambda) - f^{(n)}_{\mu_1}(\lambda),$$
we have
$$\phi^{(n)}_{\mu_1,\mu_2}(T) = f^{(n)}_{\mu_2}(T) - f^{(n)}_{\mu_1}(T),$$
and the result is an immediate consequence of Lemma 1.

REMARK. If $E_\lambda x$ is continuous at $\lambda = \mu_1$, $\lambda = \mu_2$, then
$$\phi^{(n)}_{\mu_1,\mu_2}(T)x \to (E_{\mu_2} - E_{\mu_1})x$$
as $n \to \infty$.

3. A subsidiary conjecture

In this section we obtain an estimate for $||S - T||$, and in the process verify a subsidiary conjecture of Smart, made in formula (9) of (**2**). The proof of this result involves only slight amendment of the "heuristic reasons" put forward in § 5 of (**2**) in support of the conjecture.

Let $x_0 \in \mathfrak{B}$. We shall obtain an estimate for $||Sx_0 - Tx_0||$. This will be done by approximating to S by suitable functions of T in the strong operator topology. We choose a fixed $\theta > 0$. For each integer $N = 1, 2, 3, \cdots$, let Δ_N be a dissection
$$a - \theta = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N = b + \theta$$
of the interval $[a - \theta, b + \theta]$, with the properties

(i) $\delta_N = \max (\lambda_{i+1} - \lambda_i) < 2(b - a + 2\theta)/N$;
(ii) $E_\lambda x_0$ is strongly continuous in λ at each point λ_i.

It is possible to choose such a dissection since the set of discontinuities of $E_\lambda x_0$ is at most enumerable ((**2**) Theorem A (ix), and the lemma following Theorem C).

We may now consider the operators
$$S_N = \sum_{i=0}^{N-1} \lambda_i \Delta E_{\lambda_i}$$
(where $\Delta E_{\lambda_i} = E_{\lambda_{i+1}} - E_{\lambda_i}$), and
$$S_N^{(n)} = \sum_{i=0}^{N-1} \lambda_i \phi^{(n)}_{\lambda_i,\lambda_{i+1}}(T).$$
We have
$$S_N x \to Sx \text{ as } N \to \infty \quad (x \in \mathfrak{B}),$$
where
$$S = \int \lambda dE_\lambda,$$

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the integral being defined in the sense of (**2**). Furthermore, for each fixed N, we see from the remark following Lemma 2 that

$$S_N^{(n)} x_0 \to \sum_{i=0}^{N-1} \lambda_i \Delta E_{\lambda_i} x_0 = S_N x_0$$

as $n \to \infty$.

The operator $T - S_N^{(n)}$ may be written in the form $\psi_N^{(n)}(T)$, where $\psi_N^{(n)}(\lambda)$ is the absolutely continuous function of λ defined by

$$\psi_N^{(n)}(\lambda) = \lambda - \sum_{i=0}^{N-1} \lambda_i \phi^{(n)}_{\lambda_i, \lambda_{i+1}}(\lambda).$$

We now require an estimate for $\| \psi_N^{(n)} \|_J$. We may restrict our attention to those n for which $n\theta > 1$, and assume that the interval $[\lambda_0 + 1/n, \lambda_N - 1/n]$, or (what is the same) $[a - \theta + 1/n, b + \theta - 1/n]$, contains $J = [a, b]$. We may also suppose that n is sufficiently large to ensure that

$$\lambda_i + 1/n < \lambda_{i+1} - 1/n \quad (i = 0, 1, \cdots, N-1).$$

Now $\psi_N^{(n)}(\lambda)$ is linear on each of the intervals

$$[\lambda_i + 1/n, \lambda_{i+1} - 1/n] \quad (i = 0, 1, \cdots, N-1), \text{ and}$$
$$[\lambda_i - 1/n, \lambda_i + 1/n] \quad (i = 1, 2, \cdots, N-1),$$

while

$$\psi_N^{(n)}(\lambda_i - 1/n) = \lambda_i - \lambda_{i-1} - 1/n \quad (i = 1, 2, \cdots, N),$$
$$\psi_N^{(n)}(\lambda_i + 1/n) = 1/n \quad (i = 0, 1, \cdots, N-1).$$

For sufficiently large n all these values are positive and we have

$$\sup_J |\psi_N^{(n)}(\lambda)| \leqq \max (\lambda_i - \lambda_{i-1}) = \delta_N,$$
$$\operatorname{var}_J \psi_N^{(n)}(\lambda) \leqq (2N-1)\delta_N$$
$$\leqq 4(b - a + 2\theta).$$

The last inequality follows from assumption (i) about the dissection Δ_N. We now have

$$\| \psi_N^{(n)} \| \leqq 4(b - a + 2\theta) + \delta_N,$$

and hence

$$(5) \qquad \begin{aligned} ||T - S_N^{(n)}|| &= ||\psi_N^{(n)}(T)|| \\ &\leqq 4K(b - a + 2\theta) + \delta_N K. \end{aligned}$$

Since $S_N^{(n)} x_0 \to S_N x_0$ as $n \to \infty$, we obtain

$$(6) \qquad \begin{aligned} ||Tx_0 - S_N x_0|| &\leqq \limsup_{n-\infty} ||Tx_0 - S_N^{(n)} x_0|| \\ &\leqq ||x_0|| \limsup ||T - S_N^{(n)}|| \\ &\leqq ||x_0|| \{4K(b - a + 2\theta) + \delta_N K\}. \end{aligned}$$

The last inequality is a consequence of (5). Upon letting $N \to \infty$ in (6) we obtain

$$||Tx_0 - Sx_0|| \leqq 4K||x_0||(b - a + 2\theta).$$

Since this holds for every $\theta > 0$, while x_0 is an arbitrary element of $\mathfrak{B}$,

$$||T - S|| \leqq 4K(b - a). \tag{7}$$

We note that, in (7), K is the constant occurring in (1). Hence we have the following result.

LEMMA 3. *Let T be a bounded linear operator acting in a reflexive Banach space $\mathfrak{B}$, and let $J = [a, b]$ be a finite closed interval of the real line. Suppose that there is a constant K such that*

$$||p(T)|| \leqq K\|p\|_J$$

for every polynomial $p(\lambda)$. Then $||S - T|| \leqq 4K(b - a)$, where S is the scalar operator associated with T.

The above lemma gives an estimate for $||S - T||$. Our aim in the remainder of this section is to sharpen this result by obtaining a similar estimate for the bound of the operator obtained by restricting $S - T$ to the subspace $(E_\beta - E_\alpha)\mathfrak{B}$, where α, β are arbitrary real numbers. Before we can do this we need the following auxiliary result.

LEMMA 4. *Let $\phi(\lambda)$ be an absolutely continuous function vanishing on an interval $[\alpha, \beta]$. Then $\phi(T)(E_\beta - E_\alpha) = 0$.*

PROOF. We first consider the special case in which, for some $\theta > 0$, $\phi(\lambda)$ vanishes on the interval $[\alpha, \beta + \theta]$.

For any c in (α, β) and d in $(\beta, \beta + \theta)$ we have, for all sufficiently large n,

$$\alpha < c - \frac{1}{n} < d + \frac{1}{n} < \beta + \theta.$$

For such n, $\phi(\lambda)\phi_{c,d}^{(n)}(\lambda) \equiv 0$ (where $\phi_{c,d}^{(n)}(\lambda)$ is defined as in Lemma 2). From that lemma we deduce that, if $x \in \mathfrak{B}$, then

$$\begin{aligned} &\tfrac{1}{2}\phi(T)(E_d + E_{d-} - E_c - E_{c-})x \\ &= \lim_{n\to\infty} \phi(T)\phi_{c,d}^{(n)}(T)x \\ &= 0. \end{aligned}$$

When $c \to \alpha + 0$ and $d \to \beta + 0$ we obtain

$$\phi(T)(E_\beta - E_\alpha)x = 0 \quad (x \in \mathfrak{B}).$$

This proves the required result in the special case. In the general case we define

$$\phi_\theta(\lambda) = \begin{cases} \phi(\lambda) & (\lambda < \alpha), \\ 0 & (\alpha \leqq \lambda \leqq \beta + \theta), \\ \phi(\lambda - \theta) & (\lambda > \beta + \theta). \end{cases}$$

It is easily verified that ϕ_θ is an absolutely continuous function which vanishes on the interval $[\alpha, \beta + \theta]$, and that $\boldsymbol{|}\phi - \phi_\theta\boldsymbol{|} \to 0$ as $\theta \to 0+$. We deduce that $\phi_\theta(T)(E_\beta - E_\alpha) = 0$, and that

$$\begin{aligned} \|\phi(T)(E_\beta - E_\alpha)\| &= \|\{\phi(T) - \phi_\theta(T)\}(E_\beta - E_\alpha)\| \\ &\leqq \|\phi(T) - \phi_\theta(T)\| \cdot \|E_\beta - E_\alpha\| \\ &\leqq K\boldsymbol{|}\phi - \phi_\theta\boldsymbol{|} \cdot \|E_\beta - E_\alpha\| \\ &\to 0 \end{aligned}$$

as $\theta \to 0+$. Since the left hand side is independent of θ, this completes the proof of the lemma.

We now state and prove one of the subsidiary conjectures made in (**2**).

LEMMA 5. *Let T be a bounded linear operator acting in a reflexive Banach space $\mathfrak{B}$, let $J = [a, b]$ be a finite closed interval of the real line, and suppose that there is a constant K such that*

$$\|p(T)\| \leqq K\boldsymbol{|}p\boldsymbol{|}_J$$

for every polynomial $p(\lambda)$. Let α, β be real numbers, and denote by $\mathfrak{B}_0$ the space $(E_\alpha - E_\beta)\mathfrak{B}$, and by S_0, T_0 the restrictions to $\mathfrak{B}_0$ of S, T (where S is the scalar operator associated with T). Then

$$\|S_0 - T_0\| \leqq 4K|\beta - \alpha|.$$

PROOF. We may suppose that $\alpha < \beta$. Since $\|p(T_0)\| \leqq \|p(T)\|$ for every polynomial $p(\lambda)$, we have

$$\|p(T_0)\| \leqq K\boldsymbol{|}p\boldsymbol{|}_J \tag{8}$$

for all polynomials, and, in fact, for all absolutely continuous functions. We now wish to show that, if J_0 denotes the interval $[\alpha, \beta]$, then J may be replaced by J_0 in (8). Given an absolutely continuous function $p(\lambda)$, let $p_0(\lambda)$ be the function equal to $p(\lambda)$ on J_0 and constant on each of the intervals $(-\infty, \alpha]$ and $[\beta, \infty)$. Lemma 4 implies that $p(T_0) = p_0(T_0)$, and we have

$$\|p(T_0)\| = \|p_0(T_0)\| \leqq K\boldsymbol{|}p_0\boldsymbol{|}_J \leqq K\boldsymbol{|}p\boldsymbol{|}_{J_0}. \tag{9}$$

Hence T_0 is a well-bounded operator. The family of projections in $\mathfrak{B}_0$, which is obtained by restricting to $\mathfrak{B}_0$ the projections E_λ, has all the properties which are required by Theorem A of (**2**) for the spectral family of projections associated with T_0. Hence, by the uniqueness clause of that theorem, we

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see that the two families coincide. It follows that S_0 is the scalar operator associated with T_0. We now deduce from (9), and from Lemma 3 applied to T_0, that

$$||S_0 - T_0|| \leqq 4K(\beta - \alpha).$$

4. The proof that $S = T$

We shall use Lemma 5 and the following result.

LEMMA 6. *Let $\mathfrak{X}$ be a Banach space, x_λ a function of a real variable λ, taking values in $\mathfrak{X}$. Suppose that x_λ satisfies a Lipschitz condition*

$$(10) \qquad ||x_\mu - x_\lambda|| \leqq M|\mu - \lambda|.$$

Then (i) *given an element f in the dual space $\mathfrak{X}^*$, there is a subset $\Lambda(f)$ of the real line, which is of measure zero and such that*

$$\lim_{\mu \to \lambda} f\left(\frac{x_\mu - x_\lambda}{\mu - \lambda}\right)$$

exists whenever $\lambda \notin \Lambda(f)$;

(ii) *if in addition $\mathfrak{X}$ is reflexive then there is a set Λ of measure zero which contains all the sets $\Lambda(f)$ $(f \in \mathfrak{X}^*)$; and when $\lambda \notin \Lambda$,*

$$\frac{x_\mu - x_\lambda}{\mu - \lambda}$$

converges weakly to an element y_λ of $\mathfrak{X}$ as $\mu \to \lambda$;

(iii) *if, further, $y_\lambda = 0$ $(\lambda \notin \Lambda)$, then x_λ is constant.*

PROOF. (i) Given $f \in \mathfrak{X}^*$, let

$$\phi_f(\lambda) = f(x_\lambda).$$

Since

$$\begin{aligned} |\phi_f(\lambda) - \phi_f(\mu)| &= |f(x_\lambda - x_\mu)| \\ &\leqq M||f||\,|\lambda - \mu|, \end{aligned}$$

it follows that the function $\phi_f(\lambda)$ is absolutely continuous and therefore differentiable for almost all λ. Let $\Lambda(f)$ denote the exceptional set (of measure zero). Then when $\lambda \notin \Lambda(f)$,

$$f\left(\frac{x_\mu - x_\lambda}{\mu - \lambda}\right) \to \phi_f'(\lambda)$$

as $\mu \to \lambda$.

(ii) In the first instance we make the additional assumption that $\mathfrak{X}$ is separable. Since $\mathfrak{X}$ is reflexive, it follows that $\mathfrak{X}^*$ is also separable (Zaanen (3) 152, Theorem 10, with $E = \mathfrak{X}^*$). Let (f_n) be a sequence of elements forming a dense subset of $\mathfrak{X}^*$, and define

$$\Lambda = \bigcup_n \Lambda(f_n).$$

Clearly Λ is a set of measure zero. We now choose and fix a real number λ not in Λ, and when $\mu \neq \lambda$ we define

$$x_{\mu,\lambda} = \frac{x_\mu - x_\lambda}{\mu - \lambda}. \tag{11}$$

From (10) it follows that

$$\|x_{\mu,\lambda}\| \leqq M. \tag{12}$$

The limit

$$\psi(f) = \lim_{\mu\to\lambda} f(x_{\mu,\lambda})$$

exists when f is a member of the sequence (f_n). Since the members of this sequence form a dense subset of $\mathfrak{X}^*$, while the elements $x_{\mu,\lambda}$ are uniformly bounded, it follows that the limit $\psi(f)$ exists for every $f \in \mathfrak{X}^*$, and that ψ is a bounded linear functional on $\mathfrak{X}^*$ ((1) 25, Theorem 2.12.1). Since $\mathfrak{X}$ is reflexive there is an element y_λ of $\mathfrak{X}$ such that

$$\psi(f) = f(y_\lambda) \quad (f \in \mathfrak{X}^*).$$

We have

$$\begin{aligned} f\left(\frac{x_\mu - x_\lambda}{\mu - \lambda}\right) &= f(x_{\mu,\lambda}) \\ &\to \psi(f) \\ &= f(y_\lambda), \end{aligned}$$

as required.

If $\mathfrak{X}$ is not separable, we denote by $\mathfrak{X}_0$ the closed subspace generated by elements of the form x_λ. Then $\mathfrak{X}^0$, considered as a Banach space in its own right, is reflexive, and is also separable (being generated by elements x_λ with *rational* λ). We may consider x_λ as a function taking values in $\mathfrak{X}_0$. Since the weak topology on $\mathfrak{X}_0$ is exactly the topology induced on $\mathfrak{X}_0$ by the weak topology on $\mathfrak{X}$, the required result may now be deduced from the special case in which $\mathfrak{X}$ is separable.

(iii) Let $f \in \mathfrak{X}^*$. The function $f(x_\lambda)$ is absolutely continuous, and when $\lambda \notin \Lambda$ we have

$$\begin{aligned} \frac{d}{d\lambda}[f(x_\lambda)] &= \lim_{\mu\to\lambda} f\left(\frac{x_\mu - x_\lambda}{\mu - \lambda}\right) \\ &= f(y_\lambda) \\ &= 0. \end{aligned}$$

Hence the function $f(x_\lambda)$ has zero derivative almost everywhere, and is therefore constant. Since this holds for every $f \in \mathfrak{X}^*$, we deduce that x_λ is constant.

We are now in a position to prove our main result.

THEOREM. *Let T be a well-bounded operator acting in a reflexive Banach space $\mathfrak{B}$, and let S be the associated scalar operator. Then $S = T$.*

PROOF. Let $x \in \mathfrak{B}$, and set

$$x_\lambda = (S - T)E_\lambda x.$$

We recall from (**2**) that $||E_\lambda|| \leqq 2K$. Since

$$(E_\lambda - E_\mu)x \in (E_\lambda - E_\mu)\mathfrak{B},$$

it follows from Lemma 5 that

$$\begin{aligned} ||x_\mu - x_\lambda|| &\leqq 4K|\mu - \lambda| \, ||E_\lambda x - E_\mu x|| \\ &\leqq 16K^2||x|| \cdot |\mu - \lambda|. \end{aligned}$$

The conditions of Lemma 6 (i) and (ii) are therefore satisfied. We deduce that, for almost all λ, the element

$$x_{\mu,\lambda} = \frac{x_\mu - x_\lambda}{\mu - \lambda}$$

of $\mathfrak{B}$ tends weakly, as $\mu \to \lambda$, to a vector y_λ in $\mathfrak{B}$. We now prove that y_λ, when it exists, is zero. For if $\rho > \mu > \lambda$, we have

$$\begin{aligned} (E_\rho - E_\lambda)x_{\mu,\lambda} &= (\mu - \lambda)^{-1}(E_\rho - E_\lambda)(E_\mu - E_\lambda)(S - T)x \\ &= (\mu - \lambda)^{-1}(E_\mu - E_\lambda)(S - T)x \\ &= x_{\mu,\lambda}. \end{aligned}$$

Here we have used the fact that the projections E_λ commute with S and T, and satisfy the relations $E_\alpha E_\beta = E_\beta E_\alpha = E_\alpha$ $(\alpha \leqq \beta)$. Allowing μ to decrease to λ, we obtain

$$(E_\rho - E_\lambda)y_\lambda = y_\lambda.$$

Finally, letting ρ decrease to λ, and using the right continuity of E_λ, we find that

$$0 = y_\lambda.$$

From Lemma 6 (iii) it follows that x_λ is constant, and by choosing $\lambda < a$ we deduce that x_λ vanishes identically. When $\lambda > b$ we obtain

$$(S - T)x = 0.$$

This holds for every $x \in \mathfrak{B}$. Hence $S = T$.

I am indebted to Dr. Smart for a number of suggestions which have simplified some of the proofs in this note.

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