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Maximum Principles for Subharmonic Functions Via Local Semi-Dirichlet Forms

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Abstract. Maximum principles for subharmonic functions in the framework of quasi-regular local semi-Dirichlet forms admitting lower bounds are presented. As applications, we give weak and strong maximum principles for (local) subsolutions of a second order elliptic differential operator on the domain of Euclidean space under conditions on coefficients, which partially generalize the results by Stampacchia.

1 Introduction

In this paper, we show maximum principles for $\mathcal{E}$-subharmonic functions in the framework of quasi-regular local semi-Dirichlet forms $(\mathcal{E}, \mathcal{F})$ with lower semiboundedness. The maximum principle for (sub)harmonic functions with respect to Laplacian on the domain of Euclidean space has played an important role in partial differential equations, spectral geometry and so on. In particular, the weak maximum principle assures the uniqueness of the solution of a Dirichlet boundary value problem. Consider a bounded open domain G in $\mathbb{R}^d$ and the second order elliptic operator L defined by $-Lu = -(1/2)\,\mathrm{div}(a\nabla u) + \langle b, \nabla u\rangle_{\mathbb{R}^d} + \mathrm{div}(u\widehat{b}) + cu$ with a uniformly elliptic symmetric matrix valued measurable function a on G and bounded coefficients $b, \widehat{b}, c$ on G with $c - \mathrm{div}\,\widehat{b} \geq 0$ on G in the distributional sense. In Gilbarg–Trudinger [22], maximum principles for the subsolution $u \in H^1(G)$ of $Lu = 0$ are presented. In the case of the second order elliptic operator L with coefficients $b \in L^p(G \to \mathbb{R}^d)$, $\widehat{b} \in L^q(G \to \mathbb{R}^d)$, $c \in L^{q-d/2}(G)$ for $p, q \geq d \geq 3$ and uniformly elliptic a on G, a weak (resp. strong) maximum principle for the subsolution (resp. solution) $u \in \widehat{H}^1(G)$ of $Lu = 0$ is proved if $p = q = d$ and $c - \mathrm{div}\,\widehat{b} \geq c_0$ on G with positive constant c_0 (resp. $p = d$, $q > d$ and $c = \mathrm{div}\,\widehat{b} = 0$ on G) by Stampacchia [42]. Here $\widehat{H}^1(G)$ is the completion of $C^1(\overline{G})$ with respect to the norm of $H^1(G)$. R.-M. Hervé and M. Hervé [23] also gave a version of a generalized (weak) maximum principle in the framework of Stampacchia under mild conditions. It is

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well known that the generalized (weak) maximum principle is a consequence of the weak maximum principle. Their generalized (weak) maximum principle is weaker than a usual generalized maximum principle and is described on a neighborhood on the boundary (see [23, Théorème 4]). After that, Chen and Wu [7] showed Stampacchia's weak maximum principle for the subsolution $u \in H^1(G)$ of $Lu = 0$ under the same conditions without assuming $c - \operatorname{div} \widehat{b} \geq c_0 > 0$ on G, but they assume the boundedness of G and the bound of the sum of norms of $|b|, |\widehat{b}| \in L^d(G)$ and $c \in L^{d/2}(G)$ is the same as half of the upper bound of a. Edmunds–Evans [11] also presented a weak maximum principle under the coercivity of forms. They also considered the case for $d = 1, 2$. Putting $b_0 := b - \widehat{b}$, their condition is that $|b_0| \in L^p(G)$ for $p > 1$ if $d = 2$, or for $p \geq 1$ if $d = 1$.

The second purpose of this paper is the application of our maximum principles and showing an extension of Stampacchia's weak maximum principles without assuming the coercivity of forms. More precisely, under $d \geq 1$, the finiteness of the volume of G and a milder condition than the integrability of coefficients, we prove a weak maximum principle for the local subsolution $u \in H^1_0(G)_{\mathrm{loc}}$ of $Lu = 0$ having an ($\mathcal{E}$-)upper bounded (strictly $\mathcal{E}$-quasi-)continuous extension for (not necessarily bounded) open set G, and also give strong maximum principles (see Theorems 8.1 and 8.4 below). However, our strong maximum principle does not completely cover Stampacchia's (see Remark 8.5). We also give the complete extension of Stampacchia's (also Chen and Wu's) weak maximum principle for the subsolution $u \in H^1(G)$ of $Lu = 0$ under the same conditions as above, but without assuming the finiteness of the volume of G (see Theorem 8.2 and Remark 8.4). Our conditions for coefficients $b, \widehat{b}, c$ are related to the classical Hardy inequality, *cf.* [12, 14].

Let us state our framework and main theorems. Let X be a separable metric space and m a σ-finite Borel measure on X. We consider a quasi-regular local semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X; m)$ $(\gamma \geq 0)$. Under the quasi-regularity of $(\mathcal{E}, \mathcal{F})$, we may assume that X is a Lusin topological space (see Remark 4.3 below). Here $(\mathcal{E}, \mathcal{F})$ is said to be a *semi-Dirichlet form with a lower bound* $-\gamma$ *on* $L^2(X; m)$ if $(\mathcal{E}_\gamma, \mathcal{F})$ is a non-negative definite coercive closed bilinear form on $L^2(X; m)$ and for $u \in \mathcal{F}$, $u^+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$, where $\mathcal{E}_\gamma(u, v) := \mathcal{E}(u, v) + \gamma(u, v)_m$, $u, v \in \mathcal{F}$. In this definition, $(\mathcal{E}_\gamma, \mathcal{F})$ on $L^2(X; m)$ is also a semi-Dirichlet form in the usual sense as in [32]. See the definitions of semi-Dirichlet form and its quasi-regularity in Section 3 below. We fix a non-empty open set G with non-empty boundary and consider the part space $(\mathcal{E}_G, \mathcal{F}_G)$ on $L^2(G; m)$ (see Definition 3.1 below), which is again a local quasi-regular semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(G; m)$. Let $(\dot{\mathcal{F}}_G)_{\mathrm{loc}}$ be the family of functions locally in $\mathcal{F}_G$ (Definition 3.2) and $u \in (\dot{\mathcal{F}}_G)_{\mathrm{loc}}$ is said to be $\mathcal{E}_G$-*subharmonic* on G if there exists an exhaustion $\{G_i\}$ of $\mathcal{E}$-quasi-open sets with $u|_{G_i} \in \mathcal{F}_G|_{G_i}$ and $\bigcup_{i=1}^\infty G_i = G$ $\mathcal{E}_G$-q.e. such that $\mathcal{E}(u, v) \leq 0$ for $v \in \bigcup_{i=1}^\infty \mathcal{F}^+_{G_i}$. If X is locally compact and m is a Radon measure on X with full support, then for $u \in (\mathcal{F}_G)_{\mathrm{loc}}$, a function locally in $\mathcal{F}_G$ in the ordinary sense (see the argument before Proposition 3.2), u is $\mathcal{E}_G$-subharmonic on

G if and only if $\mathcal{E}(u,v) \le 0$ for all $v \in \mathcal{F}_G \cap C_0^+(G)$. We further need the following assumption.

Assumption 1.1 (i) There exists an m-Hunt diffusion process $\mathbf{M} = (\Omega, X_t, P_x)$ associated with $(\mathcal{E}, \mathcal{F})$.

(ii) $\mathbf{M}$ satisfies $P_x(\tau_G < \infty) = 1$ m-a.e. $x \in X$, where $\tau_G := \inf\{t > 0 \mid X_t \notin G\}$ is the first exit time from G.

See Definition 4.2 for the definition of an m-Hunt diffusion process. We expose several criteria for Assumption 1.1(ii).

Proposition 1.1 *Take a finely open (nearly) Borel set O. Under Assumption* 1.1(i), *the following are sufficient conditions for $P_x(\tau_O < \infty) = 1$ m-a.e. $x \in X$.*

(i) $\mathbf{M}$ *is a transient doubly Feller process and O is relatively compact open.*

(ii) *$X \setminus O$ is non $\mathcal{E}$-polar, $(\mathcal{E}, \mathcal{F})$ is irreducible and* $\mathbf{M}$ *is recurrent, that is, $Rf = 0$ or $= \infty$ for any nonegative Borel function f on X.*

(iii) $\mathbf{M}$ *is transient, $m(O) < \infty$ and the dual form $(\widehat{\mathcal{E}}, \mathcal{F})$ of $(\mathcal{E}, \mathcal{F})$ is also a semi-Dirichlet form with the same lower bound $-\gamma$ on $L^2(X;m)$.*

(iv) *$(\mathcal{E}, \mathcal{F})$ is a symmetric Dirichlet form with a lower bound* 0 *on $L^2(X;m)$, $m(O) < \infty$, and assume one of the following:*

   (a) *$(\mathcal{E}_O, \mathcal{F}_O)$ is transient;*

   (b) $\mathbf{M}$ *admits a symmetric heat kernel $p_t(x,y)$ satisfying the Nash-type estimate $\sup_{x,y \in X} p_t(x,y) \le Ct^{-\nu/2}$ for some $\nu > 0$, $C > 0$.*

Remark 1.1 Assumption 1.1(i) is satisfied if X is a locally compact separable metric space, m is a positive Radon measure with full topological support and $(\mathcal{E}, \mathcal{F})$ is a local regular semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$. This was noted by Carrillo–Menendez [6]. It should be true that Assumption 1.1(i) holds if $(\mathcal{E}, \mathcal{F})$ is a strictly quasi-regular local semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$. This was shown by Albeverio, Ma, and Röckner [2] for $\gamma = 0$. For general $\gamma \ge 0$, the proof should be described.

Under Assumption 1.1, we have the following.

Theorem 1.1 **(Weak maximum principle I)** *Suppose that Assumption* 1.1 *holds. Let $u \in (\dot{\mathcal{F}}_G)_{\mathrm{loc}} \cap C(\overline{G})$ be an upper bounded $\mathcal{E}_G$-subharmonic function on G.*

(i) *We have $\sup_{\overline{G}} u \le \sup_{\partial G} u^+$; in particular, $\sup_{\overline{G}} u = \sup_{\partial G} u$ if $u \ge 0$ on ∂G.*

(ii) *If* 1 *is $\mathcal{E}_G$-harmonic on G, then $\sup_{\overline{G}} u = \sup_{\partial G} u$.*

Theorem 1.2 **(Comparison principle, Harnack's first theorem)** *Suppose Assumption* 1.1 *holds. Then the following assertions hold.*

(i) *Let $u \in (\dot{\mathcal{F}}_G)_{\mathrm{loc}}$ be an upper bounded $\mathcal{E}_G$-subharmonic function on G which has a continuous extension u on $\overline{G}$. Then $u \le 0$ on ∂G implies $u \le 0$ on G.*

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(ii) *Suppose that* 1 *is* $\mathcal{E}_G$*-harmonic on* G *and* $\gamma = 0$. *Let* $\{u_n\}_{n\in\mathbb{N}} \subset (\dot{\mathcal{F}}_G)_{\mathrm{loc}} \cap C_b(\overline{G})$ *be a family of bounded* $\mathcal{E}_G$*-harmonic functions on* G *possessing continuous extensions on* $\overline{G}$. *Assume that* u_n *converges uniformly on* ∂G. *Assume that* G *is either relatively compact or* $u_n \in \mathcal{F}|_G$. *Then there exists an* $\mathcal{E}_G$*-harmonic function* $u \in (\dot{\mathcal{F}}_G)_{\mathrm{loc}} \cap C_b(\overline{G})$ *such that* u_n *converges to* u *uniformly on* $\overline{G}$.

The next corollary is a slightly extended version of the above theorem.

Corollary 1.1 **(Weak maximum principle II)** *Let* G *be an* $\mathcal{E}$*-quasi-open set with a non* $\mathcal{E}$*-polar* $\mathcal{E}$*-quasi-boundary* $\mathcal{E}$*-*∂G. *Suppose that Assumption* 1.1 *holds. Let* $u \in (\dot{\mathcal{F}}_G)_{\mathrm{loc}}$ *be an* $\mathcal{E}_G$*-subharmonic function on* G *which has a strictly* $\mathcal{E}$*-quasi-continuous extension* $\widetilde{u}$ *on* X *such that* $\widetilde{u}$ *is* $\mathcal{E}$*-upper bounded on* $\overline{G}^{\mathcal{E}}$. *Then*

(i) *We have* $\mathcal{E}\text{-}\sup_{\overline{G}^{\mathcal{E}}} \widetilde{u} \le \mathcal{E}\text{-}\sup_{\mathcal{E}\text{-}\partial G} \widetilde{u}^+$. *In particular,* $\mathcal{E}\text{-}\sup_{\overline{G}^{\mathcal{E}}} \widetilde{u} = \mathcal{E}\text{-}\sup_{\mathcal{E}\text{-}\partial G} \widetilde{u}$ *if* $\widetilde{u} \ge 0$ $\mathcal{E}$*-q.e. on* ∂G.

(ii) *If* 1 *is* $\mathcal{E}_G$*-harmonic on* G, *then* $\mathcal{E}\text{-}\sup_{\overline{G}^{\mathcal{E}}} \widetilde{u} = \mathcal{E}\text{-}\sup_{\mathcal{E}\text{-}\partial G} \widetilde{u}$.

Here $\mathcal{E}$-$\sup_A$ means the $\mathcal{E}$-quasi-essentially supremum on A defined by

$$\mathcal{E}\text{-}\sup_A f := \inf\{k \in \mathbb{R} \mid \{f > k\} \cap A \text{ is } \mathcal{E}\text{-polar}\};$$

f is said to be $\mathcal{E}$*-upper bounded on* A if $\mathcal{E}\text{-}\sup_A f < \infty$; and f is called *strictly* $\mathcal{E}$*-quasi-continuous on* X if there exists a strictly $\mathcal{E}$-nest $\{F_n\}$ of closed sets such that $f|_{F_n \cup \{\Delta\}}$ is continuous on $F_n \cup \{\Delta\}$. Note that if f is $\mathcal{E}$-quasi-continuous on an $\mathcal{E}$-quasi-open set A, then $\mathcal{E}$-$\sup_A f$ coincides with the m-essentially supremum on A (see the definitions of (strictly) $\mathcal{E}$-nest and $\mathcal{E}$-quasi-continuity, and $\mathcal{E}$-quasi-open sets in Section 3 below).

Now we state our strong maximum principle.

Consider a quasi-regular local semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X; m)$ again. We assume that there exists a Borel right process $\mathbf{M}^\gamma$ associated with the semi-Dirichlet form $(\mathcal{E}_\gamma, \mathcal{F})$ in the usual sense. Denote by $C_f(X)$ the family of Borel finely continuous functions with respect to $\mathbf{M}^\gamma$ (see the definitions of fine continuity, m-tightness and m-special standardness in Definition 4.2). We say that $\mathbf{M}^\gamma$ satisfies *the absolute continuity condition with respect to m* if the transition kernel $p_t^\gamma(x, dy)$ of $\mathbf{M}^\gamma$ is absolutely continuous with respect to $m(dy)$ for any $t > 0$ and $x \in X$. And $(\mathcal{E}_\gamma, \mathcal{F})$ is called *irreducible* if any Borel set B with the property that $I_B u \in \mathcal{F}$ for $u \in \mathcal{F}$ always satisfies $m(B) = 0$ or $m(B^c) = 0$. We have the following.

Theorem 1.3 **(Strong maximum principle I)** *Suppose that* $\mathbf{M}^\gamma$ *satisfies the absolute continuity condition with respect to m and* $(\mathcal{E}_\gamma, \mathcal{F})$ *is irreducible. Let* $u \in \dot{\mathcal{F}}_{\mathrm{loc}} \cap C_f(X)$ *be an* $\mathcal{E}$*-subharmonic Borel function on* X.

(i) *If u attains its maximum at some* $x_0 \in X$, *then we have* $u^+ \equiv u^+(x_0)$.

(ii) *Suppose that* 1 *is* $\mathcal{E}$*-harmonic on* X. *If u attains its maximum at some* $x_0 \in X$, *then we have* $u \equiv u(x_0)$.

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The next corollary is an easy consequence of the above theorem.

Corollary 1.2 *Under the same conditions as in the above theorem, we have the following. Let $u \in \dot{\mathcal{F}}_{\rm loc} \cap C_f(X)$ be an $\mathcal{E}$-subharmonic function on X. Assume that u is not constant on X.*

(i) *If u attains its maximum at $x_0 \in X$, then $u(x_0) < 0$, namely, u has no non-negative maximum in X.*

(ii) *Suppose that 1 is $\mathcal{E}$-harmonic on X. Then u does not attain its maximum in X.*

Theorem 1.4 **(Strong maximum principle II)** *Suppose that $\mathbf{M}^\gamma$ satisfies the absolute continuity condition with respect to m and $(\mathcal{E}_\gamma, \mathcal{F})$ is irreducible. Let $u \in \dot{\mathcal{F}}_{\rm loc}$ be an $\mathcal{E}$-subharmonic finely upper semi continuous Borel function on X with respect to $\mathbf{M}^\gamma$ and G a nonempty finely open set.*

(i) *If u attains its maximum at any $x_0 \in G$, then we have $u^+ \equiv u^+(x_0)$.*

(ii) *Suppose that 1 is $\mathcal{E}$-harmonic on X. If u attains its maximum at any $x_0 \in G$, then we have $u \equiv u(x_0)$.*

The constitution of this paper is as follows: in Section 2, we summarize the basic facts on coercive closed forms which are needed to analyze positivity preserving or semi-Dirichlet forms later. In Section 3, we build up several useful tools and properties of quasi-regular local positivity preserving forms and also quasi-regular local semi-Dirichlet forms. In Section 4, we analyze the (Borel) right processes on Radon spaces associated with quasi-regular semi-Dirichlet forms with lower bounds. In Section 5, we give an irreducibility criterion for quasi-regular semi-Dirichlet forms with lower bounds and a criterion for connectedness of the fine topology of corresponding right process, which are utilized in the proof of our strong maximum principle. In Section 6, we investigate the structure of $\mathcal{E}$-subharmonic functions for quasi-regular local positivity preserving/semi-Dirichlet forms with lower bounds. In Section 7, we give the proofs of Theorem 1.1, Corollary 1.1, and Theorem 1.2, and finally we prove Theorems 1.3 and 1.4. In Section 8, we first apply our maximum principles to the framework of the second order elliptic equation with *Hardy class* coefficients as noted above and extend Stampacchia's weak maximum principles. Secondly, we show that a regular strongly local symmetric Dirichlet form associated with a doubly Feller diffusion admitting a continuous heat kernel satisfies the strong maximum principle in our sense.

2 Coercive Closed Forms

Throughout this paper, we basically assume that X is a separable metric space and m is a σ-finite Borel measure on X. Denote by $\mathcal{B}(X)$ the topological σ-field or Borel functions on X and by $\mathcal{B}_b(X)$ the bounded Borel functions. We take another point Δ and endow $X_\Delta := X \cup \{\Delta\}$ with a topology of one point compactification if X is locally compact; otherwise Δ is added as an isolated point. Let $\mathcal{B}(X_\Delta)$ be the topological σ-field on X_Δ. The measure m can be extended on $(X_\Delta, \mathcal{B}(X_\Delta))$ by setting $m(\{\Delta\}) = 0$. Let $L^0(X;m)$ be the totality of m-measurable real functions on X, $L^p(X;m)$ the totality of p-th power m-integrable functions on X for $p > 0$, and $L^\infty(X;m)$ the totality of bounded m-measurable functions on X. We let

$$\mathcal{K}(X) := \{\varphi \in L^1(X;m) \mid 0 < \varphi \leq 1 \ m\text{-a.e. on } X\}.$$

Every positive Borel measure μ on X admits the *support of* μ defined by $\mathrm{supp}[\mu] := \{x \in X \mid \mu(G) > 0 \text{ for any open neighborhood } G \text{ of } x\}$. For $u \in L^0(X;m)$, we set $\mathrm{supp}[u] := \mathrm{supp}[|u|m]$. For functions u, v on X, we write $u \vee v := \max\{u, v\}$, $u \wedge v := \min\{u, v\}$, $u^+ := u \vee 0$, $u^- := (-u) \vee 0$. For a subfamily $\mathcal{A}$ of $L^0(X;m)$, we denote $\mathcal{A}_b := \mathcal{A} \cap L^\infty(X;m)$, $\mathcal{A}_{\mathrm{cpt}} := \{u \in \mathcal{A} \mid \mathrm{supp}[u] \text{ is compact}\}$, $\mathcal{A}^+$ (or $\mathcal{A}_+$) $:= \{u \in \mathcal{A} \mid u \geq 0 \ m\text{-a.e.}\}$, $\mathcal{A}^-$(or $\mathcal{A}_-$) $:= \{u \in \mathcal{A} \mid u \leq 0 \ m\text{-a.e.}\}$. For subfamilies $\mathcal{A}_1, \mathcal{A}_2$ of $L^0(X;m)$, we set $\mathcal{A}_1 \wedge \mathcal{A}_2 := \{u \wedge v \mid u \in \mathcal{A}_1, v \in \mathcal{A}_2\}$ and $\mathcal{A}_1 \vee \mathcal{A}_2 := \{u \vee v \mid u \in \mathcal{A}_1, v \in \mathcal{A}_2\}$. Note that $\mathcal{A} \vee \{0\} \neq \mathcal{A}^+$ in our notation. Let $\mathcal{E}$ be a bilinear form with domain $\mathcal{F}$ on the real Hilbert space $L^2(X;m)$ with inner product $(\,\cdot\,,\,\cdot\,)_m$ and norm $\|\cdot\|_2$. We set $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_m$, $\alpha \geq 0$, $\widehat{\mathcal{E}}(u,v) := \mathcal{E}(v,u)$, $\mathcal{E}^\circ(u,v) := (1/2)\{\mathcal{E}(u,v) + \widehat{\mathcal{E}}(u,v)\}$ and $\check{\mathcal{E}}(u,v) := (1/2)\{\mathcal{E}(u,v) - \widehat{\mathcal{E}}(u,v)\}$ for $u, v \in \mathcal{F}$. We call $\mathcal{E}^\circ$, $\check{\mathcal{E}}$ the symmetric, anti-symmetric part of $\mathcal{E}$, respectively. We simply write $\mathcal{E}(u) := \mathcal{E}(u,u)$, $\mathcal{E}_\alpha(u) := \mathcal{E}_\alpha(u,u)$ for $u \in \mathcal{F}, \alpha > 0$. Fix $\gamma \geq 0$. A bilinear form $(\mathcal{E}, \mathcal{F})$ with dense domain $\mathcal{F}$ in $L^2(X;m)$ is called a *coercive closed form with a lower bound* $-\gamma$ *on* $L^2(X;m)$ if the following conditions hold:

- $(\mathcal{E}^\circ_\gamma, \mathcal{F})$ is non-negative definite and closed on $L^2(X;m)$.
- (Weak sector condition) for each $\alpha > \gamma$, there exists a constant $K_\alpha > 0$ such that $|\mathcal{E}_\alpha(u,v)| \leq K_\alpha \mathcal{E}_\alpha(u,u)^{1/2} \mathcal{E}_\alpha(v,v)^{1/2}$ for any $u, v \in \mathcal{F}$.

When $\gamma = 0$, we may omit the phrase "with a lower bound 0". If $(\mathcal{E}, \mathcal{F})$ is a coercive closed form with a lower bound $-\gamma$ on $L^2(X;m)$, then clearly $(\mathcal{E}_\gamma, \mathcal{F})$ is a coercive closed form on $L^2(X;m)$.

The following projection theorem is due to Stampacchia.

Theorem 2.1 **([33, Ch. I. Theorem 2.6])** *Let $(\mathcal{E}, \mathcal{F})$ be a coercive closed form with a lower bound $-\gamma$ on $L^2(X;m)$ and Γ a non-empty closed convex subset of $\mathcal{F}$. Let J be a continuous linear functional on $\mathcal{F}$ and $\alpha > \gamma$. Then there exists a unique $v \in \Gamma$ such that $\mathcal{E}_\alpha(v,w) \geq J(w)$ for all $w \in \Gamma - v$. Here $\Gamma - v := \{w - v \mid w \in \Gamma\}$. In particular, if Γ is a closed subspace of $\mathcal{F}$, then $\mathcal{E}_\alpha(v,w) = J(w)$ for all $w \in \Gamma$.*

Corollary 2.1 **(α-projection)** *Let $(\mathcal{E}, \mathcal{F})$ be a coercive closed form with a lower bound $-\gamma$ on $L^2(X;m)$ and Γ a non-empty closed convex subset of $\mathcal{F}$. For any $u \in \mathcal{F}$ and $\alpha > \gamma$, there exists a unique $v \in \Gamma$ such that $\mathcal{E}_\alpha(u - v, w) \leq 0$ for all $w \in \Gamma - v$. In particular, if Γ is a closed subspace of $\mathcal{F}$, then $\mathcal{E}_\alpha(u - v, w) = 0$ for all $w \in \Gamma$.*

Proof It suffices to set $J(v) := \mathcal{E}_\alpha(u,v)$ in the previous theorem. This completes the proof. ■

For any coercive closed form $(\mathcal{E},\mathcal{F})$ (resp. the dual form $(\widehat{\mathcal{E}},\mathcal{F})$) with a lower bound $-\gamma$ on $L^2(X;m)$ and $\alpha>\gamma$ we say that v specified in the above corollary with respect to $(\mathcal{E},\mathcal{F})$ (resp. $(\widehat{\mathcal{E}},\mathcal{F})$) is the *$\alpha$-projection* (resp. *$\alpha$-coprojection*) of u on Γ denoted by $\Pi^{\alpha}_{\Gamma}(u)$ (resp. $\widehat{\Pi}^{\alpha}_{\Gamma}(u)$).

Let $(\mathcal{E},\mathcal{F})$ be a coercive closed form with a lower bound $-\gamma$ on $L^2(X;m)$. Applying Theorem 2.1 to $(\mathcal{E},\mathcal{F})$ for $J(w):=(f,w)_m$, $f\in L^2(X;m)$, and $\Gamma:=\mathcal{F}$, we get a family $(G_\alpha)_{\alpha>\gamma}$ (resp. $(\widehat{G}_\alpha)_{\alpha>\gamma}$) of strongly continuous resolvents (resp. coresolvents) on $L^2(X;m)$, a family $(T_t)_{t>0}$ (resp. $(\widehat{T}_t)_{t>0}$) of strongly continuous semigroups (resp. cosemigroups) on $L^2(X;m)$ and a closed operator L (resp. $\widehat{L}$) on $L^2(X;m)$ such that for $\alpha>\gamma$

- $\mathcal{E}_\alpha(G_\alpha f,v)=\mathcal{E}_\alpha(v,\widehat{G}_\alpha f)=(f,v)_m$ for $f\in L^2(X;m)$, $v\in\mathcal{F}$,
- $G_\alpha f=\int_0^\infty e^{-\alpha s}T_s f\,ds$, $\widehat{G}_\alpha f=\int_0^\infty e^{-\alpha s}\widehat{T}_s f\,ds$ for $f\in L^2(X;m)$,
- L (resp. $\widehat{L}$) is the generator of T_t (resp. $\widehat{T}_t$): $T_t=e^{tL},\ t>0$ (resp. $\widehat{T}_t=e^{t\widehat{L}},\ t>0$)

with the property that $(\alpha-\gamma)G_\alpha$ (resp. $(\alpha-\gamma)\widehat{G}_\alpha$) and $e^{-\gamma t}T_t$ (resp. $e^{-\gamma t}\widehat{T}_t$) are contractive operators on $L^2(X;m)$. It is known that there is a one-to-one correspondence among $(\mathcal{E},\mathcal{F})$, $(G_\alpha)_{\alpha>\gamma}$, $(T_t)_{t>0}$ and L (resp. $(\widehat{\mathcal{E}},\mathcal{F})$, $(\widehat{G}_\alpha)_{\alpha>\gamma}$, $(\widehat{T}_t)_{t>0}$ and $\widehat{L}$).

Proposition 2.1 *Let $(\mathcal{E},\mathcal{F})$ be a coercive closed form with a lower bound $-\gamma$ on $L^2(X;m)$ and Γ a closed subspace of $\mathcal{F}$. For any $u,v\in\mathcal{F}$ and $\alpha>\gamma$, we have*

$$\mathcal{E}_\alpha(\Pi^{\alpha}_{\Gamma}(u),v)=\mathcal{E}_\alpha(\Pi^{\alpha}_{\Gamma}(u),\widehat{\Pi}^{\alpha}_{\Gamma}(v))=\mathcal{E}_\alpha(u,\widehat{\Pi}^{\alpha}_{\Gamma}(v)).$$

In particular, $(\Pi^{\alpha}_{\Gamma}(G_\alpha f),g)=(f,\widehat{\Pi}^{\alpha}_{\Gamma}(\widehat{G}_\alpha g))$ for any $f,g\in L^2(X;m)$.

A coercive closed form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$ is called a *positivity preserving form* if in addition

- (Positivity preserving property) for every $u\in\mathcal{F}$, $u^+\in\mathcal{F}$ and $\mathcal{E}(u^+,u^-)\le 0$.

The next proposition is shown in [34].

Proposition 2.2 **([34])** *Suppose that $(\mathcal{E},\mathcal{F})$ is a coercive closed form with a lower bound $-\gamma$ on $L^2(X;m)$. Let $(G_\alpha)_{\alpha>\gamma}$ (resp. $(T_t)_{t>0}$) be the associated resolvent (resp. semigroup) on $L^2(X;m)$. Then the following are equivalent.*

(i) *For all $u\in\mathcal{F}$, $u^+,u^-\in\mathcal{F}$ and $\mathcal{E}(u^+,u^-)\le 0$.*
(ii) *For all $u\in\mathcal{F}$, $u^-\in\mathcal{F}$ and $\mathcal{E}(u,u^-)\le\gamma\|u^-\|_2^2$.*
(iii) *For all $u\in\mathcal{F}$, $u^+\in\mathcal{F}$ and $\mathcal{E}(u,u^+)\ge-\gamma\|u^+\|_2^2$.*
(iv) *For $\alpha>\gamma$ and $f\in L^2(X;m)$, $0\le f$ implies $0\le\alpha G_\alpha f$.*
(v) *For $t>0$ and $f\in L^2(X;m)$, $0\le f$ implies $0\le T_t f$.*

Hence, a coercive closed form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$ is positivity preserving if and only if for $\alpha>\gamma$ (resp. $t>0$), $G_\alpha f\ge 0$ m-a.e. (resp. $T_t f\ge 0$ m-a.e.) if $f\ge 0$ m-a.e.

For any positivity preserving form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$, $(\widehat{\mathcal{E}},\mathcal{F})$ and $(\mathcal{E}^\circ,\mathcal{F})$ are also positivity preserving forms with a lower bound $-\gamma$ on

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$L^2(X;m)$ (see [34, Remark 1.4(i)]). In particular, for any positivity preserving form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$, $\mathcal{F}$ is a vector lattice, namely,

$$u,v\in\mathcal{F}\Longrightarrow u\wedge v\in\mathcal{F},\quad \mathcal{E}(u\wedge v)\leq\mathcal{E}(u)+\mathcal{E}(v). \tag{2.1}$$

A coercive closed form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$ is called a *semi-Dirichlet form* if it satisfies

- (Semi-Dirichlet property) for every $u\in\mathcal{F}$,

$$u^+\wedge 1\in\mathcal{F}\quad\text{and}\quad \mathcal{E}(u^+\wedge 1, u-u^+\wedge 1)\geq 0.$$

The next proposition is due to Kunita [24] which is a generalization of [33, Ch. I, Proposition 4.3, Theorem 4.4]. We will give a proof for the reader's convenience.

Proposition 2.3 **([24])** *Suppose that $(\mathcal{E},\mathcal{F})$ is a coercive closed form with a lower bound $-\gamma$ on $L^2(X;m)$. Let $(G_\alpha)_{\alpha>\gamma}$ (resp. $(T_t)_{t>0}$) be the associated resolvent (resp. semigroup) on $L^2(X;m)$. Then the following are equivalent.*

(i) *For all $u\in\mathcal{F}$ and $\alpha\geq 0$, $u\wedge\alpha\in\mathcal{F}$ and $\mathcal{E}(u\wedge\alpha, u-u\wedge\alpha)\geq 0$.*
(ii) *For all $u\in\mathcal{F}$, $u^+\wedge 1\in\mathcal{F}$ and $\mathcal{E}(u^+\wedge 1, u-u^+\wedge 1)\geq 0$.*
(iii) *For all $u\in\mathcal{F}$, $u^+\wedge 1\in\mathcal{F}$ and $\mathcal{E}(u+u^+\wedge 1, u-u^+\wedge 1)\geq -\gamma\|u-u^+\wedge 1\|_2^2$.*
(iv) *For $\alpha>\gamma$ and $f\in L^2(X;m)$, $0\leq f\leq 1$ implies $0\leq\alpha G_\alpha f\leq 1$.*
(v) *For $t>0$ and $f\in L^2(X;m)$, $0\leq f\leq 1$ implies $0\leq T_t f\leq 1$.*

Proof Except (iii) $\Rightarrow$ (iv) the proof is the same as in [33, Ch I. Proposition 4.3, Theorem 4.4]. We only prove (iii) $\Rightarrow$ (iv). Let $f\in L^2(X;m)$ with $0\leq f\leq 1$ and set $u:=\alpha G_\alpha f$ for $\alpha>\gamma$. Then $(u^+\wedge 1-f, u-u^+\wedge 1)_m\geq 0$ as proved in [33, Ch. I. Theorem 4.4]. We have

$$\begin{aligned}
0&\geq -\mathcal{E}(u+u^+\wedge 1, u-u^+\wedge 1)-\mathcal{E}_{2\gamma}(u-u^+\wedge 1, u-u^+\wedge 1)\\
&=-2\mathcal{E}(u,u-u^+\wedge 1)-2\gamma\|u-u^+\wedge 1\|_2^2\\
&=2(\alpha-\gamma)\|u-u^+\wedge 1\|_2^2+2\alpha(u^+\wedge 1-f, u-u^+\wedge 1)_m\\
&\geq 2(\alpha-\gamma)\|u-u^+\wedge 1\|_2^2\geq 0,
\end{aligned}$$

which implies $\|u-u^+\wedge 1\|_2=0$, hence $0\leq u\leq 1$. ■

Hence, a coercive closed form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$ is a semi-Dirichlet form if and only if for $\alpha>\gamma$ (resp. $t>0$), $0\leq\alpha G_\alpha f\leq 1$ m-a.e. (resp. $0\leq T_t f\leq 1$ m-a.e.) if $0\leq f\leq 1$ m-a.e. [33, Ch. I. Proposition 4.3].

A coercive closed form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$ is called a *Dirichlet form (with a lower bound $-\gamma$)* if both $(\mathcal{E},\mathcal{F})$ and $(\widehat{\mathcal{E}},\mathcal{F})$ are semi-Dirichlet forms (with a lower bound $-\gamma$ on $L^2(X;m)$).

For any semi-Dirichlet form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$, $(\mathcal{E},\mathcal{F})$, $(\widehat{\mathcal{E}},\mathcal{F})$ and $(\mathcal{E}^\circ,\mathcal{F})$ are positivity preserving forms with the same lower bound [34, Remark 1.4(iii)]. So $\mathcal{F}$ is a vector lattice.

A coercive closed form $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$ with lower bound $-\gamma$ is said to be *local* (resp. *left-strongly local*) if and only if $\mathcal{E}(u,v)=0$ if $\mathrm{supp}[u]\cap\mathrm{supp}[v]=\varnothing$ (resp. u is a constant m-a.e. on a neighborhood of $\mathrm{supp}[v]$) for any $u,v\in\mathcal{F}$ with compact supports.

Definition 2.1 **(Excessive functions in L^2)** Let $(\mathcal{E},\mathcal{F})$ be a coercive closed form with a lower bound $-\gamma$ on $L^2(X;m)$ and $(T_t)_{t>0}$ the associated strongly continuous semigroup on $L^2(X;m)$. Fix $\alpha\geq 0$. A function $u\in L^2(X;m)$ is said to be *α-excessive with respect to* $(\mathcal{E},\mathcal{F})$ if $u\geq 0$ m-a.e. and $e^{-\alpha t}T_tu\leq u$ m-a.e. on X for all $t>0$. We simply say *excessive* instead of 0-excessive. Remark that for $u\in L^2(X;m)$ with $e^{-\alpha t}T_tu\leq u$ m-a.e. on X for all $t>0$, $u\geq 0$ m-a.e. on X automatically holds if $\alpha>\gamma$.

The next lemma is shown in [34] under the positivity preserving property of forms. We give another proof, somewhat irrelevant, to this property.

Lemma 2.1 *Fix an $\alpha>\gamma$. Let $(\mathcal{E},\mathcal{F})$ be a coercive closed form with a lower bound $-\gamma$ on $L^2(X;m)$ and $u\in L^2(X;m)$ satisfies that $e^{-\alpha t}T_tu\leq u$ m-a.e. for all $t>0$. If $v\in\mathcal{F}$ and $u\leq v$ m-a.e., then $u\in\mathcal{F}$. Further assume the positivity preserving property of $(\mathcal{E},\mathcal{F})$. Then the same conclusion holds for any α-excessive function u and $\alpha\in[0,\gamma]$.*

Proof First we take $\alpha>\gamma$. It suffices to show that $\sup_{\beta>0}\mathcal{E}^{(\beta,\alpha)}(u,u)<\infty$ in view of [33, Ch. I, Theorem 2.13(i)], where $\mathcal{E}^{(\beta,\alpha)}(f,g):=\beta(f-\beta G_{\alpha+\beta}f,g)_m$ for $f,g\in L^2(X;m)$ is the approximating form for $(\mathcal{E}_\alpha,\mathcal{F})$. We can see that $\mathcal{E}^{(\beta,\alpha)}(f,g)=\mathcal{E}_\alpha(\beta G_{\beta+\alpha}f,g)$ and $\mathcal{E}_\alpha(\beta G_{\beta+\alpha}f,\beta G_{\beta+\alpha}f)\leq\mathcal{E}^{(\beta,\alpha)}(f,f)$ for $f\in L^2(X;m)$, $g\in\mathcal{F}$. Hence $|\mathcal{E}^{(\beta,\alpha)}(f,g)|\leq K_\alpha\mathcal{E}_\alpha(g,g)^{1/2}\mathcal{E}^{(\beta,\alpha)}(f,f)^{1/2}$. Thus

$$\mathcal{E}^{(\beta,\alpha)}(u,u)\leq\mathcal{E}^{(\beta,\alpha)}(u,v)\leq K_\alpha\mathcal{E}_\alpha(v,v)^{1/2}\mathcal{E}^{(\beta,\alpha)}(u,u)^{1/2}.$$

We obtain $\sup_{\beta>0}\mathcal{E}^{(\beta,\alpha)}(u,u)\leq K_\alpha^2\mathcal{E}_\alpha(v,v)<\infty$. Next we prove the conclusion for an α-excessive u satisfying $u\leq v$ with $v\in\mathcal{F}$ and $\alpha\in[0,\gamma]$. In this case, u is automatically β-excessive for any $\beta>\gamma$, because of the positivity preserving property of $(T_t)_{t>0}$. So the assertion follows from the first argument. This completes the proof. ■

The next lemma is a generalization of [18, Theorem 2.2.1] and [34, Remark 3.4].

Lemma 2.2 *Let $(\mathcal{E},\mathcal{F})$ be a positivity preserving form with a lower bound $-\gamma$ on $L^2(X;m)$. For $u\in\mathcal{F}$ and $\alpha\geq 0$, the following are equivalent to each other.*

(i) *$u\leq e^{-\alpha t}T_tu$ m-a.e. for any $t>0$.*

(ii) *$u\leq\beta G_{\alpha+\beta}u$ m-a.e. for any $\beta>\gamma-\alpha$.*

(iii) $\mathcal{E}_\alpha(u,v) \le 0$ *for* $v \in \mathcal{F}^+$.

Proof The case for $\alpha > \gamma$ is shown in [34, Remark 3.4]. The implication (i) $\Rightarrow$ (ii) is clear. Noting $\mathcal{E}_\alpha(u, \widehat{G}_{\alpha+\beta}v) = (u - \beta G_{\alpha+\beta}u, v)_m$ with the positivity preserving property of the dual form, we see the equivalence (ii) $\Leftrightarrow$ (iii), where we use the fact that $(p-\gamma)\widehat{G}_p v \to v$ in $\mathcal{E}^{1/2}_{\gamma+1}$-norm as $p \to \infty$. Next we prove (iii) $\Rightarrow$ (i). Suppose $v \in L^2_+(X;m)$. Setting $T^{(\alpha)}_t := e^{-\alpha t}T_t$ and $\widehat{T}^{(\alpha)}_t := e^{-\alpha t}\widehat{T}_t$, we have

$$
\begin{aligned}
(u - e^{-\alpha t}T_t u, v)_m &= -\int_0^t \frac{d}{ds}(T^{(\alpha)}_s u, v)_m\, ds = \int_0^t ((\alpha - L)T^{(\alpha)}_s u, v)_m\, ds \\
&= \int_0^t \mathcal{E}_\alpha(T^{(\alpha)}_s u, v)\, ds = \int_0^t \mathcal{E}_\alpha(u, \widehat{T}^{(\alpha)}_s v)\, ds \le 0.
\end{aligned}
$$

This completes the proof. ■

Let $\mathcal{H}(X)$ be a family of m-measurable real functions on X as follows:

$$
\mathcal{H}(X) := \{h \in L^0(X;m) \mid h > 0 \ m\text{-a.e. on } X \\
\text{and } h^2 m \text{ is a } \sigma\text{-finite Borel measure on } X\}.
$$

Definition 2.2 **(h-transform)** Let $(\mathcal{E},\mathcal{F})$ be a coercive closed form on $L^2(X;m)$. For $h \in \mathcal{H}(X)$, we define

$$
\mathcal{F}^h := \{u \in L^2(X;h^2m) \mid uh \in \mathcal{F}\}, \quad \mathcal{E}^h(u,v) := \mathcal{E}(uh, vh) \text{ for } u, v \in \mathcal{F}^h.
$$

Then $(\mathcal{E}^h,\mathcal{F}^h)$ is a coercive closed form on $L^2(X;h^2m)$, and $(\mathcal{E}^h,\mathcal{F}^h)$ is called the *h-transform* of $(\mathcal{E},\mathcal{F})$. Note that $(\mathcal{E},\mathcal{F})$ is positivity preserving if and only if $(\mathcal{E}^h,\mathcal{F}^h)$ is also, and $(\mathcal{E},\mathcal{F})$ is local if and only if $(\mathcal{E}^h,\mathcal{F}^h)$ is local.

Let $(\mathcal{E},\mathcal{F})$ be a positivity preserving form on $L^2(X;m)$ and $(G_\alpha)_{\alpha>0}$ the associated resolvent on $L^2(X;m)$. It is essentially shown in [34] that for 1-excessive $h \in \mathcal{H}(X) \cap L^2(X;m)$ with respect to $(\mathcal{E},\mathcal{F})$, $(\mathcal{E}^h_1,\mathcal{F}^h)$ is a semi-Dirichlet form on $L^2(X;h^2m)$. In particular, if $h := G_1\varphi$ with $\varphi \in \mathcal{K}(X)$, then $h \in \mathcal{H}(X)$ [34, Lemma 3.6]. Hence $(\mathcal{E}^h_1,\mathcal{F}^h)$ is a semi-Dirichlet form on $L^2(X;h^2m)$. Let $(\mathcal{E}^\circ,\mathcal{F})$ be the symmetric part of a positivity preserving form $(\mathcal{E},\mathcal{F})$ and $(G^\circ_\alpha)_{\alpha>0}$ the associated resolvent on $L^2(X;m)$. Take an $h^\circ := G^\circ_1\varphi$ with $\varphi \in \mathcal{K}(X)$. Then $((\mathcal{E}^\circ)^{h^\circ}_1, \mathcal{F}^{h^\circ})$ is a symmetric Dirichlet form on $L^2(X;(h^\circ)^2m)$.

3 Potential Theories on Positivity Preserving and Semi-Dirichlet Forms

Throughout this section we treat the case $\gamma = 0$. The case $\gamma > 0$ can be reduced to this case if we replace $\mathcal{E}$ with $\mathcal{E}_\gamma$. Let $(\mathcal{E},\mathcal{F})$ be a coercive closed form on $L^2(X;m)$. For

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a closed subset F of X, we set $\mathcal{F}_F := \{u \in \mathcal{F} \mid u = 0 \ m\text{-a.e. on } X \setminus F\}$. An increasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed subsets of X is said to be an $\mathcal{E}$-*nest* or *generalized nest* if $\bigcup_{n=1}^{\infty} \mathcal{F}_{F_n}$ is $\mathcal{E}_1^{1/2}$-dense in $\mathcal{F}$. A subset N of X is said to be $\mathcal{E}$-*polar* or $\mathcal{E}$-*exceptional* if there exists an $\mathcal{E}$-nest $\{F_n\}_{n\in\mathbb{N}}$ such that $N \subset \bigcap_{n=1}^{\infty}(X \setminus F_n)$. A statement $P = P(x)$ depending on $x \in X$ is said to be "$P\mathcal{E}$-q.e." if there exists an $\mathcal{E}$-polar set N such that $P(x)$ holds for $x \in X \setminus N$. A function u is said to be $\mathcal{E}$-*quasi-continuous* if there exists an $\mathcal{E}$-nest $\{F_n\}_{n\in\mathbb{N}}$ such that $u|_{F_n}$ is continuous on F_n for each $n \in \mathbb{N}$. A subset E of X is said to be $\mathcal{E}$-*quasi-open* if there exists an $\mathcal{E}$-nest $\{F_n\}_{n\in\mathbb{N}}$ such that $E \cap F_n$ is open with respect to the relative topology on F_n for each $n \in \mathbb{N}$. $\mathcal{E}$-quasi-closedness can be similarly defined. For two subsets A, B of X, we write $A \subset B$ $\mathcal{E}$-q.e. if $I_A \le I_B$ $\mathcal{E}$-q.e. and $A = B$ $\mathcal{E}$-q.e. if $I_A = I_B$ $\mathcal{E}$-q.e. If a function u has an $\mathcal{E}$-quasi-continuous m-version, we denote it by $\widetilde{u}$. We shall recall the notion of quasi-regularity of a positivity preserving form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ as follows [34, Definition 4.9]:

(QR1) There exists an $\mathcal{E}$-nest of compact sets.

(QR2) There exists an $\mathcal{E}_1^{1/2}$-dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-quasi-continuous m-versions.

(QR3) There exist an $\mathcal{E}$-polar set $N \subset X$ and $u_n \in \mathcal{F}$, $n \in \mathbb{N}$ having $\mathcal{E}$-quasi-continuous m-versions $\widetilde{u}_n$, $n \in \mathbb{N}$ such that $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ separates the points of $X \setminus N$.

(QR4) There exists an $\mathcal{E}$-q.e. strictly positive $\mathcal{E}$-quasi-continuous m-version h of an α-excessive function in $\mathcal{F}$ for some $\alpha \in \,]0, \infty[$.

Under the conditions (QR1), (QR2) and (QR3), the last condition (QR4) is equivalent to the following (QR4$'$) and (QR4$''$) (see [34, Proposition 4.11; Lemma 4.12]):

(QR4$'$) There exists an m-a.e. strictly positive m-version of an α-excessive function h in $\mathcal{F}$ for some $\alpha \in \,]0, \infty[$ and an $\mathcal{E}$-quasi-continuous function g such that $h \le g$ m-a.e. on X.

(QR4$''$) One can choose $\{u_n\}_{n\in\mathbb{N}}$ and $N \subset X$ in (QR3) with the property that $X \setminus N \subset \bigcup_{n=1}^{\infty}\{\widetilde{u}_n \neq 0\}$.

We also remark that (QR2), and (QR4$'$) or (QR4) together imply that every $u \in \mathcal{F}$ has an $\mathcal{E}$-quasi-continuous m-version [34, Lemma 4.12(i)].

Lemma 3.1 *Let $(\mathcal{E}, \mathcal{F})$ be a positivity preserving form on $L^2(X; m)$. Then $(\mathcal{E}, \mathcal{F})$ is quasi-regular if and only if so is the symmetric part $(\mathcal{E}^\circ, \mathcal{F})$ of $(\mathcal{E}, \mathcal{F})$. In particular, $(\mathcal{E}, \mathcal{F})$ is quasi-regular if and only if so is the dual form.*

Proof Note that for an increasing sequence $\{F_n\}$ of closed sets, $\{F_n\}$ is an $\mathcal{E}$-nest if and only if it is an $\mathcal{E}^\circ$-nest. Suppose that $(\mathcal{E}, \mathcal{F})$ is quasi-regular. Then (QR1)–(QR3) and (QR4$''$) hold for $(\mathcal{E}^\circ, \mathcal{F})$. We set $h^\circ := G_1^\circ \varphi$ with $\varphi \in \mathcal{K}(X)$. Then $h^\circ \in \mathcal{F}$ is 1-excessive with respect to $(\mathcal{E}^\circ, \mathcal{F})$ and $h^\circ > 0$ m-a.e. on X (see [34, Lemma 3.6]). Owing to the quasi-regularity of $(\mathcal{E}, \mathcal{F})$, h° has an $\mathcal{E}^\circ$-quasi-continuous m-version. Hence (QR4$'$) holds for $(\mathcal{E}^\circ, \mathcal{F})$. The converse is similar. This completes the proof. ■

Proposition 3.1 **([34, Theorems 3.5, 4.14])** *Let $(\mathcal{E}, \mathcal{F})$ be a positivity preserving form on $L^2(X; m)$ and $(G_\alpha)_{\alpha>0}$ be the resolvent associated with $(\mathcal{E}, \mathcal{F})$. Let h be an m-a.e. strictly positive α-excessive function in $\mathcal{F}$ for some $\alpha \in]0, \infty[$. Then $(\mathcal{E}, \mathcal{F})$ is quasi-regular if and only if so is $(\mathcal{E}^h, \mathcal{F}^h)$, equivalently $(\mathcal{E}^h_\alpha, \mathcal{F}^h)$ is a quasi-regular semi-Dirichlet form on $L^2(X; h^2 m)$.*

Corollary 3.1 *Let $(\mathcal{E}^\circ, \mathcal{F})$ be the symmetric part of a positivity preserving form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ and $(G^\circ_\alpha)_{\alpha>0}$ be the resolvent associated with $(\mathcal{E}^\circ, \mathcal{F})$. We set $h^\circ := G^\circ_1 \varphi$ with $\varphi \in \mathcal{K}(X)$. Then $(\mathcal{E}, \mathcal{F})$ is quasi-regular if and only if $((\mathcal{E}^\circ)^{h^\circ}_1, \mathcal{F}^{h^\circ})$ is a quasi-regular symmetric Dirichlet form on $L^2(X; (h^\circ)^2 m)$.*

Proof It is easy to see the assertion by Lemma 3.1 and Proposition 3.1. This completes the proof. ∎

The following lemma is essentially shown in [34]. (See [34, Corollary 4.5] and [26, Lemmas 3.1, 3.2].) We omit the details.

Lemma 3.2 *Let $(\mathcal{E}, \mathcal{F})$ be a positivity preserving form on $L^2(X; m)$.*

(i) *Let u be an $\mathcal{E}$-quasi-continuous function and E an $\mathcal{E}$-quasi-open set. If $u \geq 0$ m-a.e. on E, then $u \geq 0$ $\mathcal{E}$-q.e. on E.*

(ii) *Any m-negligible $\mathcal{E}$-quasi-open sets are $\mathcal{E}$-polar.*

(iii) *Let $\{\{F^k_n\}_{n\in\mathbb{N}}\}_{k\in\mathbb{N}}$ be a countable family of $\mathcal{E}$-nests. Then there exists a subsequence $\{n(l,k)\}_{l\in\mathbb{N}}$ of $\{n\}$ depending on $k \in \mathbb{N}$ with $n(l,k) \geq l$ such that $F_l := \bigcap_{k=1}^\infty F^k_{n(l,k)}$ makes an $\mathcal{E}$-nest. In particular, for a countable family $\{f_j\}$ (resp. $\{A_j\}$) of $\mathcal{E}$-quasi-continuous functions (resp. $\mathcal{E}$-quasi-closed sets), we can take common $\mathcal{E}$-nest $\{F_n\}$ such that $f_j|_{F_n}$ is continuous on F_n (resp. $A_j \cap F_n$ is closed) for all $j, n \in \mathbb{N}$. Hence a countable intersection (resp. union) of $\mathcal{E}$-quasi-closed (resp. -open) sets is $\mathcal{E}$-quasi-closed (resp. -open).*

If $(\mathcal{E}, \mathcal{F})$ is a semi-Dirichlet form on $L^2(X; m)$ which satisfies (QR1)–(QR3), then (QR4) is automatically satisfied (see Remarks at pp. 834 and 4.10 in [34]). If a coercive closed form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ is a quasi-regular semi-Dirichlet form, namely, the conditions (QR1)–(QR3) hold for $(\mathcal{E}, \mathcal{F})$, then there exists an m-equivalence class $\mathbf{M}/\sim$ of m-tight special standard processes $\mathcal{E}$-properly associated with $(\mathcal{E}, \mathcal{F})$ (see Definition 4.2(viii), (ix), (x) for the notions of m-equivalence, m-tight, (m-)special standardness below). Conversely if a Borel right m-tight m-special standard process $\mathbf{M}$ is associated with a semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$, then $(\mathcal{E}, \mathcal{F})$ is quasi-regular and the right process is $\mathcal{E}$-properly associated with $(\mathcal{E}, \mathcal{F})$. More generally, by Fitzsimmons [13], for any right process on a co-Souslin space associated with a semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$, $(\mathcal{E}, \mathcal{F})$ is quasi-regular and the right process is $\mathcal{E}$-properly associated with $(\mathcal{E}, \mathcal{F})$. The $\mathcal{E}$-proper association with $(\mathcal{E}, \mathcal{F})$ of a special standard process $\mathbf{M} = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \zeta, P_x)$ means that $x \mapsto \int_\Omega f(X_t(\omega)) P_x(d\omega)$ is an $\mathcal{E}$-quasi-continuous m-version of $T_t f$ for $f \in \mathcal{B}_+(X) \cap L^2(X; m)$.

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Lemma 3.3 *Let $(\mathcal{E},\mathcal{F})$ be a quasi-regular positivity preserving form on $L^2(X;m)$. The following are equivalent.*

(i) *$(\mathcal{E},\mathcal{F})$ is local.*

(ii) *$\mathcal{E}(u,v)=0$ if $u,v\in\mathcal{F}$ have disjoint supports.*

(iii) *$\mathcal{E}(u,v)=0$ if $u,v\in\mathcal{F}$ satisfy $uv=0$ m-a.e.*

Proof The implication (ii) $\Rightarrow$ (i) is trivial. The equivalence (ii) $\Leftrightarrow$ (iii) is due to Schmuland [39]. The proof of the implication (i) $\Rightarrow$ (ii) is the same as in the proof of (i) $\Rightarrow$ (ii) of [33, Ch. V. Proposition 1.2]. This completes the proof. ■

Definition 3.1 **(Part space)** Let $(\mathcal{E},\mathcal{F})$ be a coercive closed form on $L^2(X;m)$ which satisfies the condition that every $u\in\mathcal{F}$ has an $\mathcal{E}$-quasi-continuous m-version $\widetilde{u}$. Let E be a subset of X. We define

$$\mathcal{F}_E:=\{u\in\mathcal{F}\mid\widetilde{u}=0\ \mathcal{E}\text{-q.e. on } X\setminus E\},\quad \mathcal{E}_E(u,v):=\mathcal{E}(u,v)\text{ for } u,v\in\mathcal{F}_E.$$

If E is a closed set, then $\mathcal{F}_E$ is a closed subspace of $\mathcal{F}$. Under the condition that $\mathcal{F}_E$ is a closed subspace of $\mathcal{F}$, we can consider the resolvent $(G^E_\alpha)_{\alpha>0}$ on $L^2(X;m)$ associated with $(\mathcal{E}_E,\mathcal{F}_E)$ by way of Theorem 2.1, that is, for each $f\in L^2(X;m)$ there exists $G^E_\alpha f\in\mathcal{F}_E$ such that $\mathcal{E}_\alpha(G^E_\alpha f,v)=(f,v)_m$ for any $v\in\mathcal{F}_E$. Then we see that $\Pi^\alpha_{\mathcal{F}_E}(G_\alpha f)=G^E_\alpha f$ for $f\in L^2(X;m)$ under the closedness of $\mathcal{F}_E$ in $\mathcal{F}$. If E is $\mathcal{E}$-quasi-open, $(\mathcal{E}_E,\mathcal{F}_E)$ is called the *part space of $(\mathcal{E},\mathcal{F})$ on E.*

The following lemma is easy to check.

Lemma 3.4 *Let $(\mathcal{E},\mathcal{F})$ be a coercive closed form on $L^2(X;m)$. Let $\{F_n\}$ be an increasing sequence of closed subsets of X and take an $h\in\mathcal{H}(X)$.*

(i) *$\{F_n\}_{n\in\mathbb{N}}$ is an $\mathcal{E}$-nest if and only if it is an $\mathcal{E}^h$-nest/$\mathcal{E}^h_1$-nest. In particular, $\mathcal{E}$-polarity (resp. $\mathcal{E}$-quasi-upper-semi-continuity) is equivalent to $\mathcal{E}^h$-polarity/$\mathcal{E}^h_1$-polarity (resp. $\mathcal{E}^h$-quasi-upper-semi-continuity/$\mathcal{E}^h_1$-quasi-upper-semi-continuity).*

(ii) *Suppose that every $u\in\mathcal{F}$ has an $\mathcal{E}$-quasi-continuous m-version. Let E be an $\mathcal{E}$-quasi-open set. Assume that h is an $\mathcal{E}$-q.e. strictly positive $\mathcal{E}$-quasi-continuous function on X. Then $(\mathcal{F}_E)^h=(\mathcal{F}^h)_E$ and $(\mathcal{E}^h_1)_E(u,v)=(\mathcal{E}_{E,1})^h(u,v)$ for $u,v\in(\mathcal{F}_E)^h=(\mathcal{F}^h)_E$.*

Proof (i) is trivial. We show (ii). It is easy to see

$$\begin{aligned}(\mathcal{F}_E)^h&:=\{u\in L^2(X;h^2m)\mid uh\in\mathcal{F}_E\}\\&=\{u\in L^2(X;h^2m)\mid uh\in\mathcal{F}\text{ and }\widetilde{u}h=0\ \mathcal{E}\text{-q.e. on }E^c\}\\&=\{u\in\mathcal{F}^h\mid\widetilde{u}=0\ \mathcal{E}^h\text{-q.e. on }E^c\}=:(\mathcal{F}^h)_E\end{aligned}$$

and for $u,v\in(\mathcal{F}^h)_E$,

$$(\mathcal{E}^h_1)_E(u,v):=\mathcal{E}^h_1(u,v)=\mathcal{E}(uh,vh)+(uh,vh)_m=(\mathcal{E}_{E,1})^h(u,v).$$

This completes the proof. ■

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Proposition 3.2 **(Quasi-regularity of part spaces)** *Let $(\mathcal{E},\mathcal{F})$ be a positivity preserving form on $L^2(X;m)$ and E an $\mathcal{E}$-quasi-open set. Then the following assertions hold.*

(i) *If $(\mathcal{E},\mathcal{F})$ is quasi-regular, then $(\mathcal{E}_E,\mathcal{F}_E)$ is a quasi-regular positivity preserving form on $L^2(E;m)$.*

(ii) *Suppose that $(\mathcal{E},\mathcal{F})$ is quasi-regular. For $N\subset E$, N is $\mathcal{E}_E$-polar if and only if N is $\mathcal{E}$-polar, and $G\subset E$, G is $\mathcal{E}_E$-quasi-open if and only if G is $\mathcal{E}$-quasi-open.*

Proof First we show (i). Let $(\mathcal{E},\mathcal{F})$ be a quasi-regular positivity preserving form on $L^2(X;m)$. By (QR4), there exists an $\mathcal{E}$-q.e. strictly positive $\mathcal{E}$-quasi-continuous m-version h of an α-excessive function in $\mathcal{F}$. Then $(\mathcal{E}^h_\alpha,\mathcal{F}^h)$ is a quasi-regular semi-Dirichlet form on $L^2(X;h^2m)$. By [26, Lemma 3.4(ii)] and Lemma 3.4,

$$((\mathcal{E}^h_\alpha)_E,(\mathcal{F}^h)_E)=((\mathcal{E}_E)^h_\alpha,(\mathcal{F}_E)^h)$$

is a quasi-regular semi-Dirichlet form on $L^2(E;h^2m)$. In particular, $(\mathcal{F}^h)_E$ is dense in $L^2(E;h^2m)$ and $(\mathcal{E}^h_\alpha)^{1/2}_E$-complete, hence $\mathcal{F}_E$ is dense in $L^2(E;m)$ and $(\mathcal{E}_E)^{1/2}_\alpha$-complete. Then $(\mathcal{E}_E,\mathcal{F}_E)$ is a coercive closed form on $L^2(E;m)$. The positivity preserving property of $(\mathcal{E}_E,\mathcal{F}_E)$ is clear. Applying Lemma 3.4(i) to $(\mathcal{E}_E,\mathcal{F}_E)$ with $h|_E\in\mathcal{H}(E)$, for any increasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed subsets of E, $\{F_n\}$ is an $(\mathcal{E}_E)^h_\alpha$-nest if and only if it is an $\mathcal{E}_E$-nest. Hence (QR1) holds for $(\mathcal{E}_E,\mathcal{F}_E)$ and every $v\in(\mathcal{F}_E)^h=(\mathcal{F}^h)_E$ has an $\mathcal{E}_E$-quasi-continuous m-version. Applying [26, Lemma 3.4] to $(\mathcal{E}^h_\alpha,\mathcal{F}^h)$, we can conclude that every $\mathcal{E}$-polar subset of E is $(\mathcal{E}_E)^h_\alpha$-polar, hence $\mathcal{E}_E$-polar and for every $\mathcal{E}$-quasi-continuous function u on X, $u|_E$ is $(\mathcal{E}_E)^h_\alpha$-quasi-continuous on E, hence $\mathcal{E}_E$-quasi-continuous on E. Since h is $\mathcal{E}^h_\alpha$-quasi-continuous and $\mathcal{E}^h_\alpha$-q.e. strictly positive on X, $h|_E$ is $(\mathcal{E}_E)^h_\alpha$-quasi-continuous and $(\mathcal{E}_E)^h_\alpha$-q.e. strictly positive on E by [26, Lemma 3.4(ii)]. Consequently $h|_E$ is $\mathcal{E}_E$-quasi-continuous and $\mathcal{E}_E$-q.e. strictly positive on E. Note that for $u\in\mathcal{F}_E$, $u/h\in(\mathcal{F}_E)^h$ has an $\mathcal{E}_E$-quasi-continuous m-version. Therefore we can conclude that every $u\in\mathcal{F}_E$ has an $\mathcal{E}_E$-quasi-continuous m-version, namely, (QR2) holds for $(\mathcal{E}_E,\mathcal{F}_E)$. In particular, $h^E:=G^E_\alpha\varphi$ with $\varphi\in\mathcal{K}(E)$ satisfies that $h^E\in\mathcal{F}_E$, $h^E>0$ m-a.e. on E and h^E has an $\mathcal{E}_E$-quasi-continuous m-version. Hence (QR4)$'$ holds for $(\mathcal{E}_E,\mathcal{F}_E)$. Note that (QR3) and (QR4)$''$ hold for $((\mathcal{E}_E)^h_\alpha,(\mathcal{F}_E)^h)$. There exist an $(\mathcal{E}_E)^h_\alpha$-polar set $N(\subset E)$ and $u_n\in(\mathcal{F}_E)^h, n\in\mathbb{N}$ having $(\mathcal{E}_E)^h_\alpha$-quasi-continuous m-versions $\widetilde{u}_n$ such that $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ separates the points in $E\setminus N$ and $E\setminus N\subset\bigcup_{n=1}^\infty\{\widetilde{u}_n\neq 0\}$. Set $\widehat{N}:=N\cup\{x\in E\mid h=0\}$. Then $\widehat{N}$ is $\mathcal{E}_E$-polar and $\widetilde{u}_nh\in\mathcal{F}_E, n\in\mathbb{N}$ are $\mathcal{E}_E$-quasi-continuous functions satisfying that $\{\widetilde{u}_nh\}_{n\in\mathbb{N}}$ separates the points in $E\setminus\widehat{N}$ and $E\setminus\widehat{N}\subset\bigcup_{n=1}^\infty\{\widetilde{u}_nh\neq 0\}$. Therefore (QR3) and (QR4)$''$ hold for $(\mathcal{E}_E,\mathcal{F}_E)$.

Finally we show (ii). The assertion is proved in the case that $(\mathcal{E},\mathcal{F})$ is a quasi-regular semi-Dirichlet form on $L^2(X;m)$ (see [26, Lemma 3.5]). Combining Proposition 3.1 with Lemma 3.4, we can confirm (ii). This completes the proof. ■

Definition 3.2 **(The space of functions locally in $\mathcal{F}$)** Let $(\mathcal{E},\mathcal{F})$ be a quasi-regular positivity preserving form on $L^2(X;m)$. We define a family of sequences of $\mathcal{E}$-quasi-

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open sets denoted by Ξ as follows:

$$\Xi := \{\{G_n\}_{n\in\mathbb{N}} \mid G_n \text{ is } \mathcal{E}\text{-quasi-open for all } n \in \mathbb{N},$$
$$G_n \subset G_{n+1}\ \mathcal{E}\text{-q.e. and } X = \bigcup_{n=1}^{\infty} G_n\ \mathcal{E}\text{-q.e.}\},$$
$$\Xi_{\rm cpt} := \{\{G_n\}_{n\in\mathbb{N}} \in \Xi \mid G_n \text{ is relatively compact for all } n \in \mathbb{N}\}.$$

Then we let

$$\dot{\mathcal{F}}_{\rm loc} := \{u \in L^0(X;m) \mid \exists\{E_n\}_{n\in\mathbb{N}} \in \Xi \text{ and } \exists u_n \in \mathcal{F}$$
$$\text{such that } u = u_n\ m\text{-a.e. on } E_n\}.$$

The space $\dot{\mathcal{F}}_{\rm loc}$ is called *the space of functions locally in $\mathcal{F}$ in the broad sense.*

For $u \in \dot{\mathcal{F}}_{\rm loc}$, there exists a $\{G_i\} \in \Xi$ and $\{u_i\} \subset \mathcal{F}$ such that $u = u_i$ m-a.e. on G_i. Then we say that such $\{G_i\}$ *is attached to* $u \in \dot{\mathcal{F}}_{\rm loc}$. For $u \in \dot{\mathcal{F}}_{\rm loc}$, we set

$$\Xi(u) := \{\{G_n\}_{n\in\mathbb{N}} \in \Xi \mid \{G_n\} \text{ is attached to } u\},$$
$$\Xi_{\rm cpt}(u) := \Xi(u) \cap \Xi_{\rm cpt}.$$

Let E be an $\mathcal{E}$-quasi-open set. We can similarly define Ξ_E, $\Xi_{E,cpt}$ for $(\mathcal{E}_E, \mathcal{F}_E)$, and $\Xi_E(u)$, $\Xi_{E,cpt}(u)$ for $u \in (\dot{\mathcal{F}}_E)_{\rm loc}$. Recall that the $\mathcal{E}_E$-quasi-open subset of E is $\mathcal{E}$-quasi-open. Further we define

$$\dot{\mathcal{F}}_{E,\rm loc} := \Big\{ u \in L^0(E;m) \mid \exists\{E_i\}_{i\in\mathbb{N}} \in \Xi_E \text{ and } \exists u_i \in \mathcal{F}$$
$$\text{such that } u = u_i\ m\text{-a.e. on } E_i \Big\}.$$

Remark 3.1 If $(\mathcal{E}, \mathcal{F})$ is a quasi-regular semi-Dirichlet form on $L^2(X;m)$, $1 \in \dot{\mathcal{F}}_{\rm loc}$ (see [26, Theorem 4.1]). This property cannot be expected in the general framework of quasi-regular positivity preserving forms.

Lemma 3.5 *Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular positivity preserving form on $L^2(X;m)$.*

(i) *For an $\mathcal{E}$-quasi-open set E, each $u \in \dot{\mathcal{F}}_{E,\rm loc}$ admits an ($\mathcal{E}_E$-q.e. finite) $\mathcal{E}_E$-quasi-continuous m-version $\widetilde{u}$ on E and $\dot{\mathcal{F}}_{E,\rm loc} = (\dot{\mathcal{F}}_E)_{\rm loc}$. In particular, $\mathcal{F}|_E \subset (\dot{\mathcal{F}}_E)_{\rm loc}$.*

(ii) *For $u \in \dot{\mathcal{F}}_{\rm loc}$, there exists a $\{G_i\} \in \Xi(u)$ such that u is bounded on each G_i.*

Proof (i) Owing to the quasi-regularity of $(\mathcal{E}_E, \mathcal{F}_E)$ on $L^2(E;m)$, $h^E := G_\alpha^E\varphi$ with $\varphi \in \mathcal{K}(E)$ has an $\mathcal{E}_E$-quasi-continuous m-version $\widetilde{h^E}$ which is $\mathcal{E}_E$-q.e. strictly positive on E by [34, Proposition 4.13]. Then the same proof as in [26, Lemma 4.1; Theorem 4.2] works in the present context.

(ii) Since $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$ admits an $\mathcal{E}$-quasi-continuous m-version, we may assume the $\mathcal{E}$-quasi-continuity of u. By assumption, there is an $\{E_i\} \in \Xi(u)$. Then

$$G_i := \{x \in E_i \mid |u(x)| < i\}$$

satisfies the desired assertion. This completes the proof. ■

The following is proved similarly to [33, Ch. III, Proposition 1.5(i),(ii)].

Proposition 3.3 *Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular positivity preserving form on $L^2(X; m)$ and h a function on X having an $\mathcal{E}$-quasi-continuous m-version $\widetilde{h}$. For each subset B of X, we set $\mathcal{L}_{h,B} := \{w \in \mathcal{F} \mid \widetilde{w} \geq \widetilde{h}\ \mathcal{E}\text{-q.e. on } B\}$. Suppose that $\mathcal{L}_{h,B} \neq \varnothing$ is a closed convex subset of $\mathcal{F}$. Then there exists $h_B^\alpha, \widehat{h}_B^\alpha \in \mathcal{L}_{h,B}$ such that $\mathcal{E}_\alpha(h_B^\alpha, w) \geq \mathcal{E}_\alpha(h_B^\alpha, h_B^\alpha)$ and $\mathcal{E}_\alpha(w, \widehat{h}_B^\alpha) \geq \mathcal{E}_\alpha(\widehat{h}_B^\alpha, \widehat{h}_B^\alpha)$ for all $w \in \mathcal{L}_{h,B}$. Further if $h \in \mathcal{F}$ and $\mathcal{F}_{B^c}$ is a closed subspace of $\mathcal{F}$, then $h_B^\alpha = h - \Pi_{\mathcal{F}_{B^c}}^\alpha h$ and $\widehat{h}_B^\alpha = h - \widehat{\Pi}_{\mathcal{F}_{B^c}}^\alpha h$, in particular,*

$$(G_\alpha f)_B^\alpha = G_\alpha f - G_\alpha^{B^c} f \quad \textit{and} \quad \widehat{(\widehat{G}_\alpha f)}_B^{\alpha} = \widehat{G}_\alpha f - \widehat{G}_\alpha^{B^c} f$$

for $f \in L^2(X; m)$.

Definition 3.3 **(Weighted capacity, [2, 33, 34])** Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular positivity preserving form on $L^2(X; m)$. Let h (resp. g) be a 1-excessive (resp. 1-co-excessive) function in $\mathcal{F}$. For each open subset O of X, $\mathcal{L}_{h,O}$ and $\mathcal{L}_{g,O}$ are closed in $\mathcal{F}$, hence one can consider h_O^1 and $\widehat{g}_O^1$ constructed in the above proposition and set $\mathrm{Cap}_{h,g}(O) := \mathcal{E}_1(h_O^1, \widehat{g}_O^1)$. Take $h := G_1\varphi$ and $g := \widehat{G}_1\varphi$ with $\varphi \in \mathcal{K}(X)$. Then $\mathrm{Cap}_{h,g}(O) = (h_O^1, \varphi)_m = (\widehat{g}_O^1, \varphi)_m$. So we set $\mathrm{Cap}_h(O) := (h_O^1, \varphi)_m$. For any subset B of X $\mathrm{Cap}_h(B) := \inf\{\mathrm{Cap}_h(O) \mid B \subset O,\ O \text{ is open}\}$.

Further we set for $g := \widehat{G}_1\varphi$ with $\varphi \in \mathcal{K}(X)$

$$\mathrm{Cap}_{1,g}(O) := \sup\{\mathrm{Cap}_{u,g}(O) \mid u \in \mathcal{F} \text{ is 1-excessive and } u \leq 1\}.$$

For any subset B of X, $\mathrm{Cap}_{1,g}(B) := \inf\{\mathrm{Cap}_{1,g}(O) \mid B \subset O,\ O \text{ is open}\}$.

Then both Cap_h and $\mathrm{Cap}_{1,g}$ are Choquet capacities and for an increasing sequence of closed sets $\{F_n\}$, it is an $\mathcal{E}$-nest if and only if $\lim_{n\to\infty} \mathrm{Cap}_h(X \setminus F_n) = 0$ (see [34, Theorem 4.4]). By [34, Proposition 4.8], $\mathcal{F}_{B^c}$ and $\mathcal{L}_{h,B}$ is closed in $\mathcal{F}$. Combining [34, Proposition 4.13] and [32, proof of Theorem 2.10], we can confirm $\mathrm{Cap}_h(B) = (h_B^1, \varphi)_m$ for any subset B of X.

Definition 3.4 **($s.\mathcal{E}$-quasi-notions, [2, 33])** We say that an increasing sequence of closed sets $\{F_n\}$ is said to be *strictly $\mathcal{E}$-nest* (write $s.\mathcal{E}$-nest) if

$$\lim_{n\to\infty} \mathrm{Cap}_{1,g}(X \setminus F_n) = 0.$$

A subset N is said to be *strictly* $\mathcal{E}$*-polar* (write $s.\mathcal{E}$-polar) if there exists an $s.\mathcal{E}$-nest $\{F_n\}$ such that $N \subset \bigcap_{n=1}^{\infty} F_n^c$ and a function u on X_Δ is said to be *strictly* $\mathcal{E}$*-quasi-continuous* (write $s.\mathcal{E}$-quasi-continuous) if there exists an $s.\mathcal{E}$-nest $\{F_n\}$ such that $u|_{F_n\cup\{\Delta\}}$ is continuous on $F_n \cup \{\Delta\}$.

Lemma 3.6 *Let $(\mathcal{E},\mathcal{F})$ be a quasi-regular positivity preserving form on $L^2(X;m)$, $\{E_n\}$ an $\mathcal{E}$-q.e. increasing sequence of $\mathcal{E}$-quasi-open sets and E an $\mathcal{E}$-quasi-open set. Then the following are equivalent.*

(i) $\{E_n\} \in \Xi_E$.
(ii) $\bigcup_{n=1}^{\infty} \mathcal{F}_{E_n}$ *is* $\mathcal{E}_1^{1/2}$*-dense in* $\mathcal{F}_E$.
(iii) $\bigcup_{n=1}^{\infty} \mathcal{F}_{E_n}^+$ *is* $\mathcal{E}_1^{1/2}$*-dense in* $\mathcal{F}_E^+$.
(iv) $\lim_{n\to\infty} \mathcal{E}_1(G_1^E f - G_1^{E_n} f) = 0$ *for* $f \in L^2(X;m)$.

Proof In view of Proposition 3.2(i), we may assume $X = E$. In order to prove the equivalence (i) $\Leftrightarrow$ (ii), it suffices to show the case that $(\mathcal{E},\mathcal{F})$ is a quasi-regular symmetric Dirichlet form on $L^2(X;m)$ by Corollary 3.1. We set $h := G_1\varphi$ with $\varphi \in \mathcal{K}(X)$. Then for any increasing sequence $\{F_n\}$ of closed set, $\{F_n\}$ is an $\mathcal{E}$-nest if and only if $\mathrm{Cap}_h(X \setminus F_n) = 0$, hence for $N \subset X$, N is $\mathcal{E}$-polar if and only if $\mathrm{Cap}_h(N) = 0$ [33, Ch. III, Theorem 2.11]. By Proposition 3.3, we can see that (ii) is equivalent to $\lim_{n\to\infty} \mathrm{Cap}_h(X \setminus E_n) = 0$ in the same way as [33, Ch. III, proof of Theorem 2.11]. On the other hand, (QR1) implies that every $\mathcal{E}$-quasi-closed set is quasi-compact with respect to Cap_h (see [15]). Then [15, Theorem 2.10] tells us that (i) is equivalent to $\lim_{n\to\infty} \mathrm{Cap}_h(X \setminus E_n) = 0$.

Next we show (ii) $\Leftrightarrow$ (iii). The implication (ii) $\Rightarrow$ (ii) is trivial. Suppose (ii). Owing to the weak sector condition and the positivity preserving property, there exists $K_1 > 0$ such that for all $u \in \mathcal{F}$,

$$\mathcal{E}_1(u^+) \le K_1^2 \mathcal{E}_1(u). \tag{3.1}$$

Take a $u \in \mathcal{F}^+$. Then there exists a sequence $\{u_i\} \subset \bigcup_{n=1}^{\infty} \mathcal{F}_{E_n}$ which $\mathcal{E}_1^{1/2}$-converges to u as $i \to \infty$. By (3.1), $\{u_i^+\}$ is $\mathcal{E}_1^{1/2}$-bounded, so the Banach–Saks theorem implies (iii).

Finally we show (ii) $\Leftrightarrow$ (iv). The implication (iv) $\Rightarrow$ (ii) is easy. We only prove (ii) $\Rightarrow$ (iv). We may assume $f \in L_+^2(X;m)$. Let $h = G_1 f$ and recall $h^1_{E_n^c} = G_1 f - G_1^{E_n} f$. Since $\{E_n\}$ is $\mathcal{E}$-q.e. increasing, $\{h^1_{E_n^c}\}$ is m-a.e. decreasing in view of a version of [33, Ch. III, Proposition 1.5(iv)]. Then there exists $h_\infty := \lim_{n\to\infty} h^1_{E_n^c}$ in $L^2(X;m)$, $h_\infty \in \mathcal{F}$ and $h^1_{E_n^c} \to h_\infty$ in $\mathcal{E}_1$-weakly as $n \to \infty$ (see the proof of [33, Theorem 2.11]). Suppose (ii). Then we have $h_\infty = 0$ similarly to [33, Ch. III. Theorem 2.11 (2.5)]. Therefore $\mathcal{E}_1(h^1_{E_n^c}, h^1_{E_n^c}) \le \mathcal{E}_1(h^1_{E_n^c}, G_1 f) \to 0$ as $n \to \infty$. This completes the proof. ■

Let $(\mathcal{E},\mathcal{F})$ be a quasi-regular positivity preserving form on $L^2(X;m)$ and set $h^\circ := G_1^\circ\varphi$, $\varphi \in \mathcal{K}(X)$ as before. Then $((\mathcal{E}^\circ)_1^{h^\circ}, \mathcal{F}^{h^\circ})$ is a quasi-regular symmetric Dirichlet form on $L^2(X;(h^\circ)^2 m)$. By Lemma 3.4(i), for an increasing sequence $\{F_n\}$

of closed sets, $\{F_n\}$ is an $\mathcal{E}$-nest if and only if it is an $(\mathcal{E}^\circ)_1^{h^\circ}$-nest. Hence the $\mathcal{E}$-polarity and $\mathcal{E}$-quasi-upper-semi-continuity can be reduced to the cases with respect to $((\mathcal{E}^\circ)_1^{h^\circ}, \mathcal{F}^{h^\circ})$. So we can consider $\mathcal{E}$-*quasi-closure,* $\mathcal{E}$-*quasi-interior* and $\mathcal{E}$-*quasi-support* of a measure charging no $\mathcal{E}$-polar sets by way of the weighted capacity with respect to $((\mathcal{E}^\circ)_1^{h^\circ}, \mathcal{F}^{h^\circ})$ on $L^2(X; (h^\circ)^2 m)$ (see Fuglede [15]). For a set A, denote by $\overline{A}^{\mathcal{E}}$ (resp. $A^{\mathcal{E}\text{-int}}$) the $\mathcal{E}$-quasi-closure (resp. $\mathcal{E}$-quasi-interior) of A.

Lemma 3.7 *Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular positivity preserving form on $L^2(X; m)$ and E an $\mathcal{E}$-quasi-open set. For an increasing sequence $\{F_n\}$ of closed sets contained in E, the following are equivalent.*

(i) *$\{F_n\}$ is an $\mathcal{E}_E$-nest of $(\mathcal{E}_E, \mathcal{F}_E)$.*

(ii) *$\{F_n^{\mathcal{E}\text{-int}}\} \in \Xi_E$.*

Proof The assertion is shown in the case that $(\mathcal{E}, \mathcal{F})$ is a quasi-regular semi-Dirichlet form (see [26, Lemma 3.3]). The present assertion is an easy consequence from this fact and the above observation. This completes the proof. ∎

Lemma 3.8 *Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular positivity preserving form on $L^2(X; m)$ and E an $\mathcal{E}$-quasi-open set. For any $\{G_i\} \in \Xi_E$, there exists an $\{E_i\} \in \Xi_E$ such that $\overline{E}_i^{\mathcal{E}} \subset E_{i+1}$ $\mathcal{E}$-q.e. and $E_i \subset G_i$ $\mathcal{E}$-q.e. for each $i \in \mathbb{N}$.*

Proof We may assume $X = E$. It suffices to set $E_i := \{x \in X \mid \widetilde{h}^{G_i} > 1/i\}$. Here $h^{G_i} := G_1^{G_i}\varphi$, $\varphi \in \mathcal{K}(X)$ and $G_1^{G_i}$ is the 1-resolvent with respect to $(\mathcal{E}_{G_i}, \mathcal{F}_{G_i})$ on $L^2(G_i; m)$. By $\widetilde{h}^{G_i} = 0$ $\mathcal{E}$-q.e. on $X \setminus G_i$, we have $E_i \subset G_i$ $\mathcal{E}$-q.e. Since $h^{G_i} = h - h^1_{G_i^c}$ with $h := G_1\varphi$, $\{\widetilde{h}^{G_i}\}$ is an $\mathcal{E}$-q.e. increasing sequence and converges to h in $\mathcal{F}$ as $i \to \infty$ by Lemma 3.6(iv). By using the Banach–Saks theorem, we can conclude that $\widetilde{h}^{G_i}$ converges to $\widetilde{h}$ as $i \to \infty$ $\mathcal{E}$-q.e. Thus

$$\overline{E}_i^{\mathcal{E}} \subset \{x \in X \mid \widetilde{h}^{G_i} \geq 1/i\} \subset \{x \in X \mid \widetilde{h}^{G_{i+1}} \geq 1/i\} \subset E_{i+1} \quad \mathcal{E}\text{-q.e.}$$

and $X = \bigcup_{i=1}^\infty E_i$ $\mathcal{E}$-q.e. because $\widetilde{h} > 0$ $\mathcal{E}$-q.e. ∎

Let X be a locally compact separable metric space and m a Radon measure with full topological support. A semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ is said to be *regular* if $\mathcal{F} \cap C_0(X)$ is $\mathcal{E}_1^{1/2}$-dense in $\mathcal{F}$ and uniformly dense in $C_0(X)$. In this context, we also consider another localized space $\mathcal{F}_{\mathrm{loc}}$, which is called *the space of functions locally in $\mathcal{F}$ in the ordinary sense*:

$$\mathcal{F}_{\mathrm{loc}} := \Big\{ u \in L^0(X; m) \mid \text{for any relatively compact open set } G \\ \text{there exists } u_G \in \mathcal{F} \text{ such that } u = u_G \ m\text{-a.e. on } G \Big\}.$$

Clearly $\mathcal{F}_{\mathrm{loc}} \subset \dot{\mathcal{F}}_{\mathrm{loc}}$.

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Proposition 3.4 **(Regularity of part spaces)** *Suppose that X is a locally compact separable metric space and m is a Radon measure with full topological support. Let $(\mathcal{E},\mathcal{F})$ be a regular semi-Dirichlet form on $L^2(X;m)$ and G a nonempty open set. Then $(\mathcal{E}_G,\mathcal{F}_G)$ is a regular semi-Dirichlet form on $L^2(G;m)$.*

Proof By the same proof of [18, Lemma 1.4.2], we can confirm that for any $u \in C_0(X)$, there exists a $u_n \in \mathcal{F}\cap C_0(X)$ such that $\operatorname{supp}[u_n] \subset \{x \in X \mid u(x) \neq 0\}$, $n \in \mathbb{N}$ and u_n is uniformly convergent to u as $n \to \infty$. Hence $\mathcal{F}_G \cap C_0(G)$ is uniformly dense in $C_0(G)$. Next we show that $\mathcal{F}_G \cap C_0(G)$ is $(\mathcal{E}_G)_1^{1/2}$-dense in $\mathcal{F}_G$. Consider an increasing sequence $\{O_i\}$ of relatively compact open sets with $\overline{O_i} \subset O_{i+1} \subset G$ for all $i \in \mathbb{N}$ and $G = \bigcup_{i=1}^{\infty} O_i$. By using Lemma 3.6, it suffices to prove that every $u \in (\mathcal{F}_{O_i})_b$ is $\mathcal{E}_1^{1/2}$-approximated by an element of $\mathcal{F}_G \cap C_0(G)$ for a fixed $i \in \mathbb{N}$. Owing to the first argument, we can take $\psi_i \in \mathcal{F}\cap C_0(X)$ with $\psi_i = 1$ on $\overline{O_i}$ and $\psi_i = 0$ on O_{i+1}^c. Let $\widehat{u}_k \in \mathcal{F}\cap C_0(X)$ be an $\mathcal{E}_1^{1/2}$-approximating sequence to u and set $u_k := (-\|u\|_\infty \psi_i) \vee \widehat{u}_k \wedge \|u\|_\infty \psi_i$. Then $u_k \in \mathcal{F}_G \cap C_0(G)$ is $\mathcal{E}_1^{1/2}$-bounded by use of (2.1), and is L^2-convergent to u. The Banach–Saks theorem tells us the assertion. This completes the proof. ■

Finally, we present the case for $\gamma > 0$.

Definition 3.5 Let $(\mathcal{E},\mathcal{F})$ be a positivity preserving or a semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$.

- An increasing sequence of closed sets $\{F_n\}$ is said to be an $\mathcal{E}$-nest (resp. $s.\mathcal{E}$-nest) if it is an $\mathcal{E}_\gamma$-nest (resp. $s.\mathcal{E}_\gamma$-nest). Hence every $\mathcal{E}_\gamma$-quasi notion (resp. $s.\mathcal{E}_\gamma$-quasi notion) concerning $(\mathcal{E}_\gamma,\mathcal{F})$ is said to be $\mathcal{E}$-quasi notion (resp. $s.\mathcal{E}_\gamma$-quasi notion) respectively.
- $(\mathcal{E},\mathcal{F})$ is said to be *quasi-regular* if $(\mathcal{E}_\gamma,\mathcal{F})$ is so.
- For a set A, $\mathcal{E}$-quasi-interior (resp. $\mathcal{E}$-quasi-closure) of A is defined to be an $\mathcal{E}_\gamma$-quasi-interior (resp. $\mathcal{E}_\gamma$-quasi-closure) of A.
- $(\mathcal{E},\mathcal{F})$ is said to be *local* if $(\mathcal{E}_\gamma,\mathcal{F})$ is so.
- Suppose X is a locally compact separable metric space and m is a positive Radon measure with full support. Then we call $(\mathcal{E},\mathcal{F})$ a *regular* semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$ if $(\mathcal{E}_\gamma,\mathcal{F})$ is a regular semi-Dirichlet form on $L^2(X;m)$.

4 Analysis on Right Processes Associated with Semi-Dirichlet Forms

To analyze right processes, we prepare the following spaces.

Definition 4.1 A topological space X is said to be a *Radon* (resp. *Lusin*) space if X is homeomorphic to a universally (resp. Borel) measurable subset of a compact metric space. A topological space X is said to be a *Souslin* (resp. *co-Souslin*) space if X is homeomorphic to (resp. a complement of) an analytic subset of a compact

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metric space. The following inclusions hold: $\{\text{Polish space}\} \subset \{\text{Lusin space}\} \subset \{\text{Souslin space}\} \cup \{\text{co-Souslin space}\} \subset \{\text{Radon space}\}$.

Throughout this section, we assume in addition that X is a Radon space and $\mathbf{M} = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \zeta, \{P_x\}_{x\in X})$ is a *right process* on X in the sense of Getoor [19] or Sharpe [40]. That is, $\mathbf{M}$ is a normal strong Markov process possessing right continuous sample paths X_t P_x-almost surely for all $x \in X$, and $h(X_t)$ is also right continuous P_x-almost surely for all $x \in X$ and any α-excessive function $h(\alpha \geq 0)$. It should be noted that $\mathbf{M}$ does not necessarily have the Borel measurability of

$$x \mapsto p_t f(x) := \int_\Omega f(X_t)\, dP_x, \quad t > 0,$$

nor of

$$x \mapsto R_\alpha f(x) := \int_0^\infty e^{-\alpha t} p_t f(x)\, dt, \quad \alpha > 0,$$

for any non-negative/bounded Borel measurable function f on X. Let $\mathcal{B}^*(X)$ be the universal completion of $\mathcal{B}(X)$. Then p_t preserves the class of $\mathcal{B}^*(X)$-measurable bounded functions. $\mathbf{M}$ is called a *Borel right process* if p_t preserves the Borel measurability for all $t > 0$. The notion of right process treated in [33] is actually a Borel right process over a Hausdorff space X with $\mathcal{B}(X) = \sigma(C(X))$. Recall that a $\mathcal{B}^*(X)$-measurable function h is said to be *α-excessive* if for any $x \in X$, $e^{-\alpha t} p_t h(x) \leq h(x)$ for all $t > 0$ and $\lim_{t\downarrow 0} e^{-\alpha t} p_t h(x) = h(x)$. Let $\mathcal{B}^e(X)$ be the σ-field generated by $\{h \mid h \text{ is } \alpha\text{-excessive for some } \alpha > 0\}$. Then $\mathcal{B}(X) \subset \mathcal{B}^e(X) \subset \mathcal{B}^*(X)$, $\sigma\{h \mid h \text{ is 0-excessive}\} \subset \mathcal{B}^e(X)$ and p_t preserves the class of $\mathcal{B}^e(X)$-measurable bounded functions.

We say that a right process $\mathbf{M}$ satisfies the *absolute continuity condition with respect to a measure m* if $P_x(X_t \in dy) \ll m(dy)$ for any $t > 0$ and $x \in X$.

Definition 4.2 Let $\mathbf{M}$ be a right process on X.

(i) A set $B(\subset X_\Delta)$ is called *nearly Borel* if there exist Borel subsets B_1, B_2 of X_Δ such that $B_1 \subset B \subset B_2$ and $P_\mu(X_t \in B_2 \setminus B_1, \exists t \in [0,\infty[) = 0$ for all $\mu \in \mathcal{P}(X_\Delta)$. Denote by $\mathcal{B}^n(X_\Delta)$ (resp. $\mathcal{B}^n(X)$) the family of nearly Borel subsets of X_Δ (resp. X).

(ii) For $B \in \mathcal{B}^n(X)$, we set

$$\sigma_B(\omega) := \inf\{t > 0 \mid X_t(\omega) \in B\},$$

$$\dot\sigma_B(\omega) := \inf\{t \geq 0 \mid X_t(\omega) \in B\},$$

$$\tau_B(\omega) := \inf\{t > 0 \mid X_t(\omega) \notin B\}.$$

Then σ_B (resp. $\dot\sigma_B$) is called the *first hitting time* (resp. *first entry time*) to B and τ_B is called the *first exit time* from B. It well known for $B \in \mathcal{B}^n(X)$, σ_B and $\dot\sigma_B$ are $\mathcal{F}_t$-stopping times and $\tau_B = \sigma_{B^c} \wedge \zeta$ P_x-a.s. for $x \in X$. We further define $H_B^\alpha f(x) :=$

$E_x[e^{-\alpha\sigma_B} f(X_{\sigma_B})] = \int_X f(y) H_B^\alpha(x, dy)$ for $\alpha \geq 0$ and with f a nearly Borel non-negative/bounded function on X. We see

$$H_B^\alpha f(x) = E_x[e^{-\alpha\tau_{B^c}} f(X_{\tau_{B^c}})] = E_x[e^{-\alpha\sigma_B} f(X_{\sigma_B}) : \sigma_B < \infty]$$
$$= E_x[e^{-\alpha\sigma_B} f(X_{\sigma_B}) : \sigma_B < \zeta],$$

because $f(\Delta) := 0$. We call $H_B^\alpha(x, dy)$ *α-hitting distribution* to B. Note that

$$H_B^\alpha R_\alpha f(x) = E_x\Big[\int_{\sigma_B}^\infty e^{-\alpha t} f(X_t)\, dt\Big].$$

(iii) A set A is called *finely open* if for each $x \in A$, there exists a $B \in \mathcal{B}^n(X)$ such that $X \setminus A \subset B$ and $P_x(\sigma_B > 0) = 1$. The family of finely open sets defines a topology on X which is called the *fine topology* of $\mathbf{M}$.

(iv) A set A is called *thin* if there exists a $B \in \mathcal{B}^n(X)$ with $A \subset B$ such that $P_x(\sigma_B = 0) = 0$ for all $x \in B$, and A is said to be *semi-polar* if $A \subset \bigcup_{n=1}^\infty A_n$ for some thin sets A_n.

(v) A set $B \in \mathcal{B}^n(X)$ is called m-polar if $P_m(\sigma_A < \infty) = 0$. A set N is said to be *exceptional* if there is an m-polar set $B \in \mathcal{B}^n(X)$ satisfying $N \subset B$. A statement $P = P(x)$ depending $x \in X$ is said to be "P holds q.e." if there exists an exceptional set N such that $P(x)$ holds for $x \in X \setminus N$.

(vi) A set A is called *finely open q.e.* if there exists a finely open set $B \in \mathcal{B}^n(X)$ such that $A \setminus B$ and $B \setminus A$ are exceptional. A function u defined q.e. on X is called *finely upper-semi-continuous q.e.* if there exists an exceptional set $N \in \mathcal{B}^n(X)$ such that $X \setminus N$ is finely open and u is $\mathcal{B}^n(X)$-measurable and finely upper-semi-continuous on $X \setminus N$. A function u defined q.e. on X is called *finely continuous q.e.* if both u and $-u$ are finely upper-semi-continuous q.e.

(vii) A set $B \in \mathcal{B}^n(X)$ is said to be $\mathbf{M}$-*invariant* if there exists $\Omega_{X\setminus B} \in \mathcal{F}_\infty$ such that

$$\Omega_{X\setminus B} \supset \{\overline{X_0^t} \cap (X \setminus B) \neq \varnothing \text{ for some } 0 \leq t < \zeta\}$$

and $P_x(\Omega_{X\setminus B}) = 0$ for all $x \in B$. Here $\overline{X_0^t}$ stands for the closure of $\{X_s(\omega) \mid s \in [0, t]\}$ in X. A set $N \in \mathcal{B}^n(X)$ is said to be *(m-)properly exceptional* if $m(N) = 0$ and $X \setminus N$ is $\mathbf{M}$-invariant.

(viii) Two right processes $\mathbf{M}_1$ and $\mathbf{M}_2$ are said to be *m-equivalent* if there exists a common m-properly exceptional set N outside of which their transition functions coincide.

(ix) Let μ be a positive measure on $(X_\Delta, \mathcal{B}(X_\Delta))$. A right process $\mathbf{M}$ is called *μ-tight* if there exists an increasing sequence $\{K_n\}$ of compact sets in X such that $P_\mu(\lim_{n\to\infty} \sigma_{X\setminus K_n} < \zeta) = 0$.

(x) Let μ be a positive measure on $(X_\Delta, \mathcal{B}(X_\Delta))$. A right process $\mathbf{M}$ is called *μ-special standard* if one (and hence all) probability measure ν on $(X_\Delta, \mathcal{B}(X_\Delta))$, which is equivalent to μ having the following properties:

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(a) (Left limits up to ζ) $X_{t-} := \lim_{s\uparrow t, s<t} X_s$ exists in X for all $t \in]0, \zeta[$ P_ν-a.s.

(b) (Quasi-left-continuity up to ζ) Let $\tau, \tau_n, n \in \mathbb{N}$ be $\mathcal{F}_t^\nu$-stopping times such that $\tau_n \uparrow \tau$ as $n \to \infty$. Then $P_\nu(\lim_{n\to\infty} X_{\tau_n} \neq X_\tau,\ \tau < \zeta) = 0$.

(c) (Special) Let τ_n, τ be as above. Then X_τ is $\sigma\{\bigcup_{n=1}^\infty \mathcal{F}_{\tau_n}^\nu\}$-measurable.

Here $\mathcal{F}_t^\nu$ is the completion of $\mathcal{F}_t$ with respect to $P_\nu := \int_{X_\Delta} P_x \nu(dx)$.

(xi) A right process is called a *special standard* if it is μ-special standard for all probability measure μ on $(X_\Delta, \mathcal{B}(X_\Delta))$.

(xii) A right process is called a *μ-Hunt process* if (a) and (b) hold with ζ replaced by ∞ and X by X_Δ for $\mu \in \mathcal{P}(X_\Delta)$, and it is called an *$m$-Hunt process* if it is a μ-Hunt process for some $\mu \in \mathcal{P}(X_\Delta)$ equivalent to m. A right process is called a *Hunt process* if it is a μ-Hunt process for all $\mu \in \mathcal{P}(X_\Delta)$

Remark 4.1 (i) Our definition of the $\mathbf{M}$-invariance is due to [33] and seems to be slightly weaker than the definition of $\mathbf{M}$-invariance treated in [18]. However, for a (Borel right) Hunt process $\mathbf{M}$, these notions are equivalent to each other. Note that the existence of the left limit up to the life time is not formulated for general (Borel) right process. If $\mathbf{M}$ is a (Borel right) standard process, then a set $B \in \mathcal{B}^n(X)$ is $\mathbf{M}$-invariant if and only if $P_x(X_t \in B \text{ for } {}^\forall t \in [0, \zeta[,\ X_{t-} \in B \text{ for } {}^\forall t \in]0, \zeta[) = 1$ for all $x \in B$. If further $\mathbf{M}$ is a (Borel right) Hunt process, then a set $B \in \mathcal{B}^n(X)$ is $\mathbf{M}$-invariant if and only if $P_x(X_t \in B_\Delta \text{ for } {}^\forall t \in [0, \infty[,\ X_{t-} \in B_\Delta \text{ for } {}^\forall t \in]0, \infty[) = 1$ for all $x \in B$.

(ii) If $\mathbf{M}$ is a Borel right process, then every α-excessive($\alpha \geq 0$) function is nearly Borel measurable (see [19, (9.4)]), in particular, $\mathcal{B}^e(X) \subset \mathcal{B}^n(X)$.

(iii) If $\mathbf{M}$ satisfies the absolute continuity condition with respect to a measure m, then $\mathcal{B}^e(X) = \mathcal{B}(X)$, in particular, $\mathbf{M}$ is a Borel right process (see [40, (10.25)]).

Definition 4.3 Let $(\mathcal{E}, \mathcal{F})$ be a semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X; m)$. A right process $\mathbf{M}$ with state space X or its transition semigroup $(p_t)_{t>0}$ is called *associated with* $(\mathcal{E}, \mathcal{F})$ if $p_t f$ is an m-version of $T_t f$ for all $t > 0$ and $f \in \mathcal{B}(X) \cap L^2(X; m)$, and $\mathbf{M}$ is called *($\mathcal{E}$-)properly associated* if in addition, $p_t f$ is an $\mathcal{E}$-quasi-continuous for all $t > 0$ and $f \in \mathcal{B}(X) \cap L^2(X; m)$.

Remark 4.2 It is essentially shown in [33, Ch. IV, Exercise 2.7] that $\mathbf{M}$ is associated with $(\mathcal{E}, \mathcal{F})$ if and only if $R_\alpha f$ is an m-version of $G_\alpha f$ for all $\alpha > \gamma$ and $f \in \mathcal{B}(X) \cap L^2(X; m)$. In [33], Borel measurability of $p_t f$ is assumed for each $t > 0$ and $f \in \mathcal{B}_b(X)$, but the above association holds without this Borel measurability.

The following is essentially due to Fitzsimmons [13, Theorem 3.22].

Theorem 4.1 *Let $\mathbf{M}$ be a right process on X. Suppose that X is co-Souslinean. If $\mathbf{M}$ is associated with a semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X; m)$, then $(\mathcal{E}, \mathcal{F})$ is quasi-regular and $\mathbf{M}$ is $\mathcal{E}$-properly associated with $(\mathcal{E}, \mathcal{F})$.*

Proof Let $\mathbf{M}^\gamma$ be the subprocess of $\mathbf{M}$ by the multiplicative functional $e^{-\gamma t}$. Then

$\mathbf{M}^\gamma$ is associated with a semi-Dirichlet form $(\mathcal{E}_\gamma, \mathcal{F})$ on $L^2(X;m)$. So by [13, Theorem 3.22], $\mathbf{M}^\gamma$ is m-tight m-special standard. Therefore, $(\mathcal{E}_\gamma, \mathcal{F})$ is quasi-regular and $\mathbf{M}^\gamma$ is $\mathcal{E}_\gamma$-properly associated with $(\mathcal{E}_\gamma, \mathcal{F})$. That is, $(\mathcal{E}, \mathcal{F})$ is quasi-regular and $\mathbf{M}$ is $\mathcal{E}$-properly associated with $(\mathcal{E}, \mathcal{F})$. This completes the proof. ■

Remark 4.3 It should be noted that if $(\mathcal{E}, \mathcal{F})$ is a quasi-regular semi-Dirichlet form on $L^2(X;m)$ (with lower bound 0), the associated Borel right m-tight special standard process is not defined on the co-Souslin space as a whole space. However, in this case, replacing the underlying space X with a countable union of compact sets derived from (QR1), we may assume that X is Lusinian, hence co-Souslinean (see [33, Ch. IV. 3.2(iii)]).

From now on to the end of this section, we fix a quasi-regular semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X;m)$ with a lower bound $-\gamma$ and a Borel right process $\mathbf{M}$.

Lemma 4.1 *Suppose that* $\mathbf{M}$ *is associated with* $(\mathcal{E}, \mathcal{F})$. *Fix* $\alpha \geq 0$ *and a (nearly) Borel function* $f \in L^2(X;m)$. *Suppose that* $R_\alpha f \in L^2(X;m)$. *Then* $R_\alpha f \in \mathcal{F}$ *and* $\mathcal{E}_\alpha(R_\alpha f, v) = (f, v)_m$ *for all* $v \in \mathcal{F}$.

Proof Note that for $\beta > \gamma$, $G_\beta g = R_\beta g \in \mathcal{F}$ and $\mathcal{E}_\beta(R_\beta g, v) = (g, v)_m$ for all $g \in L^2(X;m)$ and $v \in \mathcal{F}$. By resolvent equation, we see $R_\alpha f = R_\beta(f + (\beta - \alpha)R_\alpha f) \in \mathcal{F}$ and

$$
\begin{aligned}
\mathcal{E}_\alpha(R_\alpha f, v) &= \mathcal{E}_\beta(R_\alpha f, v) + (\alpha - \beta)(R_\alpha f, v)_m \\
&= \mathcal{E}_\beta(R_\beta(f + (\beta - \alpha)R_\alpha f), v) + (\alpha - \beta)(R_\alpha f, v)_m \\
&= (f + (\beta - \alpha)R_\alpha f, v)_m + (\alpha - \beta)(R_\alpha f, v)_m \\
&= (f, v)_m.
\end{aligned}
$$

This completes the proof. ■

Lemma 4.2 *Suppose that* $\mathbf{M}$ *is associated with* $(\mathcal{E}, \mathcal{F})$. *Let* G *be an open subset of* X. *Then for* $u := R_\alpha f$ *with* $f \in \mathcal{B}(X) \cap L^2_+(X;m)$, $\alpha > \gamma$, $H^\alpha_G \widetilde{u}$ *is an* m*-version of* $u^\alpha_G := u - \Pi^\alpha_{\mathcal{F}_{G^c}}(u)$.

Proof Fix $\alpha > \gamma$. Recall that u^α_G is characterized as a unique element satisfying

$$
u^\alpha_G = u \ m\text{-a.e. on } G, \quad \mathcal{E}_\alpha(u^\alpha_G, w) = 0 \text{ for all } w \in \mathcal{F}_{G^c}.
$$

Since $\mathbf{M}$ is associated with $(\mathcal{E}, \mathcal{F})$, we have $H^\alpha_G \widetilde{u}$ is α-excessive and $H^\alpha_G \widetilde{u} \leq \widetilde{u}$ m-a.e., hence $H^\alpha_G \widetilde{u} \in \mathcal{F}$ by Lemma 2.1. Note that u^α_G is also an α-reduced function of u on G, which implies u^α_G is α-excessive in $\mathcal{F}$. Hence $v := H^\alpha_G \widetilde{u} \wedge u^\alpha_G$ is α-excessive in $\mathcal{F}$. In particular, $\mathcal{E}_\alpha(v, u^\alpha_G - v) \geq 0$. On the other hand, $v - u^\alpha_G \in \mathcal{F}_{G^c}$ leads us to $\mathcal{E}_\alpha(u^\alpha_G, v - u^\alpha_G) = 0$. Therefore $\mathcal{E}_\alpha(v - u^\alpha_G, v - u^\alpha_G) = 0$, hence $u^\alpha_G \leq H^\alpha_G \widetilde{u}$ m-a.e.

Next we show the converse inequality. Let $\bar{u}_G^\alpha$ be a Borel m-version of u_G^α such that $\bar{u}_G^\alpha = \widetilde{u}$ on G. Then $\{e^{-\alpha t}\bar{u}_G^\alpha(X_t)\}$ is an $(\mathcal{F}_t, P_{\phi m})$-supermartingale for $\phi \in \mathcal{K}(X)$ with $\int_X \phi dm = 1$. We obtain the converse inequality as in the proof of [18, Lemma 4.2.1]. This completes the proof. ■

In the following lemma, without assuming the existence of dual processes, we have the same assertion as in [18, Lemma 4.1.7] and [21, Proposition 6.9].

Lemma 4.3 *Let $\alpha \geq 0$ and let $\{u_n\}$ be a decreasing sequence of α-excessive functions of* **M** *with limit u and suppose that $u = 0$ m-a.e. Then $u = 0$ q.e.*

Proof The proof of [18, Lemma 4.1.7] or [21, Proposition 6.9] depends on the assumption that X is Lusinian. We shall remark that the proof remains valid without this assumption. Note that u is finely upper-semi-continuous and $\mathcal{B}^n(X)$-measurable, hence finely upper-semi-continuous q.e. We see that $A_n := \{u \geq 1/n\}$ is finely closed and $\mathcal{B}^n(X)$-measurable. As in the proof of [18, Lemma 4.1.7], A_n is m-polar, hence $A := \{u > 0\}$ is m-polar by $\inf_{n\in\mathbb{N}} \sigma_{A_n} = \sigma_A$. This completes the proof. ■

Lemma 4.4 *Fix $\alpha \geq 0$. For any α-excessive function h of* **M***, $h < \infty$ m-a.e. on X if and only if $h < \infty$ q.e. on X.*

Proof Set $B := \{x \in X \mid h(x) = \infty\}$. Then $h_n := \frac{h}{n} \wedge 1$ is an α-excessive function decreasing to I_B as $n \to \infty$ and $I_B = 0$ m-a.e. By Lemma 4.3, we have $I_B = 0$ q.e. ■

Lemma 4.5 *Suppose that* **M** *is associated with $(\mathcal{E}, \mathcal{F})$. Take $\varphi \in \mathcal{K}(X)$ and set $h := G_{\gamma+1}\varphi$. Let Cap_h be the h-weighted capacity with respect to $(\mathcal{E}_\gamma, \mathcal{F})$.*

(i) *Let $\{G_n\}$ be a decreasing sequence of open sets. Then $\lim_{n\to\infty} \mathrm{Cap}_h(G_n) = 0$ if and only if $\lim_{n\to\infty} H_{G_n}^{\gamma+1} R_{\gamma+1}\varphi = 0$ q.e.*

(ii) *For any set N, N is exceptional if and only if N is $\mathcal{E}$-polar. In particular, the notion $\mathcal{E}$-q.e. coincides with the notion q.e.*

Proof (i) Suppose $\lim_{n\to\infty} \mathrm{Cap}_h(G_n) = 0$. Since

$$\mathrm{Cap}_h(G_n) = (h_{G_n}^{\gamma+1}, \varphi)_m = (H_{G_n}^{\gamma+1} R_{\gamma+1}\varphi, \varphi)_m$$

by Lemma 4.2, we have $\lim_{n\to\infty} H_{G_n}^{\gamma+1} R_{\gamma+1}\varphi = 0$ q.e. from Lemma 4.3. Conversely, by Lemma 4.2, $\lim_{n\to\infty} H_{G_n}^{\gamma+1} R_{\gamma+1}\varphi = 0$ q.e. gives $\mathrm{Cap}_h(G_n) = (H_{G_n}^{\gamma+1} R_{\gamma+1}\varphi, \varphi)_m \to 0$ as $n \to \infty$.

Next we show (ii). First we show the only if part. Let K be a compact m-polar set and $\{G_n\}$ a decreasing sequence of open sets with $\overline{G_{n+1}} \subset G_n$ for all $n \in \mathbb{N}$ and $K = \bigcap_{n=1}^\infty G_n$. The quasi-left-continuity up to ζ and the right continuity of sample path for **M** imply that $P_x(\lim_{n\to\infty} \sigma_{G_n} \neq \sigma_K, \lim_{n\to\infty} \sigma_{G_n} < \zeta) = 0$ for any $x \in X$.

Hence

$$\begin{aligned}\lim_{n\to\infty} H_{G_n}^{\gamma+1}R_{\gamma+1}\varphi(x) &= E_x\Big[\int_{\lim_{n\to\infty}\sigma_{G_n}}^{\infty} e^{-(\gamma+1)s}\varphi(X_s)\,ds\Big]\\ &= E_x\Big[\int_{\lim_{n\to\infty}\sigma_{G_n}}^{\infty} e^{-(\gamma+1)s}\varphi(X_s)\,ds : \lim_{n\to\infty}\sigma_{G_n}<\zeta\Big]\\ &\le E_x\Big[\int_{\sigma_K}^{\infty} e^{-(\gamma+1)s}\varphi(X_s)\,ds\Big]=0\end{aligned}$$

for m-a.e. $x\in X$ by $P_m(\sigma_K<\infty)=0$, which implies $\lim_{n\to\infty}\mathrm{Cap}_h(G_n)=0$ from (i) and Lemma 4.3. Thus we have the $\mathcal{E}$-polarity of K. Let N be a Borel m-polar set. Since $(\mathcal{E},\mathcal{F})$ is quasi-regular, there exists an $\mathcal{E}$-nest $\{K_n\}$ of compact sets. Then $\widehat{X}:=\bigcup_{n=1}^{\infty}K_n$ is a Lusin space with respect to the relative topology on $\widehat{X}$. Then the trace of Cap_h on $\widehat{X}$ is a Choquet capacity, hence Choquet's capacitability theorem tells us $\mathrm{Cap}_h(N)=\mathrm{Cap}_h(N\cap\widehat{X})=\sup_{K\subset N\cap\widehat{X}}\mathrm{Cap}_h(K)=0$. Next we prove the if part. Suppose that N is an $\mathcal{E}$-polar set. Then there exists a decreasing sequence $\{G_n\}$ of open sets with $N\subset G_n$ and $\mathrm{Cap}_h(G_n)=(H_{G_n}^{\gamma+1}R_{\gamma+1}\varphi,\varphi)_m\to 0$ as $n\to\infty$. We set $B:=\bigcap_{n=1}^{\infty}G_n$. We have $H_B^{\gamma+1}R_{\gamma+1}\varphi=0$ m-a.e. on X, by $H_{G_n}^{\gamma+1}R_{\gamma+1}\varphi\to 0$, $n\to\infty$ m-a.e. on X. Since $\varphi>0$ m-a.e., $P_m(\sigma_B<\infty)=P_m(\sigma_B<\zeta)=0$, which implies the m-polarity of B, hence the exceptionality of N. This completes the proof. ■

Lemma 4.6 *Suppose that* **M** *is associated with* $(\mathcal{E},\mathcal{F})$. *Then the following properties hold.*

(i) *Let A be an m-null $\mathcal{B}^n(X)$-measurable finely open set. Then A is m-polar.*

(ii) *Let u be a finely upper-semi-continuous q.e. function with respect to* **M**. *If $u\ge 0$ m-a.e. on a finely open set $E\in\mathcal{B}^n(X)$, then $u\ge 0$ q.e. on E.*

(iii) *For any exceptional set N, there exists a properly Borel exceptional set B containing N.*

Proof We first prove (ii) in the case that u is a finely upper-semi-continuous $\mathcal{B}^n(X)$-measurable function and $E=X$. We set $A:=\{u<0\}$ and $A_n:=\{u\le -1/n\}$. Then A is an m-negligible finely open $\mathcal{B}^n(X)$-measurable set and A_n is a finely closed $\mathcal{B}^n(X)$-measurable set. Let G_n be an open set containing A_n. We have for $f,g\in L^2_+(X;m)$ and $\alpha>\gamma$,

$$\begin{aligned}(H_{A_n}^{\alpha}R_\alpha f,g)_m &\le (H_{G_n}^{\alpha}R_\alpha f,g)_m\\ &=(G_\alpha f-\Pi^{\alpha}_{\mathcal{F}_{G_n^c}}(G_\alpha f),g)_m\\ &=(f,\widehat{G}_\alpha g-\widehat{\Pi}^{\alpha}_{\mathcal{F}_{G_n^c}}(\widehat{G}_\alpha g))_m.\end{aligned}$$

If we put $f=I_A$, we have $H_{A_n}^{\alpha}R_\beta I_A(x)=0$ m-a.e. $x\in X$ for $\beta>\alpha$. Since $\lim_{\beta\to\infty}\beta R_\beta I_A(y)=1$ for any $y\in A_n$, we get $P_m(\sigma_{A_n}<\infty)=0$, which implies $P_m(\sigma_A<\infty)=0$ by $\bigcup_{n=1}^{\infty}A_n=A$. Next we prove (i). Let $A\in\mathcal{B}^n(X)$ be an

m-negligible finely open set. Then $-I_A$ is a finely upper-semi-continuous $\mathcal{B}^n(X)$-measurable function such that $-I_A \geq 0$ m-a.e. By the first argument, we have $-I_A \geq 0$ q.e. which implies the m-polarity of A. (ii) and (iii) can be obtained from (i) as in the proofs of [18, Lemma 4.1.5; Theorem 4.1.1] by using Lemma 4.5 (i). ■

Theorem 4.2 **([13, Theorem 4.3])** *Suppose that* $\mathbf{M}$ *is associated with* $(\mathcal{E}, \mathcal{F})$*. Then any semi-polar sets of* $\mathbf{M}$ *are exceptional.*

Proof The assertion follows [13, Theorem 4.3] by noting that semi-polarity (resp. exceptionality) with respect to $\mathbf{M}$ is equivalent to the semi-polarity (resp. exceptionality) of $\mathbf{M}^\gamma$, because for any $B \in \mathcal{B}^n(X)$, we have $P_x^\gamma(\sigma_B = 0) = P_x(\sigma_B = 0)$ and $P_x^\gamma(\sigma_B < \zeta) = P_x(\sigma_B < \zeta)$ in view of the construction of the subprocess $\mathbf{M}^\gamma$ by $e^{-\gamma t}$ (see [4, Ch. III. 3]). ■

Lemma 4.7 **([18, Theorem 4.2.2, Lemma 4.2.2(i)])** *Suppose that* $\mathbf{M}$ *is associated with* $(\mathcal{E}, \mathcal{F})$*. Then the following hold.*

(i) *If* u *is* $\mathcal{E}$*-quasi-continuous, then* u *is finely continuous q.e. More specifically, there exists a Borel properly exceptional set* N *such that* u *is* $\mathcal{B}^n(X)$*-measurable on* $X \setminus N$ *and for any* $x \in X \setminus N$, $P_x(u(X_t)$ *is right continuous at* $t \in [0, \zeta[) = 1$.

(ii) *If* $u \in \mathcal{F}$ *is finely continuous q.e., then* u *is* $\mathcal{E}$*-quasi-continuous.*

Proof In view of Lemma 4.5, (i) (resp. (ii)) is similarly proved as in Theorem 4.2.2 (resp. Lemma 4.2.2) in [18]. This completes the proof. ■

Proposition 4.1 **([18, Lemma 4.3.1, Theorems 4.4.1, 4.6.1])** *Suppose that* $\mathbf{M}$ *is associated with* $(\mathcal{E}, \mathcal{F})$*. Then the following hold.*

(i) *For any set* $B \in \mathcal{B}^n(X)$, $f \in \mathcal{B}(X) \cap L^2(X; m)$ *and* $\alpha > \gamma$, $H_B^\alpha R_\alpha f$ *is an* $\mathcal{E}$*-quasi-continuous* m*-version of* $G_\alpha f - \Pi^\alpha_{\mathcal{F}_{B^c}}(G_\alpha f)$*; in particular,* $R_\alpha^{B^c} f$ *is an* $\mathcal{E}$*-quasi-continuous* m*-version of* $G_\alpha^{B^c} f := \Pi^\alpha_{\mathcal{F}_{B^c}}(G_\alpha f)$.

(ii) E *is* $\mathcal{E}$*-quasi-open if and only if* E *is finely open q.e.*

(iii) u *is* $\mathcal{E}$*-quasi-continuous if and only if* u *is finely continuous q.e.*

Proof The proof of (i) is standard as in the proof of [18, Lemma 4.3.1]. Combining (i) and Theorem 4.2, (ii) and (iii) can be obtained in the same way as the proof of [18, Theorem 4.6.1]. This completes the proof. ■

By Proposition 4.1, we recognize an $\mathcal{E}$-quasi-open (resp. $\mathcal{E}$-quasi-closed) set as a finely open, (resp. finely closed) nearly Borel, set. In particular, for an $\mathcal{E}$-quasi-open (resp. -closed) set E we define $\sigma_E := \sigma_{\widetilde{E}}$, where $\widetilde{E}$ is a finely open (resp. closed) nearly Borel $\mathcal{E}$-q.e. version of E.

For a nearly Borel set E, $\mathbf{M}_E := (\Omega, X_t^E, P_x)_{x \in E_\Delta}$, defined by $X_t^E := X_t$ if $t < \sigma_{X \setminus E}$ and $X_t^E := \Delta$ if $t \geq \sigma_{X \setminus E}$, is called the *part process* on E. For an $\mathcal{E}$-quasi-open set E, we use the same notation $\mathbf{M}_E$ for $\mathbf{M}_{\widetilde{E}}$. The transition function $(p_t^E)_{t>0}$ of $\mathbf{M}_E$ is given by $p_t^E f(x) = E_x[f(X_t) : t < \sigma_{X \setminus E}]$ (see [18, A.2]).

Theorem 4.3 **([18, Theorems 4.4.2, 4.4.3] and [13, Theorem 5.10])** *Suppose that* $\mathbf{M}$ *is associated with* $(\mathcal{E}, \mathcal{F})$*. Let* E *be an* $\mathcal{E}$*-quasi-open set. Then* $(\mathcal{E}_E, \mathcal{F}_E)$ *is associated with* $\mathbf{M}_E$ *in the sense that* $(T_t^E)_{t>0}$ *on* $L^2(E;m)$ *corresponding to* $(\mathcal{E}_E, \mathcal{F}_E)$ *is determined by the transition function* $(p_t^E)_{t>0}$ *of* $\mathbf{M}_E$*. Moreover,* $p_t^E f$ *is an* $\mathcal{E}_E$*-quasi-continuous* m*-version of* $T_t^E f$ *for any* $f \in \mathcal{B}(E) \cap L^2(E;m)$*.*

Proof By Proposition 4.1, $R_\alpha^E f$ is an $\mathcal{E}$-quasi-continuous m-version of $G_\alpha^E f$ for $\alpha > \gamma$ and $f \in \mathcal{B}(X) \cap L^2(X;m)$. The first assertion follows from [33, Ch. IV. Exercise 2.7]. The second assertion can be deduced by way of [33, Ch. IV. Proposition 2.8; Exercise 2.9]. Note that $\mathcal{B}(E) = \sigma(C(E))$. Though the proof of [33, Ch. IV. Proposition 2.8] requires the existence of the dual process (see [33, Ch. IV. Lemma 2.1]), it is possible to modify the proof in order to obtain the conclusion. In fact, if we change $\mathcal{D}(\mathbf{M})$ specified in [33, Chapter IV. 2] to be a subfamily of $\mathcal{B}_b(E) \cap L^2(E;m)$, then the function ρ satisfying [33, Ch. IV. (2.1)] can be constructed without dual process (actually $\rho := R_{\gamma+1}\varphi$ with $\varphi \in \mathcal{B}_b(E) \cap L^2(E;m)$, $\varphi > 0$ on E does the job). Then the proof of (ii)⇒(i) [33, Ch. IV, Proposition 2.8] remains valid. ■

Theorem 4.4 **([18, Theorem 4.3.1])** *Suppose that* $\mathbf{M}$ *is associated with* $(\mathcal{E}, \mathcal{F})$*. Let* B *be a nearly Borel subset of* X*. For each* $u \in \mathcal{F}$ *and* $\alpha > \gamma$*,* $H_B^\alpha \widetilde{u}$ *is an* $\mathcal{E}$*-quasi-continuous* m*-version of* $u - \Pi^\alpha_{\mathcal{F}_{B^c}} u$*. In particular, for any* $\beta > \gamma$ *and* $u \in \mathcal{F}$*, we have* $H_B^\beta \widetilde{u} \in \mathcal{F}$ *and* $\mathcal{E}_\beta(H_B^\beta \widetilde{u}, v) = 0$ *for any* $v \in \mathcal{F}_{B^c}$*. If we further assume that* $u \in \mathcal{F}$ *is bounded and* $m(B^c) < \infty$*, then the same assertion holds for* $\beta \in [0, \gamma]$*.*

Proof The proof for $\alpha > \gamma$ is similar with the proof of [18, Theorem 4.3.1]. We omit it. Next suppose $\beta \in [0, \gamma]$, $u \in \mathcal{F}_b$ and $m(B^c) < \infty$. Note that $R_\alpha^{B^c} f$ is an m-version of $G_\alpha^{B^c} f$ for $f \in L^2(X;m)$ and $\alpha > \gamma$. Applying Lemma 4.1 to $(\mathcal{E}_{B^c}, \mathcal{F}_{B^c})$, we have $R_\beta^{B^c} f \in \mathcal{F}_{B^c}$ for any $f \in L^\infty(B^c;m)$. The latter assertion is an easy consequence through $H_B^\alpha \widetilde{u} - H_B^\beta \widetilde{u} + (\alpha - \beta) R_\alpha^{B^c} H_B^\beta \widetilde{u} = 0$. This completes the proof. ■

Theorem 4.5 *Suppose that* $\mathbf{M}$ *is associated with* $(\mathcal{E}, \mathcal{F})$*. Then the following are equivalent.*

(i) $(\mathcal{E}, \mathcal{F})$ *possesses the local property.*

(ii) *For any open set* G *and* $\alpha \geq 0$*, the* α*-hitting distribution* $H_{G^c}^\alpha(x, dy)$ *to* G^c *is concentrated on the boundary* ∂G *for* $\mathcal{E}$*-q.e.* $x \in G$*.*

(iii) *For any open set* G *and* $\alpha \geq 0$*, the* α*-hitting distribution* $H_{G^c}^\alpha(x, dy)$ *to* G^c *is concentrated on the* $\mathcal{E}$*-quasi-boundary* $\mathcal{E}$*-*∂G *for* $\mathcal{E}$*-q.e.* $x \in G$*.*

(iv) *For any open set* G*,* $P_x(X_{\sigma_{G^c}} \notin \partial G, \sigma_{G^c} < \infty) = 0$ *for* $\mathcal{E}$*-q.e.* $x \in G$*.*

(v) *For any open set* G*,* $P_x(X_{\sigma_{G^c}} \notin \mathcal{E}\text{-}\partial G, \sigma_{G^c} < \infty) = 0$ *for* $\mathcal{E}$*-q.e.* $x \in G$*.*

(vi) $P_x(X_t$ *is continuous at any* $t \in [0, \zeta[) = 1$ *for* $\mathcal{E}$*-q.e.* $x \in X$*.*

Proof The implications (ii) ⇔ (iv), (iii) ⇔ (v) and (iii) ⇒ (ii) are trivial. For the proof of (i) ⇔ (vi), see [32, Remark 3.10] and [33, Ch. V. 1], or [18, Theorem 4.5.1]. We only prove (i) ⇒ (iii) and (ii) ⇒ (i). Remark that [33, Ch. V. Lemma 1.8] cannot be directly applied to show (i) ⇔ (iii). Suppose (i). In the same way as the proof of

Lemma 4.5.1(i) $\Leftrightarrow$ (ii) in [18] with the help of Lemma 3.3 and Theorem 4.4, we have $H^{\alpha}_{G^c}\widetilde{u}(x) = 0$ $\mathcal{E}$-q.e. $x \in G$ for any $u \in \mathcal{F}^+_b$ with $\mathcal{E}$-$\operatorname{supp}[u] \subset X \setminus \overline{G}^{\mathcal{E}}$ and $\alpha > \gamma$. Here we use the fact that $uv = 0$ for any $v \in \mathcal{F}_G$ and such u as above. Take a Borel function f with $0 < f \leq 1$ on X and set $u(x) := E_x[\int_0^{\sigma_G} e^{-\alpha t} f(X_t)\, dt]$. Then $u \in \mathcal{F}_b$ is $\mathcal{E}$-quasi-continuous with $0 \leq u \leq I_{X\setminus\overline{G}^{\mathcal{E}}}$ and $u > 0$ $\mathcal{E}$-q.e. on $X \setminus \overline{G}^{\mathcal{E}}$, because $\overline{G}^{\mathcal{E}} = G^r$ $\mathcal{E}$-q.e., where $G^r := \{x \in X \mid P_x(\sigma_G = 0) = 1\}$ is the set of regular points for G. Then $\mathcal{E}$-$\operatorname{supp}[u] \subset X \setminus \overline{G}^{\mathcal{E}}$. Hence we have $H^{\alpha}_{G^c} I_{X\setminus\overline{G}^{\mathcal{E}}}(x) = 0$ $\mathcal{E}$-q.e. $x \in G$, consequently (ii) holds by $P_x(X_{\sigma_{G^c}} \notin G^c, \sigma_{G^c} < \infty) = 0$ for $x \in G$. The proof of (ii) $\Rightarrow$ (i) is similar as in Lemma 4.5.1(ii) $\Rightarrow$ (i) in [18]. This completes the proof. ■

Lemma 4.8 **(Fine full support)** *If* **M** *satisfies the absolute continuity condition with respect to m, then m has hull topological support with respect to the fine topology of* **M**.

Proof Suppose that $E \in \mathcal{B}^n(X)$ is a finely open set with $m(E) = 0$. It suffices to show $E = \varnothing$. Since E is finely open, we have that $I_E(X_t)$ is lower-semi-right-continuous at 0 P_x-a.s. for $x \in X$, that is, $P_x(I_E(X_0) \leq \underline{\lim}_{t\downarrow 0} I_E(X_t)) = 1$ for any $x \in X$. Let $\{t_k\}$ be any decreasing sequence which converges to 0. Then

$$\begin{aligned} I_E(X_0) &\leq \underline{\lim}_{t\to 0} I_E(X_t) := \sup_{\delta>0} \inf_{0\leq s<\delta} I_E(X_s) \\ &= \sup_{k\geq 1} \inf_{0\leq s<t_k} I_E(X_s) \leq \sup_{k\geq 1} \inf_{\ell\geq k} I_E(X_{t_\ell}) \\ &\leq \underline{\lim}_{k\to\infty} I_E(X_{t_k}). \end{aligned}$$

Hence $I_E(x) = E_x[I_E(X_0)] \leq \underline{\lim}_{k\to 0} E_x[I_E(X_{t_k})] = \underline{\lim}_{k\to 0} P_x(X_{t_k} \in E) = 0$ for all $x \in X$, which implies $E = \varnothing$. ■

Corollary 4.1 *Suppose that* **M** *is associated with $(\mathcal{E}, \mathcal{F})$. We assume that* **M** *satisfies the absolute continuity condition with respect to m. Then the following are equivalent.*

(i) *$(\mathcal{E}, \mathcal{F})$ possesses the local property.*

(ii) *For any open set G and $\alpha \geq 0$, the α-hitting distribution $H^{\alpha}_{G^c}(x, dy)$ to G^c is concentrated on the boundary ∂G for any $x \in G$.*

(iii) *For any open set G and $\alpha \geq 0$, the α-hitting distribution $H^{\alpha}_{G^c}(x, dy)$ to G^c is concentrated on the $\mathcal{E}$-quasi-boundary $\mathcal{E}$-∂G for any $x \in G$.*

(iv) *For any open set G, $P_x(X_{\sigma_{G^c}} \notin \partial G, \sigma_{G^c} < \infty) = 0$ for any $x \in G$.*

(v) *For any open set G, $P_x(X_{\sigma_{G^c}} \notin \mathcal{E}$-$\partial G, \sigma_{G^c} < \infty) = 0$ for any $x \in G$.*

(vi) *$P_x(X_t$ is continuous at any $t \in [0, \zeta[\,) = 1$ for any $x \in X$.*

Proof Since **M** satisfies the absolute continuity condition with respect to m, for any open set G, the part process $\mathbf{M}_G$ also has the same property. Let f be a non-negative Borel function f. Then $H^{\alpha}_{G^c} f$ is α-excessive, hence finely continuous with respect to $\mathbf{M}_G$. Consequently, $H^{\alpha}_{G^c} f = 0$ $\mathcal{E}$-q.e. on G implies $H^{\alpha}_{G^c} f = 0$ on G by applying Lemma 4.8. So the proof of Theorem 4.5 remains valid. This completes the proof. ■

Theorem 4.6 *Suppose that* $\mathbf{M}$ *is associated with* $(\mathcal{E}, \mathcal{F})$*. Let* E *be a finely open (nearly) Borel set and* $\{E_n\}$ *a family of finely open (nearly) Borel sets. Then the following are equivalent.*

(i) $\{E_n\} \in \Xi_E$.

(ii) $P_x(\lim_{n\to\infty} \tau_{E_n} = \tau_E) = 1$ *for* m*-a.e.* $x \in X$.

(iii) $P_x(\lim_{n\to\infty} \tau_{E_n} = \tau_E) = 1$ *for* $\mathcal{E}$*-q.e.* $x \in X$.

Further assume that $E_n \subset E_{n+1}$ *for* $n \in \mathbb{N}$ *and* $E = \bigcup_{n=1}^{\infty} E_n$ *and* $\mathbf{M}$ *satisfies the absolute continuity condition with respect to* m*. Then* (i)–(iii) *are equivalent to*

(iv) $P_x(\lim_{n\to\infty} \tau_{E_n} = \tau_E) = 1$ *for all* $x \in X$.

Proof We may assume $\gamma = 0$ and $E = X$. In this case τ_E is changed to be ζ. Recall that $h^1_{E_n^c} = G_1\varphi - G_1^{E_n}\varphi$ for $h = G_1\varphi$, $\varphi \in L^2_+(X;m)$. For such a φ, $R_1\varphi - R_1^{E_n}\varphi$ is an $\mathcal{E}$-quasi-continuous m-version of $h^1_{E_n^c}$. Then $h^1_{E_n^c}$ m-a.e. decreases to a function $h_\infty \in \mathcal{F}$ as $n \to \infty$. Moreover, it converges in $L^2(X;m)$ and $\mathcal{E}_1$-weakly converges. Suppose (i). Then $h^1_{E_n^c}$ converges in $\mathcal{E}_1^{1/2}$-norm to $h_\infty = 0$ by Lemma 3.6. Hence $\widetilde{h}^1_{E_n^c}$ $\mathcal{E}$-quasi-uniformly converges to 0, which implies (iii). The implication (iii) $\to$ (ii) is clear. Conversely suppose (ii). Then $h_\infty = 0$, hence we have $\mathcal{E}_1(h^1_{E_n^c}, h^1_{E_n^c}) \leq \mathcal{E}_1(h^1_{E_n^c}, G_1\varphi) \to \mathcal{E}_1(h_\infty, G_1\varphi) = 0$ as $n \to \infty$. Thus we obtain (i) by Lemma 3.6. Next we prove the latter assertion. The implication (iv) $\Rightarrow$ (iii) is trivial. We shall show (i) $\Rightarrow$ (iv). Set $\tau := \lim_{n\to\infty} \tau_{E_n}$. Then we see $\tau \leq \zeta$. We should prove $P_x(\tau = \zeta) = 1$ for all $x \in X$. It suffices to prove that $R_1^\tau\varphi(x) := E_x[\int_0^\tau e^{-t}\varphi(X_t)\,dt] = R_1\varphi(x)$ for all $x \in X$ and $\varphi \in \mathcal{B}_b(X) \cap L^2_+(X;m)$ with $0 < \varphi \leq 1$. Fix such a φ. We already see that it holds for m-a.e. $x \in X$. By construction, for each $n \in \mathbb{N}$, both $R_1^\tau\varphi$ and $R_1\varphi$ are excessive with respect to $\mathbf{M}_{E_n}$, consequently finely continuous on E_n. Thus, $R_1^\tau\varphi(x) = R_1\varphi(x)$ for all $x \in E_n$ for each $n \in \mathbb{N}$ in view of Lemma 4.8. Therefore, $R_1^\tau\varphi(x) = R_1\varphi(x)$ for all $x \in X$, which implies (iv). ■

5 Fine Connectedness of Right Processes Associated with Irreducible Semi-Dirichlet Forms

We first treat a coercive closed form $(\mathcal{E}, \mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$. Let $(T_t)_{t>0}$ be a strongly continuous semigroup on $L^2(X;m)$ associated with $(\mathcal{E}, \mathcal{F})$.

Definition 5.1 **(Invariant set)** Let $(\mathcal{E}, \mathcal{F})$ be a coercive closed form with a lower bound $-\gamma$ on $L^2(X;m)$. A subset B of X is said to be *invariant* with respect to $(\mathcal{E}, \mathcal{F})$ or $(T_t)_{t>0}$ if and only if $T_t(I_B f) = I_B T_t f$ for all $t > 0$ and $f \in L^2(X;m)$.

Obviously B is invariant if and only if B^c is invariant, and B is invariant with respect to the dual form $(\widehat{\mathcal{E}}, \mathcal{F})$ or the dual semigroup $(\widehat{T}_t)_{t>0}$.

The following theorem is due to Y. Oshima [37, Theorem 1.4.1].

Theorem 5.1 *Let* $(\mathcal{E}, \mathcal{F})$ *be a coercive closed form with a lower bound* $-\gamma$ *on* $L^2(X;m)$*. Then* B *is an invariant set with respect to* $(\mathcal{E}, \mathcal{F})$ *if and only if for all* $u \in \mathcal{F}$*,* $I_B u \in \mathcal{F}$

and $\mathcal{E}(u,v) = \mathcal{E}(I_B u, I_B v) + \mathcal{E}(I_{B^c} u, I_{B^c} v)$ *for* $u, v \in \mathcal{F}$.

The next theorem is essentially due to Fukushima in the framework of regular symmetric Dirichlet forms [16, Theorem 2] and [18, Corollaries 4.6.2, 4.6.3].

Theorem 5.2 *Let* $(\mathcal{E},\mathcal{F})$ *be a quasi-regular semi-Dirichlet form with a lower bound* $-\gamma$ *on* $L^2(X;m)$. *Suppose that there exists an m-tight special standard process* $\mathbf{M}$ *associated with* $(\mathcal{E},\mathcal{F})$. *Fix an m-measurable set B.*

(i) *Suppose that B is simultaneously* $\mathcal{E}$*-quasi-open and* $\mathcal{E}$*-quasi-closed. Further assume that one of the following two conditions is satisfied:*

(a) *B is invariant with respect to* $(\mathcal{E},\mathcal{F})$.

(b) $(\mathcal{E},\mathcal{F})$ *is local, that is,* $\mathbf{M}$ *is a diffusion.*

Then there exists a properly exceptional set N such that both $B \setminus N$ *and* $B^c \setminus N$ *are* $\mathbf{M}$*-invariant. Further assume that* $\mathbf{M}$ *satisfies the absolute continuity condition with respect to m and B is finely open and finely closed Then B and* B^c *are* $\mathbf{M}$*-invariant. In particular, N can be taken to be empty.*

(ii) *The following condition is equivalent to* (a).

(c) *X can be decomposed as* $X = B_1 + B_2 + N$ *where* B_1*(resp.* B_2*) is an* $\mathbf{M}$*-invariant nearly Borel m-version of B(resp.* B^c*) and* $m(N) = 0$.

Further assume that $\mathbf{M}$ *satisfies the absolute continuity condition with respect to m. Then the following condition is equivalent to* (a).

(c′) *X can be decomposed as* $X = B_1 + B_2$ *where* B_1*(resp.* B_2*) is an* $\mathbf{M}$*-invariant nearly Borel m-version of B(resp.* B^c*).*

Moreover, condition (c′) *yields that both* B_1 *and* B_2 *specified in* (c′) *are finely open and finely closed (resp. open and closed) under the absolute continuity condition with respect to m (resp. strong Feller property).*

(iii) *If we assume* (b), *then one (hence all) of the following conditions is equivalent to* (a) *(or* (c)*).*

(d) $u \in \mathcal{F}$ *implies* $I_B u \in \mathcal{F}$.

(e) $I_B \in \dot{\mathcal{F}}_{\mathrm{loc}}$.

(f) *B has an* $\mathcal{E}$*-quasi-open and* $\mathcal{E}$*-quasi-closed m-version* $\widetilde{B}$ *of B.*

Proof By use of the results in the previous section, the proof of the first part of (i) (resp. (ii)) is similar to the proof of Lemma 4.6.3 (resp. Corollary 4.6.2) in [18]. The proof of (a) $\Leftrightarrow$ (f) is similar to the proof of [18, Corollary 4.6.3]. The proof of (e) $\Rightarrow$ (f) is clear. The proof of (a) $\Rightarrow$ (f) follows from Theorem 5.1. d We first prove (d) $\Rightarrow$ (e). Set $h := G_{\gamma+1}\varphi$ with $\varphi \in \mathcal{K}(X)$. Then $\widetilde{h} > 0$ $\mathcal{E}$-q.e. on X. Hence $I_B = I_B(nh \wedge 1)$ m-a.e. on $\{x \in X \mid \widetilde{h} > 1/n\}$. Since $I_B(nh \wedge 1) \in \mathcal{F}$ by $nh \wedge 1 \in \mathcal{F}$, we have $I_B \in \dot{\mathcal{F}}_{\mathrm{loc}}$. Next we show the second part of (i). Suppose (a) or (b) and the absolute continuity condition for $\mathbf{M}$ with respect to m, and assume that B and B^c are finely open and finely closed. Then we have that for any $\alpha > 0$, $R_\alpha I_B = I_B R_\alpha 1$ and $R_\alpha I_{B^c} = I_{B^c} R_\alpha 1$ m-a.e. on X. Both sides are finely continuous, which implies the

$\mathbf{M}$-invariance of B and B^c. Finally we show the second part of (ii). The implication (a) $\Rightarrow$ (c$'$) is essentially proved above by taking a finely open and finely closed nearly Borel m-version of the invariant set B. Conversely suppose (c$'$). Then we see that $R_\alpha I_{B_1} u = I_{B_1} R_\alpha u$ and $R_\alpha I_{B_2} u = I_{B_2} R_\alpha u$ for any $\alpha > 0$ and $u \in L^2(X; m)$. Hence B_1 and B_2 are finely open and finely closed, and B is invariant. If $\mathbf{M}$ is a strong Feller process, B_1 and B_2 are open and closed. ■

Definition 5.2 **(Irreducibility)** A coercive closed form $(\mathcal{E}, \mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X; m)$ is called *irreducible* if for any invariant set B of $(\mathcal{E}, \mathcal{F})$, $m(B) = 0$ or $m(B^c) = 0$.

Corollary 5.1 **(Inheritance of irreducibility)** *Let $(\mathcal{E}^{(1)}, \mathcal{F})$ on $L^2(X; m^{(1)})$ and $(\mathcal{E}^{(2)}, \mathcal{F})$ on $L^2(X; m^{(2)})$ be two quasi-regular semi-Dirichlet forms with the same lower bound $-\gamma$ having a common domain $\mathcal{F} \subset L^2(X; m^{(1)}) \cap L^2(X; m^{(2)})$ and suppose that $(\mathcal{E}^{(1)}, \mathcal{F})$ has the local property and there exists an m-tight special standard diffusion process $\mathbf{M}^{(1)}$ associated with $(\mathcal{E}^{(1)}, \mathcal{F})$. Here $m^{(1)}$ and $m^{(2)}$ are σ-finite Borel measures on X. Suppose that $\mathcal{E}^{(2)}(u, u) \geq \mathcal{E}^{(1)}(u, u)$ holds for $u \in \mathcal{F}$ and $m^{(2)} \geq m^{(1)}$. Then the irreducibility of $\mathcal{E}^{(1)}$ implies the same property of $\mathcal{E}^{(2)}$.*

Proof It is easy to see that every $\mathcal{E}^{(2)}$-nest is an $\mathcal{E}^{(1)}$-nest. Hence $\mathcal{E}^{(2)}$-polarity (resp. $\mathcal{E}^{(2)}$-quasi-upper-semi-continuity) implies the $\mathcal{E}^{(1)}$-polarity (resp. $\mathcal{E}^{(1)}$-quasi-upper-semi-continuity). Suppose that B is invariant with respect to $(\mathcal{E}^{(2)}, \mathcal{F})$. Then condition (f) of Theorem 5.2 holds for $(\mathcal{E}^{(2)}, \mathcal{F})$. Therefore it holds for $(\mathcal{E}^{(1)}, \mathcal{F})$. By applying Theorem 5.2(iii) again, we have the invariance of B with respect to $(\mathcal{E}^{(1)}, \mathcal{F})$. This completes the proof. ■

Theorem 5.3 **(Fine connectedness)** *Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X; m)$. Suppose that there exists an m-tight special standard process $\mathbf{M}$ associated with $(\mathcal{E}, \mathcal{F})$. Further assume that $\mathbf{M}$ satisfies the absolute continuity condition with respect to m and $(\mathcal{E}, \mathcal{F})$ is irreducible. Then $\mathbf{M}$ is finely connected; namely, if G is finely open and finely closed Borel set of $\mathbf{M}$, $G = \varnothing$ or $G = X$.*

Proof Suppose that G is a finely open and finely closed Borel set of $\mathbf{M}$. Then G is $\mathcal{E}$-quasi-open and $\mathcal{E}$-quasi-closed by Proposition 4.1(ii). We have that G is invariant by Theorem 5.2. According to the irreducibility of $(\mathcal{E}, \mathcal{F})$, $m(G) = 0$ or $m(G^c) = 0$, hence $G = \varnothing$ or $G^c = \varnothing$ by Lemma 4.8. This completes the proof. ■

6 $\mathcal{E}$-Subharmonic Functions

Throughout this section, we fix $\gamma \in [0, \infty[$.

Definition 6.1 Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular local positivity preserving form with a lower bound $-\gamma$ on $L^2(X; m)$. For $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$ and $\{G_i\} \in \Xi(u)$ and $v \in \mathcal{F}_{G_i}$, we set $\mathcal{E}(u, v) := \mathcal{E}(u_i, v)$. Here $u_i \in \mathcal{F}$ is the function specified in the definition of

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$u \in \dot{\mathcal{F}}_{\mathrm{loc}}$. Owing to the local property of $(\mathcal{E}, \mathcal{F})$, $\mathcal{E}(u, v)$ is well defined for $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$ and $v \in \bigcup_{i=1}^{\infty} \mathcal{F}_{G_i}$ with $\{G_i\} \in \Xi(u)$. Similarly we can define $\mathcal{E}_\alpha(u, v)$ for such u, v and $\alpha > 0$.

Definition 6.2 **($\mathcal{E}_\alpha$-subharmonicity)** Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular local positivity preserving form with a lower bound $-\gamma$ on $L^2(X; m)$. Fix an $\alpha \geq 0$. A function $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$ is said to be $\mathcal{E}_\alpha$-*subharmonic* relative to $\{G_i\} \in \Xi(u)$ if $\mathcal{E}_\alpha(u, v) \leq 0$ for all $v \in \bigcup_{i=1}^{\infty} \mathcal{F}_{G_i}^{+}$ and $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$ is called $\mathcal{E}_\alpha$-*subharmonic* or a *(local) subsolution of* "$(L - \alpha)w = 0$" if u is $\mathcal{E}_\alpha$-subharmonic relative to any $\{G_i\} \in \Xi(u)$. If $-u$ is $\mathcal{E}_\alpha$-subharmonic, u is said to be $\mathcal{E}_\alpha$-*superharmonic* or a *(local) supersolution of* "$(L - \alpha)w = 0$". If both u and $-u$ are $\mathcal{E}_\alpha$-subharmonic, u is said to be $\mathcal{E}_\alpha$-*harmonic* or a *(local) solution of* "$(L - \alpha)w = 0$".

Lemma 6.1 *Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular local positivity preserving form with a lower bound $-\gamma$ on $L^2(X; m)$. For $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$ and $\alpha \geq 0$, the following conditions are equivalent.*

(i) *u is $\mathcal{E}_\alpha$-subharmonic.*

(ii) *u is $\mathcal{E}_\alpha$-subharmonic relative to any $\{G_i\} \in \Xi_{\mathrm{cpt}}(u)$.*

(iii) *u is $\mathcal{E}_\alpha$-subharmonic relative to some $\{G_i\} \in \Xi_{\mathrm{cpt}}(u)$.*

(iv) *u is $\mathcal{E}_\alpha$-subharmonic relative to some $\{G_i\} \in \Xi(u)$.*

Proof The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial. We shall prove (iv) $\Rightarrow$ (i). Suppose that u is $\mathcal{E}_\alpha$-subharmonic relative to some $\{E_j\} \in \Xi(u)$. Take $\{G_i\} \in \Xi(u)$ and $v \in \bigcup_{i=1}^{\infty} \mathcal{F}_{G_i}^{+}$. Then $v \in \mathcal{F}_{G_i}^{+}$ for some $i \in \mathbb{N}$. Noting $\{E_j \cap G_i\}_{j=1}^{\infty} \in \Xi_{G_i}$, there exists $\{v_n\} \subset \bigcup_{j=1}^{\infty} \mathcal{F}_{E_j \cap G_i}^{+}$ such that $v_n \to v$ in $(\mathcal{E}_{G_i})_{\gamma+1}^{1/2}$-norm by Lemma 3.6. Hence $\mathcal{E}_\alpha(u, v) = \mathcal{E}_\alpha(u_i, v) = \lim_{n\to\infty} \mathcal{E}_\alpha(u_i, v_n) = \lim_{n\to\infty} \mathcal{E}_\alpha(u, v_n) \leq 0$. Here $u_i \in \mathcal{F}$ with $u = u_i$ m-a.e. on G_i. This completes the proof. ∎

The next proposition ensures that the present definition of $\mathcal{E}$-subharmonicity is an extension of the $\mathcal{E}$-subharmonicity in the framework of regular local semi-Dirichlet forms.

Proposition 6.1 *Suppose that X is a locally compact separable metric space and m is a positive Radon measure on X with full topological support. Fix $\alpha \geq 0$. Assume that $(\mathcal{E}, \mathcal{F})$ is a regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X; m)$. Recall that $\mathcal{F}_{\mathrm{loc}}$ is the family of functions locally in $\mathcal{F}$ in the ordinary sense. For $u \in \mathcal{F}_{\mathrm{loc}}$, the following conditions are equivalent.*

(i) *u is $\mathcal{E}_\alpha$-subharmonic.*

(ii) *$\mathcal{E}_\alpha(u, v) \leq 0$ for $v \in \mathcal{F}_{\mathrm{cpt}}^{+}$.*

(iii) *$\mathcal{E}_\alpha(u, v) \leq 0$ for $v \in \mathcal{F}^{+} \cap C_0(X)$.*

Proof Take an increasing sequence $\{O_n\}$ of relatively compact open sets with $\overline{O_n} \subset O_{n+1}$ for each $n \in \mathbb{N}$. Then $\{O_n\} \in \Xi(u)$ and $\mathcal{F}_{\mathrm{cpt}} = \bigcup_{n=1}^{\infty} \mathcal{F}_{O_n}$. So the equivalence (i) $\Leftrightarrow$ (ii) is clear from the previous lemma. To show the equivalence (ii) $\Leftrightarrow$ (iii), it suffices to show that $\mathcal{F}_G \cap C_0(G)$ is $\mathcal{E}_{\gamma+1}^{1/2}$-dense in $\mathcal{F}_G$ for any open set G, which is proved in Proposition 3.4. This completes the proof. ■

Lemma 6.2 *Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular local positivity preserving form with a lower bound $-\gamma$ on $L^2(X; m)$. For each $\alpha \geq 0$ and $u \in L^0(X; m)$, the following are equivalent.*

(i) *$u \in \dot{\mathcal{F}}_{\mathrm{loc}}$ and it is $\mathcal{E}_\alpha$-subharmonic.*

(ii) *There exists a $\{G_i\} \in \Xi$ such that $u|_{G_i} \in \mathcal{F}|_{G_i}$ and it is $(\mathcal{E}_{G_i})_\alpha$-subharmonic for all $i \in \mathbb{N}$.*

(iii) *There exists an $\{E_i\} \in \Xi$ such that $u|_{E_i} \in (\dot{\mathcal{F}}_{E_i})_{\mathrm{loc}}$ and it is $(\mathcal{E}_{E_i})_\alpha$-subharmonic for all $i \in \mathbb{N}$.*

Proof The implication (ii) $\Rightarrow$ (iii) is trivial. First we prove (i) $\Rightarrow$ (ii). It suffices to show that for an $\mathcal{E}$-quasi-open set G with $u|_G \in \mathcal{F}|_G$, $u|_G$ is $(\mathcal{E}_G)_\alpha$-subharmonic. Take such G and $\{G_i\} \in \Xi(u)$. Then $\{G \cap G_i\} \in \Xi_G(u)$ by Lemma 3.5(i). Since $\mathcal{F}_{G\cap G_i} \subset \mathcal{F}_{G_i}$, $\mathcal{E}_\alpha(u, w) \leq 0$ for $w \in \bigcup_{i=1}^\infty \mathcal{F}^+_{G\cap G_i}$, which implies the $(\mathcal{E}_G)_\alpha$-subharmonicity of u by Lemma 6.1.

Next we show (iii) $\Rightarrow$ (i). Take an $\{O^i_j\} \in \Xi_{E_i}(u)$. Let $\{F_n\}$ be a common $\mathcal{E}_\gamma$-nest of compact sets such that $\overline{E_i}^{\mathcal{E}} \cap F_n$ is compact and $O^i_j \cap F_n$ is open in F_n for all $i, j, n \in \mathbb{N}$. In view of the observation before Lemma 3.7, such an $\mathcal{E}_\gamma$-nest can be constructed as in the case of quasi-regular semi-Dirichlet form. We may assume $\overline{E_i}^{\mathcal{E}} \subset E_{i+1}$, $O^i_j \subset O^i_{j+1}$ and $E_i = \bigcup_{j=1}^\infty O^i_j$ for all $i, j \in \mathbb{N}$ by deleting adequate $\mathcal{E}$-polar sets. Then there exists a $j = j(n, i)$ such that $\overline{E_i}^{\mathcal{E}} \cap F_n \subset O^{i+1}_j \cap F_n$. Therefore there exists a $u_{i,j} \in \mathcal{F}_{E_{i+1}}$ such that $u = u_{i,j}$ m-a.e. on $E_i \cap F_n^{\mathcal{E}\text{-int}}$. Putting $G_i := E_i \cap F_i^{\mathcal{E}\text{-int}}$, we have $\{G_i\} \in \Xi(u)$, which implies $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$. By assumption, for each $i \in \mathbb{N}$, $\mathcal{E}_\alpha(u, w) \leq 0$ for all $w \in \bigcup_{j=1}^\infty \mathcal{F}^+_{O^i_j}$. Take $w \in \bigcup_{i=1}^\infty \mathcal{F}^+_{G_i} \subset \bigcup_{i=1}^\infty \mathcal{F}^+_{E_i}$. Then there is an $i \in \mathbb{N}$ with $w \in \mathcal{F}^+_{E_i}$ and $w_n \in \bigcup_{j=1}^\infty \mathcal{F}^+_{O^i_j}$ such that $\{w_n\}$ is $\mathcal{E}_{\gamma+1}^{1/2}$-convergent to w. Hence $\mathcal{E}_\alpha(u, w) = \lim_{n\to\infty} \mathcal{E}_\alpha(u, w_n) \leq 0$. Applying Lemma 6.1 again, $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$ and u is $\mathcal{E}_\alpha$-subharmonic. This completes the proof. ■

Corollary 6.1 *Suppose X is a locally compact separable metric space and m is a positive Radon measure on X with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X; m)$. For each $\alpha \geq 0$ and $u \in L^\infty_{\mathrm{loc}}(X; m)$, the following are equivalent.*

(i) *$u \in \mathcal{F}_{\mathrm{loc}}$ and it is $\mathcal{E}_\alpha$-subharmonic.*

(ii) *For any relatively compact open set G, $u|_G \in \mathcal{F}|_G$ and it is $(\mathcal{E}_G)_\alpha$-subharmonic.*

(iii) *For any relatively compact open set E, $u|_E \in (\mathcal{F}_E)_{\mathrm{loc}}$ and it is $(\mathcal{E}_E)_\alpha$-subharmonic.*

Proof The proof is the same as in the above lemma. The local boundedness of u is only used in the proof of $u|_E \in (\mathcal{F}_E)_{\mathrm{loc}}$ in (ii) $\Rightarrow$ (iii). We omit the details. This completes the proof. ■

Let T be a bounded linear operator on $L^2(X; m)$ admitting m-a.e. defined bounded kernel t, namely, there exists a kernel $t\colon X \times \mathcal{B}(X) \to [0, \infty]$ with $t(x, X) < \infty$ for any $x \in X$ such that $Tf(x) = \int_X \widehat{f}(y)t(x,dy)$ m-a.e. $x \in X$ for any Borel m-version $\widehat{f}$ of $f \in L^2_+(X;m)$. Take $u \in L^0_+(X;m)$. We define Tu in the following way: $Tu(x) := \int_X \widehat{u}(y)t(x,dy)$ for some non-negative Borel m-version $\widehat{u}$ of u. Then Tu is m-a.e. well defined and satisfies $0 \le Tu \le \infty$ m-a.e. Note that $(G_\alpha)_{\alpha>\gamma}$ and $(T_t)_{t>0}$ associated with a quasi-regular semi-Dirichlet form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$ are families of bounded linear operators on $L^2(X;m)$ admitting m-a.e. defined bounded kernels if there exists a Borel right process $\mathbf{M}$ associated with $(\mathcal{E},\mathcal{F})$.

Definition 6.3 **(Excessive function in $L^0(X;m)$)** Let $(\mathcal{E},\mathcal{F})$ be a quasi-regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$ and $(T_t)_{t>0}$ semigroup associated with $(\mathcal{E},\mathcal{F})$. Fix $\alpha \ge 0$. A function $u \in L^0(X;m)$ is said to be α-*excessive with respect to* $(\mathcal{E},\mathcal{F})$ if $u \ge 0$ m-a.e. and $e^{-\alpha t}T_t u \le u$ m-a.e. for all $t \ge 0$.

Lemma 6.3 *Let $(\mathcal{E},\mathcal{F})$ be a quasi-regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$ and G an $\mathcal{E}$-quasi-open set. Assume that there exists a Borel right process $\mathbf{M}$ associated with $(\mathcal{E},\mathcal{F})$. Take $\alpha > \gamma$ and $u \in \mathcal{F}^+|_G$. Suppose that u is $(\mathcal{E}_G)_\alpha$-superharmonic on G. Then u is α-excessive with respect to $(\mathcal{E}_G,\mathcal{F}_G)$. If we further assume $u \in \mathcal{F}^+_b|_G$ and $m(G) < \infty$, then the same assertion holds for $\alpha \in [0,\gamma]$.*

Proof First suppose $\alpha > \gamma$ and $u \in \mathcal{F}^+|_G$. Take $v \in \mathcal{F}^+$ with $u = v$ m-a.e. on G. Suppose that u is $(\mathcal{E}_G)_\alpha$-superharmonic on G. We have that $v - H^\alpha_{G^c}\widetilde{v} \in \mathcal{F}_G$ is $(\mathcal{E}_G)_\alpha$-superharmonic by Theorem 4.4. According to Lemma 2.2, $v - H^\alpha_{G^c}\widetilde{v}$ satisfies $e^{-\alpha t}p^G_t(v - H^\alpha_{G^c}\widetilde{v}) \le v - H^\alpha_{G^c}\widetilde{v}$ m-a.e. on G for all $t > 0$. Noting that $e^{-\alpha t}p^G_t H^\alpha_{G^c}\widetilde{v} \le H^\alpha_{G^c}\widetilde{v}$ m-a.e. on G for all $t > 0$, we have $e^{-\alpha t}p^G_t v \le v$ m-a.e. on G for all $t > 0$. Next suppose $\alpha \in [0,\gamma]$, $u \in \mathcal{F}^+_b|_G$ and $m(G) < \infty$. Then the same proof works as above by way of Theorem 4.4. This completes the proof. ■

The following theorem extends [44, Lemma 3(a),(b)] and Lemma 2.2.

Theorem 6.1 *Let $(\mathcal{E},\mathcal{F})$ be a quasi-regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$. Furthermore, we assume that there exists a Borel right process $\mathbf{M}$ associated with $(\mathcal{E},\mathcal{F})$. For $u \in \dot{\mathcal{F}}^+_{\rm loc}$ and $\alpha \ge 0$, the following are equivalent to each other.*

(i) $e^{-\alpha t}T_t u \le u$ *for any* $t > 0$.
(ii) $\beta G_{\alpha+\beta}u \le u$ *for any* $\beta > \gamma - \alpha$.
(iii) u *is* $\mathcal{E}_\alpha$*-superharmonic.*

Proof The implication (i) ⇒ (ii) is clear. We show the implication (ii) ⇒ (iii). We may assume $\alpha > 0$, because the case for $\alpha = 0$ can be obtained from this. First we assume $\alpha > \gamma$. We set $G_i := \{x \in X \mid \widetilde{h} > 1/i, \widetilde{u}(x) < i\}$, where $h := G_{\gamma+1}\varphi$ with $\varphi \in L^2(X;m)$, $0 < \varphi \le 1$ m-a.e. Then $\{G_i\} \in \Xi$ with $m(G_i) < \infty$ for each $i \in \mathbb{N}$. Further we set $E_i := \{x \in X \mid \widetilde{h}^{G_i} > 1/i\}$, where $h^{G_i} := G^{G_i}_{\gamma+1}\varphi$. Then $\{E_i\} \in \Xi$ and $E_i \subset G_i$ by the proof of Lemma 3.8. Set $\mathcal{L}^{G_i}_{1,E_i} := \{u \in \mathcal{F}_{G_i} \mid \widetilde{u} \ge 1\ \mathcal{E}\text{-q.e. on } E_i\}$. Then $ih^{G_i} \wedge 1 \in \mathcal{L}^{G_i}_{1,E_i} \ne \emptyset$. By Stampacchia's projection theorem,

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there exists a unique $e^\alpha_{E_i} \in \mathcal{L}^{G_i}_{1,E_i}$ such that $\widetilde{e}^\alpha_{E_i} = 1$ $\mathcal{E}$-q.e. on E_i and $\mathcal{E}_\alpha(e^\alpha_{E_i}, w) \geq 0$ for $w \in \mathcal{F}^+_{G_i}$. In particular, $e^\alpha_{E_i}$ is $(\mathcal{E}_{G_i})_\alpha$-superharmonic. We may take $\{G_i\}, \{E_i\}$ as finely open Borel sets. Further, $E.[e^{-\alpha\sigma_{E_i}} : \sigma_{E_i} < \tau_{G_i}]$ is an $\mathcal{E}$-quasi-continuous m-version of $e^\alpha_{E_i}$. For general $\alpha > 0$, we set $e^\alpha_{E_i} := E_x[e^{-\alpha\sigma_{E_i}} : \sigma_{E_i} < \tau_{G_i}]$. Then by $e^\alpha_{E_i} - e^\beta_{E_i} + (\alpha - \beta)R^{G_i \setminus E_i}_\beta e^\alpha_{E_i} = 0$, $e^\alpha_{E_i}$ is an α-excessive function in $\mathcal{F}_{G_i}$ with respect to $(\mathcal{E}_{G_i}, \mathcal{F}_{G_i})$ for all $\alpha > 0$ by way of Lemma 2.2. Here we apply Lemma 4.1 to $R^{G_i \setminus E_i}_\beta e^\alpha_{E_i} \in L^2(G_i; m)$, because $m(G_i \setminus E_i) \leq m(G_i) < \infty$. From (ii), $u_i := u \wedge i e^\alpha_{E_i}$ satisfies $\beta G^{G_i}_{\beta+\alpha} u_i \leq u_i$ for all $\beta > \alpha - \gamma$, hence $u_i \in \mathcal{F}_{G_i}$ and $\{E_i\} \in \Xi(u)$ by Lemma 2.1. Therefore u_i is $(\mathcal{E}_{G_i})_\alpha$-superharmonic by Lemma 2.2. Take $v \in \bigcup_{i=1}^\infty \mathcal{F}^+_{E_i}$. Then $\mathcal{E}_\alpha(u, v) = \mathcal{E}_\alpha(u_i, v) \geq 0$ for $v \in \mathcal{F}^+_{E_i}$ and some $i \in \mathbb{N}$.

Next we show the implication (iii) $\Rightarrow$ (i). We may assume $\alpha > 0$ and the $\mathcal{E}$-quasi-continuity of u. Set $G_i := \{x \in X \mid \widetilde{h} > 1/i,\ \widetilde{u}(x) < i\}$ as in (ii) $\Rightarrow$ (iii). We can retake G_i so that $\{G_i\} \in \Xi(u)$ and for each $i \in \mathbb{N}$ there exists $u_i \in \mathcal{F}_b$ with $u = u_i$ m-a.e. on G_i. We may take $\{G_i\}$ as finely open Borel sets. Since u is $\mathcal{E}_\alpha$-superharmonic, $\mathcal{E}_\alpha(u, v) \geq 0$ for all $v \in \bigcup_{i=1}^\infty \mathcal{F}^+_{G_i}$. In particular, u_i on G_i is $(\mathcal{E}_{G_i})_\alpha$-superharmonic. Then by Lemma 6.3, we have $e^{-\alpha t} p^{G_i}_t u \leq u$ m-a.e. on X. Letting $i \to \infty$, we obtain the assertion. This completes the proof. ■

Corollary 6.2 *Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X; m)$. Furthermore, we assume that there exists a Borel right process* $\mathbf{M}$ *associated with $(\mathcal{E}, \mathcal{F})$. Then 1 is $\mathcal{E}$-superhamonic.*

Proof It suffices to show $1 \in \dot{\mathcal{F}}_{\mathrm{loc}}$. Fix $\varphi \in L^2(X; m)$ with $0 < \varphi \leq 1$ and set $h := G_\alpha \varphi$ for $\alpha > \gamma$. Then $nh \wedge 1$ is an α-excessive function in $\mathcal{F}$ by Lemma 2.1. We let $E_n := \{\widetilde{h} > 1/n\}$. We see $\{E_n\} \in \Xi$ and $1 = nh \wedge 1$ m-a.e. on E_n. ■

Theorem 6.2 *Suppose that $(\mathcal{E}, \mathcal{F})$ is a regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X; m)$ in the framework of locally compact separable metric space X having a positive Radon measure m with full support. Furthermore, we assume that there exists a Borel right process* $\mathbf{M}$ *associated with $(\mathcal{E}, \mathcal{F})$. Fix $\alpha \geq 0$. Let u be an α-excessive function with respect to $(\mathcal{E}, \mathcal{F})$. Suppose $u \in L^\infty_{\mathrm{loc}}(X; m)$, or that there exists a $v \in \mathcal{F}_{\mathrm{loc}}$ such that $u \leq v$ m-a.e. on X. Then u is an $\mathcal{E}_\alpha$-superharmonic function in $\mathcal{F}^+_{\mathrm{loc}}$.*

Proof In view of the previous theorem, it suffices to show $u \in \mathcal{F}_{\mathrm{loc}}$ under the present condition. First we assume that $u \in L^\infty_{\mathrm{loc}}(X; m)$. Let $\{A_i\}$ be an increasing sequence of relatively compact open sets with $\overline{A_i} \subset A_{i+1}$, $i \in \mathbb{N}$ and $\bigcup_{i=1}^\infty A_i = X$. We set $\mathcal{L}^{A_{i+1}}_{1,A_i} := \{u \in \mathcal{F}_{A_{i+1}} \mid u \geq 1 m\text{-a.e. on } A_i\}$. Owing to the regularity of $(\mathcal{E}_{A_{i+1}}, \mathcal{F}_{A_{i+1}})$ on $L^2(A_{i+1}; m)$, $\mathcal{L}^{A_{i+1}}_{1,A_i} \neq \emptyset$ for each $i \in \mathbb{N}$. By assumption, u is β-excessive for any $\beta > \gamma$, hence, we may assume $\alpha > \gamma$. Then there exists a unique $e^\alpha_{A_i} \in \mathcal{L}^{A_{i+1}}_{1,A_i}$ such that $\mathcal{E}_\alpha(e^\alpha_{A_i}, w) \geq \mathcal{E}_\alpha(e^\alpha_{A_i}, e^\alpha_{A_i})$ for all $w \in \mathcal{L}^{A_{i+1}}_{1,A_i}$. We see $\mathcal{E}_\alpha(e^\alpha_{A_i}, w) \geq 0$ for $w \in \mathcal{F}^+_{A_{i+1}}$ and $e^\alpha_{A_i} = 1$ m-a.e. on A_i. In particular, $e^\alpha_{A_i}$ is an α-excessive function in $\mathcal{F}_{A_{i+1}}$ with respect to $(\mathcal{E}_{A_{i+1}}, \mathcal{F}_{A_{i+1}})$ (see [33, Ch. III. Proposition 1.5]). We set $u_i := u \wedge \|u\|_{L^\infty(A_i; m)} e^\alpha_{A_i}$.

By the same reason as in the proof of (ii) $\Rightarrow$ (iii) in Theorem 6.1, u_i is an α-excessive function in $\mathcal{F}_{A_{i+1}}$. Since $u = u_i m$-a.e. on A_i, we can conclude $u \in \mathcal{F}_{\mathrm{loc}}$. Next we treat the latter case. Suppose that there exists $v \in \mathcal{F}_{\mathrm{loc}}$ such that $u \leq v$. Let $\{A_i\}$ as above. We see $\mathcal{L}_{u,A_i} := \{w \in \mathcal{F} \mid w \geq um\text{-a.e. on } A_i\} \neq \varnothing$. Then there exists a unique $u^{\alpha}_{A_i} \in \mathcal{F}$ such that $\mathcal{E}_{\alpha}(u^{\alpha}_{A_i}, w) \geq \mathcal{E}_{\alpha}(u^{\alpha}_{A_i}, u^{\alpha}_{A_i})$ for all $w \in \mathcal{L}_{u,A_i}$. We see $\mathcal{E}_{\alpha}(u^{\alpha}_{A_i}, w) \geq 0$ for $w \in \mathcal{F}^{+}$. In particular, $u^{\alpha}_{A_i}$ is an α-excessive function in $\mathcal{F}$ with respect to $(\mathcal{E}, \mathcal{F})$. We then have $u^{\alpha}_{A_i} = u$ m-a.e. on A_i, because $u \wedge u^{\alpha}_{A_i}$ is an α-excessive function in $\mathcal{F}$ by Lemma 2.1. This implies $u \in \mathcal{F}_{\mathrm{loc}}$. This completes the proof. ■

Theorem 6.3 **(Weierstrass Type Theorem)** *Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X; m)$. Assume that there exists a Borel right process* $\mathbf{M}$ *associated with $(\mathcal{E}, \mathcal{F})$. Further assume that $1 \in \dot{\mathcal{F}}_{\mathrm{loc}}$ is $\mathcal{E}$-harmonic. Take $\{v_n\} \subset \dot{\mathcal{F}}_{\mathrm{loc}}$. Then we have the following:*

(i) *Suppose that there exist common $\{G_i\} \in \bigcap_{n=1}^{\infty} \Xi(v_n)$ and $v \in L^0(X; m)$ such that v_n is uniformly bounded on G_i and it converges to v uniformly on G_i as $n \to \infty$ for each $i \in \mathbb{N}$. If v_n is $\mathcal{E}$-superharmonic for all $n \in \mathbb{N}$, then $v \in \dot{\mathcal{F}}_{\mathrm{loc}}$ and it is $\mathcal{E}$-superharmonic.*

(ii) *Suppose that $v_n \in L^{\infty}(X; m)$ is uniformly convergent to some $v \in L^{\infty}(X; m)$. If v_n is $\mathcal{E}$-superharmonic for all $n \in \mathbb{N}$, then $v \in \dot{\mathcal{F}}_{\mathrm{loc}}$ and it is $\mathcal{E}$-superharmonic.*

(iii) *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X; m)$, where X is a locally compact separable metric space and m is an everywhere dense positive Radon measure m on X. Suppose that $v_n \in \mathcal{F}_{\mathrm{loc}} \cap L^{\infty}_{\mathrm{loc}}(X; m)$ is $L^{\infty}_{\mathrm{loc}}(X; m)$-convergent to some $v \in L^{\infty}_{\mathrm{loc}}(X; m)$. If v_n is $\mathcal{E}$-superharmonic for all $n \in \mathbb{N}$, then $v \in \mathcal{F}_{\mathrm{loc}}$ and it is $\mathcal{E}$-superharmonic.*

Proof (ii) and (iii) are clear from (i) except the assertion $v \in \mathcal{F}_{\mathrm{loc}}$ in (iii). First we show (i). By assumption, there exists a constant $M_i > 0$ such that

$$w_n := v_n + M_i \geq 0, \quad w := v + M_i \geq 0$$

m-a.e. on each G_i. We get $T^{G_i}_t w_n \leq w_n$ m-a.e. on G_i by Theorem 6.1, hence $T^{G_i}_t w \leq w$ m-a.e. on G_i, which implies that $v|_{G_i} \in (\dot{\mathcal{F}}_{G_i})_{\mathrm{loc}}$ and its $\mathcal{E}_{G_i}$-superharmonicity by using the $\mathcal{E}$-harmonicity of 1 again. Owing to Lemma 6.2, we have the assertion. Finally we show $v \in \mathcal{F}_{\mathrm{loc}}$ in (iii). Let G be a relatively compact open set. We easily see that $v|_G \in (\mathcal{F}_G)_{\mathrm{loc}}$ and its $\mathcal{E}_G$-superharmonicity in the same way as above. Hence we have the assertion by Corollary 6.1. This completes the proof. ■

Lemma 6.4 *Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X; m)$. Assume that there exists a Borel right process* $\mathbf{M}$ *associated with $(\mathcal{E}, \mathcal{F})$. Fix an $\mathcal{E}$-quasi-open set O. Take $\alpha > \gamma$ and $v \in \mathcal{F}$. Then the following are equivalent.*

(i) *v is $(\mathcal{E}_O)_{\alpha}$-subharmonic.*

(ii) *$v - H^{\alpha}_{O^c}\widetilde{v} \leq e^{-\alpha t} p^O_t (v - H^{\alpha}_{O^c}\widetilde{v})$, m-a.e. on O for all $t > 0$.*

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(iii) $v \le e^{-\alpha t} p_t^O v + E.[e^{-\alpha\sigma_{O^c}} \widetilde{v}(X_{\sigma_{O^c}}) : \sigma_{O^c} \le t]$, *m-a.e. on O for all* $t > 0$.

In particular, if v is $(\mathcal{E}_O)_\alpha$-subharmonic (resp. $(\mathcal{E}_O)_\alpha$-superharmonic) for $\alpha > \gamma$, then $v \le H^\alpha_{O^c}\widetilde{v}$ (resp. $v \ge H^\alpha_{O^c}\widetilde{v}$) m-a.e. on O.

Moreover, if $v \in \mathcal{F}_b$ and $m(O) < \infty$, then the equivalence above holds for the case $\alpha \in [0,\gamma]$. In particular, if $v \in \mathcal{F}_b$, $m(O) < \infty$ and $P_x(\tau_O < \infty) = 1$ m-a.e. $x \in O$, then $(\mathcal{E}_O)_\alpha$-subharmonicity (resp. $(\mathcal{E}_O)_\alpha$-superharmonicity) of v implies $v \le H^\alpha_{O^c}\widetilde{v}$ (resp. $v \ge H^\alpha_{O^c}\widetilde{v}$) m-a.e. on O for $\alpha \in [0,\gamma]$.

Proof The equivalence (ii) $\Leftrightarrow$ (iii) is an easy calculation. Take $\alpha \ge 0$. If $\alpha \in [0,\gamma]$, we assume $v \in \mathcal{F}_b$ and $m(O) < \infty$. As in the proof of Lemma 6.3, we have that $(\mathcal{E}_O)_\alpha$-subharmonicity of v is equivalent to the $(\mathcal{E}_O)_\alpha$-subharmonicity of $v - H^\alpha_{O^c}\widetilde{v} \in \mathcal{F}_O$, because of the $(\mathcal{E}_O)_\alpha$-harmonicity of $H^\alpha_{O^c}\widetilde{v}$, which is equivalent to (ii) by Lemma 2.2, so we show the latter assertion. Let $v \in \mathcal{F}$ be an $(\mathcal{E}_O)_\alpha$-subharmonic function. Suppose $\alpha > \gamma$. Since v and $H^\alpha_{O^c}\widetilde{v}$ are in $L^2(X;m)$, we see that $e^{-\gamma t} p_t^O(v - H^\alpha_{O^c}\widetilde{v})$ is bounded in $L^2(O;m)$ with respect to $t > 0$, which implies the $L^2(O;m)$-convergence of $e^{-\alpha t} p_t^O(v - H^\alpha_{O^c}\widetilde{v})$ to 0 as $t \to \infty$. Taking a subsequence of t, we have the desired assertion. Next suppose $\alpha \in [0,\gamma]$, $v \in \mathcal{F}_b$, $m(O) < \infty$ and $P_x(\tau_O < \infty) = 1$ m-a.e. Then, by way of the proof of Lemma 6.3 again, we have (ii). By assumption, $e^{-\alpha t} p_t^O(v - H^\alpha_{O^c}\widetilde{v})$ is estimated above by $2\|v\|_\infty P_x(t < \tau_O)$ for each $x \in O$. Letting $t \to \infty$, we obtain the result. Finally we note that the boundedness of v is also used to apply Theorem 4.4 to $H^\alpha_{O^c}\widetilde{v}$ for $\alpha \in [0,\gamma]$. This completes the proof. ■

Theorem 6.4 *Let $(\mathcal{E},\mathcal{F})$ be a quasi-regular local semi-Dirichlet form with a lower bound $-\gamma$ on $L^2(X;m)$. Assume that there exists a Borel right process* **M** *associated with $(\mathcal{E},\mathcal{F})$. Take $\alpha \ge 0$.*

(i) *Let η be a convex function satisfying $\eta(0) \le 0$. Suppose that $\eta \circ f - \eta(0) \in \mathcal{F}$ for any $f \in \mathcal{F}$. Then for every $\mathcal{E}$-harmonic function u, the function $\eta \circ u$ is $\mathcal{E}$-subharmonic.*
(ii) *Take $p \ge 1$. Suppose that $|f|^p \in \mathcal{F}$ for any $f \in \mathcal{F}_b$. Then for every $\mathcal{E}_\alpha$-harmonic function u and $p \ge 1$ the function $|u|^p$ is $\mathcal{E}_\alpha$-subharmonic.*
(iii) *Let $u, v \in \dot{\mathcal{F}}_{\rm loc}$ be $\mathcal{E}_\alpha$-subharmonic functions. Then $u \vee v$ is also $\mathcal{E}_\alpha$-subharmonic.*

Proof We first show (i). Take an $\mathcal{E}$-harmonic function u. Let $\{G_i\} \in \Xi(u)$ and $u_i, v_i \in \mathcal{F}$ such that $u = u_i$ m-a.e. on G_i. We may assume that for each $i \in \mathbb{N}$, u_i is bounded on X and $m(G_i) < \infty$. By assumption, we see $w_i := \eta \circ u_i - \eta(0) \in \mathcal{F}$. By Lemma 6.4, $\mathcal{E}_{G_i}$-harmonicity of u_i is equivalent to

$$u_i = p_t^{G_i} u_i + E.[\widetilde{u}_i(X_{\sigma_{G_i^c}}) : \sigma_{G_i^c} \le t]$$

m-a.e. on G_i. Let $K_t^i(x,\cdot)$ be a kernel defined by

$$K_t^i(x,A) := p_t^{G_i}(x,A) + E_x[I_A(X_{\sigma_{G_i^c}}) : \sigma_{G_i^c} \le t].$$

Then $K_t^i(x,\cdot)$ is a Markov kernel. Indeed, $\sigma_{G_i^c} \le t$ implies $\sigma_{G_i^c} < \zeta$, hence we see $K_t^i(x,X) = 1$. Applying Jensen's inequality to η, we have

$$\eta(u_i(x)) = \eta(K_t^i u_i(x)) \le K_t^i(\eta \circ u_i)(x),$$

which implies $w_i \le p_t^{G_i} w_i + E_\cdot[\widetilde{w}_i(X_{\sigma_{G_i^c}}) : \sigma_{G_i^c} \le t]$ m-a.e. on G_i. Hence we obtain the $\mathcal{E}_{G_i}$-subharmonicity of $w_i \in \mathcal{F}$ by Lemma 6.4. Therefore $\eta \circ u$ is $\mathcal{E}$-subharmonic by Lemma 6.2, $\eta(0) \le 0$ and Corollary 6.2.

Next we show (ii). The proof is quite similar to (i) by replacing $K_t^i(x,\cdot)$ with $K_t^{i,\alpha}(x,\cdot)$ defined by $K_t^{i,\alpha}(x,A) := e^{-\alpha t} p_t^{G_i}(x,A) + E_x[e^{-\alpha\sigma_{G_i^c}} I_A(X_{\sigma_{G_i^c}}) : \sigma_{G_i^c} \le t]$. Note that $K_t^{i,\alpha}(x,\cdot)$ is not necessarily a Markov kernel, but a sub-Markov kernel. Let u be an $\mathcal{E}_\alpha$-harmonic function and u_i the $(\mathcal{E}_{G_i})_\alpha$-harmonic function as discussed. By Hölder's inequality, we have

$$\begin{aligned} |u_i(x)|^p &= |K_t^{i,\alpha} u_i(x)|^p \le (K_t^{i,\alpha} 1(x))^{p-1} K_t^{i,\alpha} |u_i|^p(x) \\ &\le K_t^{i,\alpha} |u_i|^p(x), \end{aligned}$$

which imlpies $|u_i|^p \le e^{-\alpha t} p_t^{G_i} |u_i|^p + E_\cdot[e^{-\alpha\sigma_{G_i^c}} |\widetilde{u}_i|^p(X_{\sigma_{G_i^c}}) : \sigma_{G_i^c} \le t]$ m-a.e. on G_i. Hence we obtain the $(\mathcal{E}_{G_i})_\alpha$-subharmonicity of $|u_i|^p$. Therefore $|u|^p$ is $\mathcal{E}_\alpha$-subharmonic.

Finally we show (iii). Let $\{G_i\} \in \Xi(u) \cap \Xi(v)$ and $u_i, v_i \in \mathcal{F}$ such that $u = u_i$, $v = v_i$ m-a.e. on G_i. We may assume that for each $i \in \mathbb{N}$, u_i and v_i are bounded on X and $m(G_i) < \infty$. By Lemma 6.4, $(\mathcal{E}_{G_i})_\alpha$-subharmonicity of u_i is equivalent to $u_i \le p_t^{G_i} u_i + E_\cdot[\widetilde{u}_i(X_{\sigma_{G_i^c}}) : \sigma_{G_i^c} \le t]$ m-a.e. on G_i. Thus we have the $(\mathcal{E}_{G_i})_\alpha$-subharmonicity of $u_i \vee v_i$. Consequently, $u \vee v$ is $\mathcal{E}_\alpha$-subharmonic. This completes the proof. ■

Remark 6.1 If the dual form $(\widehat{\mathcal{E}}, \mathcal{F})$ is also a semi-Dirichlet form on $L^2(X;m)$ with the same lower bound $-\gamma$, then any Lipschitz continuous function η satisfies that $\eta \circ f - \eta(0) \in \mathcal{F}$ for any $f \in \mathcal{F}$. Consequently $|f|^p \in \mathcal{F}$ for any $f \in \mathcal{F}_b$, $p \ge 1$.

7 Proofs of Maximum Principles

We fix an open set G with non-empty boundary ∂G and a quasi-regular local semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ with a lower bound $-\gamma$ on $L^2(X;m)$.

Proof of Proposition 1.1 (i) For any relatively compact open neighborhood U of O, $\mathbf{M}_U$ is also transient. Under the doubly Feller property of $\mathbf{M}$, $\mathbf{M}_U$ is a strong Feller process. Then we see

$$E_x[\tau_O] = E_x\Big[\int_0^{\tau_O} I_O(X_t)\,dt\Big] \le R^U I_O(x) < \infty$$

in view of [20, Corollary 2.3]. Hence we have the assertion.

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(ii) Owing to the irreducibility of $(\mathcal{E},\mathcal{F})$, $\mathbf{M}$ is either transient or recurrent in view of the ergodic decomposition of $\mathbf{M}$ (see [28]). If $X\setminus O$ is non $\mathcal{E}$-polar, then the irreducibility of $(\mathcal{E},\mathcal{F})$ implies $P_x(\sigma_{X\setminus O}<\infty)>0$ m-a.e. $x\in X$. Then the recurrence of $\mathbf{M}$ implies the assertion.

(iii) Since the dual form $(\widehat{\mathcal{E}},\mathcal{F})$ has Markov property, $\widehat{X}_d$ the corresponding dissipative part m-a.e. coincides with X_d (see [28]). Here $\widehat{X}_d:=\{x\in X\mid \widehat{G}f(x)<\infty\}$ for some $f\in L^1_+(X;m)$ and $\widehat{X}_d$ is m-a.e. invariant under the choice of such f. Under the transience of $\mathbf{M}$, we have $\widehat{X}_d=X_d=X$ m-a.e. by [28, Theorem 1.3]. So $I_O\in L^1(X;m)$ yields $RI_O<\infty$ m-a.e., which implies the assertion.

(iv) We set $u(x):=P_x(\tau_O=\infty)$ for $x\in X$. Then u satisfies $u=p_t^O u$ m-a.e. on X. By $m(O)<\infty$, $u=p_t^O u\in L^2(O;m)$, hence $u\in\mathcal{F}_O$. Owing to Lemma 2.2, u is a non-negative $\mathcal{E}_O$-subharmonic function on O. In particular, $\mathcal{E}(u,u)=0$, which implies $u=0$ m-a.e. on O in view of the transience of $(\mathcal{E}_O,\mathcal{F}_O)$. Suppose that the Nash-type estimate holds. Then the following Nash inequality holds (see [5] or [38, Theorem 4.3]): there exists $C>0$ such that

$$\|f\|_2^{2(1+2/\nu)}\le C\mathcal{E}(f,f)\|f\|_1^{4/\nu},\quad \forall f\in\mathcal{F}.$$

From this and $\|u\|_1\le m(O)<\infty$, we have $\|u\|_2=0$, hence u vanishes m-a.e. on X. ∎

Proof of Theorem 1.1 First we prove (i). By Theorem 4.3, the part process $\mathbf{M}_G=(\Omega,X_t^G,P_x)$ on G is associated with $(\mathcal{E}_G,\mathcal{F}_G)$. Take a $\{G_n\}\in\Xi_G(u)$ and $u_n\in\mathcal{F}_G$ such that $u=u_n$ m-a.e. on G_n and $\mathcal{E}(u,\phi)\le 0$ for any $\phi\in\bigcup_{n=1}^{\infty}\mathcal{F}_{G_n}^+$. We may assume that G_n is a finely open Borel set, $m(G_n)<\infty$ and u is bounded on G_n. In particular, we may assume that $u_n\in(\mathcal{F}_G)_b$ is an $\mathcal{E}_{G_n}$-subharmonic function. Take $\ell>\sup_{\partial G}u^+$. Then $u_\ell:=(u-\ell)^+$ is also $\mathcal{E}_G$-subharmonic by Theorem 6.4, because $\ell>0$ and the $\mathcal{E}_G$-superharmonicity of 1. Similarly $(u_n-\ell)^+$ is also $\mathcal{E}_{G_n}$-subharmonic. By assumption, $P_x(\tau_{G_n}<\infty)=1$ m-a.e. $x\in X$ for each $n\in\mathbb{N}$. Then by Lemma 6.4, we have $(u_n-\ell)^+\le H_{G_n^c}(\widetilde{u}_n-\ell)^+$ m-a.e. on G_n. Thus $u_\ell\le H_{G_n^c}u_\ell$ m-a.e. on G_m for all $n>m$ by noting $u=\widetilde{u}_n$ q.e. $x\in\overline{G}_n^{\mathcal{E}}$. We extend $u_\ell\in C(\overline{G})$ to $\widehat{u}_\ell\in C(X_\Delta)$ with $\widehat{u}_\ell(\Delta)=0$ and $\widehat{u}_\ell=u_\ell$ on $\overline{G}$. Note that $P_x(\lim_{n\to\infty}\tau_{G_n}=\tau_G)=1$ m-a.e. $x\in X$ by Theorem 4.6 and $H_{G_n^c}u_\ell(x)=E_x[\widehat{u}_\ell(X_{\tau_{G_n}})]$. Owing to the quasi-left-continuity up to ∞, we have $u_\ell\le H_{G^c}u_\ell$ m-a.e. on G_m, which implies $(u-\ell)^+\le H_{G^c}(u-\ell)^+$ m-a.e. on G. Then $(u(x)-\ell)^+\le E_x[(u(X_{\sigma_{G^c}})-\ell)^+ : \sigma_{G^c}<\zeta]=0$ m-a.e. $x\in G$. Hence $\sup_G u\le\ell$. Letting $\ell\to\sup_{\partial G}u^+$, we arrive at the result.

Next we prove (ii). Take an $\ell>\sup_{\partial G}u$. Since 1 is $\mathcal{E}_G$-harmonic, $(u-\ell)^+$ is also $\mathcal{E}_G$-subharmonic by Theorem 6.4. In the same way as in the proof of (i), we can conclude $(u-\ell)^+\le H_{G^c}(u-\ell)^+$ m-a.e. on G. Therefore, we have the conclusion. This completes the proof. ∎

Proof of Corollary 1.1 The proof is similar to that of Theorem 1.1. We first show (i). Take $\ell>\mathcal{E}\text{-}\sup_{\mathcal{E}\text{-}\partial G}\widetilde{u}^+$ and $\{G_n\}\in\Xi_G(u)$ as in the proof of Theorem 1.1 and set $u_\ell := (u-\ell)^+$. Then we have $u_\ell \le H_{G_n^c}\widetilde{u}_\ell$ m-a.e. on G_m for all $n > m$ in the same way as in the proof of Theorem 1.1. Recall that there exists a $s.\mathcal{E}$-nest $\{F_k\}$ such that $\widetilde{u}|_{F_k\cup\{\Delta\}}$ is continuous on $F_k \cup \{\Delta\}$. Since

$$\begin{aligned} H_{G_n^c}\widetilde{u}_\ell(x) &= E_x[\widetilde{u}_\ell(X_{\tau_{G_n}}) : \tau_G < \sigma_{X\setminus F_k}] + E_x[\widetilde{u}_\ell(X_{\tau_{G_n}}) : \tau_G \ge \sigma_{X\setminus F_k}] \\ &\le E_x[\widetilde{u}_\ell(X_{\tau_{G_n}}) : \tau_G < \sigma_{X\setminus F_k}] + (\mathcal{E}\text{-}\sup_G \widetilde{u})^+ P_x(\tau_G \ge \sigma_{X\setminus F_k}), \end{aligned}$$

letting $n \to \infty$, we have

$$u_\ell(x) \le E_x[\widetilde{u}_\ell(X_{\tau_G}) : \tau_G < \sigma_{X\setminus F_k}] + (\mathcal{E}\text{-}\sup_G \widetilde{u})^+ P_x(\tau_G \ge \sigma_{X\setminus F_k}), \quad m\text{-a.e. } x \in G$$

by the quasi-left-continuity up to ∞ and the continuity of $\widetilde{u}$ on $F_k \cup \{\Delta\}$. Letting $k \to \infty$ with [2, Lemma 3.4], we get $u_\ell(x) \le H_{G^c}\widetilde{u}_\ell(x) = 0$ m-a.e. $x \in G$, hence $\widetilde{u}(x) \le \mathcal{E}\text{-}\sup_{\mathcal{E}\text{-}\partial G} \widetilde{u}^+$ $\mathcal{E}$-q.e. $x \in G$. Consequently, we have $\mathcal{E}\text{-}\sup_{\overline{G}^{\mathcal{E}}} \widetilde{u} \le \mathcal{E}\text{-}\sup_{\mathcal{E}\text{-}\partial G} \widetilde{u}^+$. The proof of (ii) is similar to that of Theorem 1.1(ii). This completes the proof. ∎

Proof of Theorem 1.2 (i) is an easy consequence of Theorem 1.1(i) and Corollary 1.1(i). We shall show (ii). Note that $|u_n - u_m|$ is $\mathcal{E}_G$-subharmonic by Theorem 6.4(iii). Hence $\sup_{\overline{G}} |u_n - u_m| \le \sup_{\partial G} |u_n - u_m| \to 0$ as $n, m \to \infty$. So there exists a $u \in C_b(\overline{G})$ such that $\sup_{\overline{G}} |u_n - u| \to 0$ as $n \to \infty$. Owing to Theorem 6.3, we have $u \in (\dot{\mathcal{F}}_G)_{\mathrm{loc}}$ and its $\mathcal{E}_G$-harmonicity. This completes the proof. ∎

Proof of Theorem 1.3 The proof of (ii) is similar to the proof of (i). We only prove (i). Since $u^+(x_0) \ge 0$ and 1 is $\mathcal{E}$-superharmonic, $u^+(x_0) - u \in \dot{\mathcal{F}}_{\mathrm{loc}} \cap C_f(X)$ is a non-negative $\mathcal{E}$-superharmonic function, hence so is $v := u^+(x_0) - u^+ = (u^+(x_0) - u) \wedge u^+(x_0)$. We set $Y := \{x \in X \mid v(x) > 0\}$. Since $v \in C_f(X)$, v is also excessive with respect to $\mathbf{M}^\gamma$, so is I_Y (see [27]). In particular, I_Y is finely continuous with respect to $\mathbf{M}^\gamma$. By Theorem 5.3, we get $Y = \varnothing$ or $Y^c = \varnothing$. Since $x_0 \in Y^c$, we have $Y = \varnothing$. This completes the proof. ∎

Proof of Theorem 1.4 The proof of (ii) is similar to the proof of (i). We only prove (i). Since $u^+(x_0) \ge 0$ and 1 is $\mathcal{E}$-superharmonic, $u^+(x_0) - u \in \dot{\mathcal{F}}_{\mathrm{loc}}$ is a non-negative $\mathcal{E}$-superharmonic finely lower-semi-continuous function, hence so is

$$v := u^+(x_0) - u^+ = (u^+(x_0) - u) \wedge u^+(x_0).$$

Hence $p_t v \le v$ m-a.e. by Theorem 6.1. By absolute continuity of $p_t^\gamma(x, dy)$, $\alpha R_\alpha v(x)$ is increasing as $\alpha \to \infty$ for any $x \in X$. We put $\widehat{v}(x) :=\uparrow \lim_{\alpha\to\infty} \alpha R_\alpha v(x)$. Then $\widehat{v}$ is excessive with respect to $\mathbf{M}$. We see $\widehat{v} \le v$ m-a.e. and by using the fine lower-semi-continuity of v, $0 \le v \le \widehat{v}$ on X. On the other hand, we know $v \equiv 0$ on G. Hence $\widehat{v} = 0$ m-a.e. on G, consequently $\widehat{v} = 0$ on G. As in the argument of the proof of Theorem 1.3, we have $\widehat{v} \equiv 0$ on X or $\widehat{v} > 0$ on X. Theorefore $\widehat{v} \equiv 0$, which implies $v \equiv 0$. This completes the proof. ∎

8 Examples

Example 8.1 **([7, 42])** Let G be a nonempty open set of $\mathbb{R}^d (d \geq 1)$ with the d-dimensional Lebesgue measure $m(dx) := dx$. We set $L^p(G \to \mathbb{R}^d) := L^p(G \to \mathbb{R}^d; m)$ and $L^p(G) := L^p(G \to \mathbb{R})$ for $p > 0$. Let

$$H^1(G) := \{u \in L^2(G) \mid \nabla u \in L^2(G \to \mathbb{R}^d)\}$$

be the usual 1-order Sobolev space and $H^1_0(G)$ the completion of $C_0^\infty(G)$ with respect to the norm $\|\cdot\|_{H^1(G)}$ defined by $\|u\|^2_{H^1(G)} := \|u\|^2_{L^2(G)} + \|\nabla u\|^2_{L^2(G)}$. Let $G \ni x \mapsto a(x) := (a_{i,j}(x))_{i,j=1}^d$ be a symmetric $\mathbb{R}^d \otimes \mathbb{R}^d$-valued measurable function and $G \ni x \mapsto b(x), \widehat{b}(x)$ $\mathbb{R}^d$-valued measurable functions with expressions $b(x) = (b^1(x), b^2(x), \dots, b^d(x)), \widehat{b}(x) = (\widehat{b}^1(x), \widehat{b}^2(x), \dots, \widehat{b}^d(x))$, and $G \ni x \mapsto c(x)$ a measurable function on G. We assume a is uniformly elliptic on G: there exist constants $\Lambda_G \geq \lambda_G > 0$ such that

$$\lambda_G |\xi|^2 \leq \langle a(x)\xi, \xi\rangle_{\mathbb{R}^d} \leq \Lambda_G |\xi|^2 \text{ for all } \xi \in \mathbb{R}^d, x \in G.$$

Here $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ stands for the Euclidean inner product with $|\xi| := \langle \xi, \xi\rangle^{1/2}_{\mathbb{R}^d}$. We further assume $\nu := c - \operatorname{div} \widehat{b} \geq 0$ on G in Schwartz distributional sense, hence ν is a Radon measure on G. Then $a, b, \widehat{b}, c, \nu$ can be extended on $\mathbb{R}^d$ by putting

$$a := \{(\Lambda_G + \lambda_G)/2\}(\delta_{i,j})_{i,j=1}^d,$$

$b, \widehat{b}, c := 0$ on $\mathbb{R}^d \setminus G$ and $\nu(A) := \nu(A \cap G)$ for $A \in \mathcal{B}(\mathbb{R}^d)$. For $u, v \in C_0^\infty(\mathbb{R}^d)$, we define

$$\mathcal{E}^a(u, v) := \frac{1}{2}\int_{\mathbb{R}^d} \langle a(x)\nabla u(x), \nabla v(x)\rangle_{\mathbb{R}^d}\, dx.$$

Then $(\mathcal{E}^a, C_0^\infty(G))$ is closable on $L^2(G)$ and its domain of the closure is $H^1_0(G)$. We denote by $(\mathcal{E}^a, H^1_0(G))$ the closure of $(\mathcal{E}^a, C_0^\infty(G))$ on $L^2(G)$. Let $\mathbf{M}^a = (\Omega, X_t, P^a_x)$ be the diffusion process on $\mathbb{R}^d$ associated with $(\mathcal{E}^a, H^1_0(G))$.

Definition 8.1 **(Hardy class function)** A measurable function f on G is said to be *of Hardy class* with respect to $(\mathcal{E}^a, H^1_0(G))$ (write $f \in S_H(G)$) if there exist constants $\delta(|f|) \in]0, \infty[$ and $\gamma(|f|) \in [0, \infty[$ such that

$$\int_G u^2 |f| dm \leq \delta(|f|)\mathcal{E}^a(u, u) + \gamma(|f|)\|u\|^2_{L^2(G)} \quad \text{for } u \in H^1_0(G).$$

If $G = \mathbb{R}^d$, we write S_H instead of $S_H(\mathbb{R}^d)$.

Clearly $S_H \subset S_H(G) \subset L^1_{\mathrm{loc}}(G)$ for any open set G. By [14, Example 5.1] when $d \geq 3$, $L^{d/2}(G) \subset S_H(G)$ and for $f \in L^{d/2}(G)$, $\delta(|f|)$ can be taken to be arbitrarily small.

We prepare the following.

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Assumption 8.1 $|b|^2, |\widehat{b}|^2, |c| \in S_H$ and

$$\delta_0 := \sqrt{2\delta(|b|^2)/\lambda_G} + \sqrt{2\delta(|\widehat{b}|^2)/\lambda_G} + \delta(|c|) < 1.$$

For $u, v \in C_0^\infty(\mathbb{R}^d)$, we set

$$\mathcal{E}(u,v) := \mathcal{E}^a(u,v) + \int_{\mathbb{R}^d} \langle b(x), \nabla u(x)\rangle_{\mathbb{R}^d} v(x)\,dx + \int_{\mathbb{R}^d} \langle \widehat{b}(x), \nabla v(x)\rangle_{\mathbb{R}^d} u(x)\,dx + \int_{\mathbb{R}^d} u(x)v(x)c(x)\,dx.$$

Then under Assumption 8.1, in the same way as [14], there exist positive constants $M_1, M_2, \gamma > \gamma_0 := \gamma(|b|^2)\sqrt{2/\lambda_G\delta(|b|^2)} + \gamma(|\widehat{b}|^2)\sqrt{2/\lambda_G\delta(|\widehat{b}|^2)} + \gamma(|c|) \geq 0$ such that for $u, v \in C_0^\infty(\mathbb{R}^d)$

$$|\mathcal{E}(u,v)| \leq M_1 \|u\|_{H^1(\mathbb{R}^d)} \|v\|_{H^1(\mathbb{R}^d)}$$
$$\mathcal{E}_\gamma(u,u) \geq (1-\delta_0)\mathcal{E}^a(u,u) + (\gamma - \gamma_0)\|u\|^2_{L^2(\mathbb{R}^d)} \geq M_2 \|u\|_{H^1(\mathbb{R}^d)}.$$

Hence $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d)$, with closure denoted by $(\mathcal{E}, H^1(\mathbb{R}^d))$. For $u, v \in H^1(\mathbb{R}^d)$, we set

$$\mathcal{E}^{b_0}(u,v) := \mathcal{E}^a(u,v) - \int_{\mathbb{R}^d} \langle b_0(x), \nabla u(x)\rangle v(x)\,dx,$$

where $b_0 := \widehat{b} - b$. For $u, v \in C_0^\infty(G)$, we see

$$\mathcal{E}(u,v) = \mathcal{E}^{b_0}(u,v) + \int_G u(x)v(x)\nu(dx).$$

and $(\mathcal{E}, H_0^1(G))$ (resp. $(\mathcal{E}^{b_0}, H_0^1(G))$) is a regular local semi-Dirichlet form with a lower bound $-\gamma_0$ on $L^2(G)$ (resp. $-\gamma_0 + \gamma(|c|)$) and $(\mathcal{E}^{b_0}, H_0^1(G))$ is the part space of $(\mathcal{E}^{b_0}, H^1(\mathbb{R}^d))$ on G. Let $\mathbf{M}^G = (\Omega, X_t, \zeta_G, P_x^G)$ be the Hunt diffusion process on G which is $\mathcal{E}$-properly associated with $(\mathcal{E}, H_0^1(G))$ (see [6, Théorème IV. 1.5]).

Remark 8.1 If $b \in L^p(G \to \mathbb{R}^d)$, $\widehat{b} \in L^q(G \to \mathbb{R}^d)$ and $c \in L^r(G)$ with $p, q, 2r \geq d \geq 3$, then our assumptions except $m(G) < \infty$, $\nu := c - \operatorname{div}\widehat{b} \geq 0$ on G are satisfied by using the Sobolev inequality (see [14, 41, 42]).

Subexample 8.1 Let $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d \mid x_d > 0\}$ be the upper half plane in $\mathbb{R}^d$ ($d \geq 2$). Then the following Hardy inequality holds (see [35, §2.1.6], [10, Theorem 6]):

$$\int_{\mathbb{R}^d_+} \frac{u(x)^2}{|x_d|^2}\,dx \leq 4\int_{\mathbb{R}^d_+} |\nabla u(x)|^2\,dx \quad \text{for all } u \in H_0^1(\mathbb{R}^d_+).$$

Fix $\varepsilon, C > 0$, $\lambda_{\mathbb{R}^d_+} = \Lambda_{\mathbb{R}^d_+} = 1$ and let

$$b(x) := (\sqrt{2}(4+\varepsilon)x_d)^{-1}\mathbf{e}_1, \quad \widehat{b}(x) := (\sqrt{2}(4+\varepsilon)x_d)^{-1}\mathbf{e}_2, \quad c(x) := Cx_d^{-2},$$

where $\mathbf{e}_1 := (1,0,\dots,0)$, $\mathbf{e}_2 := (0,1,0,\dots,0)$. Then our assumptions are satisfied with $\delta(|b_0|^2) = 8/(4+\varepsilon)^2 < 1/2$ and $\nu(dx) = c(x)dx$. Note $|b|, |\widehat{b}|, c \notin L^p(\mathbb{R}^d_+)$ for any $p > 0$.

Definition 8.2 **(Dynkin and Kato class functions)** A measurable function f on G is said to be of the *Dynkin* (resp. *Kato*) *class* on G with respect to $(\mathcal{E}^a, H^1_0(G))$ (write $f \in S_D(G)$ (resp. $f \in S_K(G)$) if $c_\alpha := \|R^{a,G}_\alpha f\|_{L^\infty(G)} < \infty$ for some/all $\alpha > 0$ (resp. $\lim_{\alpha\to\infty} c_\alpha = 0$). Here $R^{a,G}_\alpha f(x) := \int_0^\infty e^{-\alpha t} p^{a,G}_t f(x)\,dt$ is the α-resolvent of f with respect to $\mathbf{M}^a_G$. A measurable function f on $\mathbb{R}^d$ is said to be of the *local Dynkin* (resp. *local Kato*) *class* on G with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ (write $f \in S^{\mathrm{loc}}_D(G)$ (resp. $f \in S^{\mathrm{loc}}_K(G)$) if $I_K f \in S_D(\mathbb{R}^d)$ (resp. $I_K f \in S_K(\mathbb{R}^d)$) for any compact set K in G. When $G = \mathbb{R}^d$, we write S_D (resp. S_K) instead of $S_D(G)$ (resp. $S_K(G)$) and also write S^{loc}_D (resp. S^{loc}_K) instead of $S^{\mathrm{loc}}_D(G)$ (resp. $S^{\mathrm{loc}}_K(G)$).

Our stochastic definition of $S_K(G)$ is rather milder than that with the same notation treated in [25]. However, $S^{\mathrm{loc}}_K(G)$ is consistent with [25]. It is known that $S_K(G) \subset S_D(G) \subset S_H(G) \subset L^1_{\mathrm{loc}}(G)$ (see [14]), hence $S^{\mathrm{loc}}_K(G) \subset S^{\mathrm{loc}}_D(G) \subset L^1_{\mathrm{loc}}(G)$. Moreover, $L^r(G) \subset S_K(G)$, $L^r_{\mathrm{loc}}(G) \subset S^{\mathrm{loc}}_K(G)$ for $2r > d \geq 2$, or $r \geq d = 1$ (see [1, Theorem 1.4(iii)]). In view of [1, Theorem 4.5] or [9, Theorem 3.6], for $d \geq 3$, $f \in S_D$ (resp. $f \in S_K$) if and only if

$$\sup_{x\in\mathbb{R}^d} \int_{|x-y|<\varepsilon} \frac{|f(y)|}{|x-y|^{d-2}}\,dy < \infty \quad \text{for some/all } \varepsilon > 0 \text{ (resp. } \to 0 \text{ as } \varepsilon \to 0).$$

Therefore, for $f \in L^1(\mathbb{R}^d)$ (resp. $f \in L^1_{\mathrm{loc}}(G)$) with $d \geq 3$, $f \in S_D$ (resp. $f \in S^{\mathrm{loc}}_D(G)$) if and only if $R|f| \in L^\infty(\mathbb{R}^d)$ (resp. $R(I_K|f|) \in L^\infty(\mathbb{R}^d)$ for any compact $K \subset G$). Here

$$Rf(x) := \int_{\mathbb{R}^d}\int_0^\infty \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x-y|^2}{2t}} f(y)\,dt dy = A_d \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}}\,dy$$

with $A_d := \Gamma(\frac{d}{2}-1)/2\pi^{d/2}$.

Subexample 8.2 Suppose $d \geq 3$. Fix an $\varepsilon \in [0,1]$ and set $\varphi_\varepsilon(t) := t^2(-\log t)^{1+\varepsilon}$. Then φ_ε is increasing on $]0, r_\varepsilon]$ and decreasing on $[r_\varepsilon, 1[$ for $r_\varepsilon := e^{-(1+\varepsilon)/2}$, and $\varphi_\varepsilon(0+) = 0$. We take a radially symmetric non-negative function $f_\varepsilon \in C_0(\mathbb{R}^d \to [0,\infty]))$ as follows: $f_\varepsilon(x) := 1/\varphi_\varepsilon(|x|)$ if $|x| < r^2_\varepsilon$ and $f_\varepsilon(x) := 0$ if $|x| \geq r_\varepsilon$. Then $f_\varepsilon \in L^{d/2}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Moreover $f_\varepsilon \in S_K$ by [1, Proposition 4.10] if $\varepsilon > 0$, but $f_0 \notin S^{\mathrm{loc}}_D$ if $\varepsilon = 0$. Indeed, if $\varepsilon = 0$,

$$RI_{B_{r_0^2}(0)} f_0(0) = A_d \int_{B_{r_0^2}(0)} \frac{dy}{\varphi_0(|y|)|y|^{d-2}} = A_d \int_{B_{r_0^2}(0)} \frac{dy}{|y|^d(-\log|y|)} = \infty.$$

Lemma 8.1 *Suppose $|\widehat{b}|^2$, $|c| \in S_H(G)$. Then there exists $C_1 > 0$ such that for any $u \in C_0^\infty(G)$ $\int_G u(x)^2\nu(dx) \leq C_1\mathcal{E}_1^a(u,u)$. In particular, $H_0^1(G)$ is continuously embedded in $L^2(G;\nu)$ and for $u \in H_0^1(G)$ $\int_G \widetilde{u}(x)^2\nu(dx) \leq C_1\mathcal{E}_1^a(u,u)$.*

Proof Take $u \in C_0^\infty(G)$. Then

$$\int_G |c|u^2\,dm \leq \delta(|c|)\mathcal{E}^a(u,u) + \gamma(|c|)\|u\|^2_{L^2(G)}$$

and

$$\begin{aligned}
\int_G |\langle \widehat{b}, \nabla u^2\rangle_{\mathbb{R}^d}|\,dm &= 2\int_G |\langle \widehat{b}, \nabla u\rangle_{\mathbb{R}^d}||u|\,dm \\
&\leq 2\|\nabla u\|_{L^2(G)}\||\widehat{b}|u\|_{L^2(G)} \\
&\leq 2\|\nabla u\|_{L^2(\mathbb{R}^d)}\Big(\delta(|\widehat{b}|^2)\mathcal{E}^a(u,u) + \gamma(|\widehat{b}|^2)\|u\|^2_{L^2(G)}\Big)^{1/2} \\
&\leq 2\sqrt{2\delta(|\widehat{b}|^2)/\lambda_G}\Big(\mathcal{E}^a(u,u) + \frac{\gamma(|\widehat{b}|^2)}{\delta(|\widehat{b}|^2)}\|u\|^2_{L^2(G)}\Big).
\end{aligned}$$

This completes the proof. ∎

Corollary 8.1 *Suppose $|\widehat{b}|^2$, $|c| \in S_H(G)$. Then $\nu \in S(G)$, that is, ν is a smooth measure on G with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$.*

Proof For any relatively compact open set O with $\overline{O} \subset G$, it suffices to prove $I_O\nu \in S_0(G)$. Take $u \in C_0^\infty(G)$. Then $\int_O |u|\,d\nu \leq \nu(O)^{1/2}\sqrt{C_1\mathcal{E}_1^a(u,u)}$. This completes the proof. ∎

We have the following weak maximum principle.

Theorem 8.1 *Suppose Assumption 8.1 and $m(G) < \infty$. Let $u \in H_0^1(G)_{\mathrm{loc}} \cap C(\overline{G})$ be an upper bounded $\mathcal{E}_G$-subharmonic function on G. Then $\sup_{\overline{G}} u \leq \sup_{\partial G} u^+$. In particular, $\sup_{\overline{G}} u = \sup_{\partial G} u$ if $u \geq 0$ on ∂G. If further $\nu = 0$ on G, then $\sup_{\overline{G}} u = \sup_{\partial G} u$.*

As corollaries we have the following.

Corollary 8.2 *Suppose that $|b|^2, |\widehat{b}|^2, c \in S_K$ and $m(G) < \infty$. Let $u \in H_0^1(G)_{\mathrm{loc}} \cap C(\overline{G})$ be an upper bounded $\mathcal{E}_G$-subharmonic function on G. Then $\sup_{\overline{G}} u \leq \sup_{\partial G} u^+$. In particular, $\sup_{\overline{G}} u = \sup_{\partial G} u$ if $u \geq 0$ on ∂G. If further $\nu = 0$ on G, then $\sup_{\overline{G}} u = \sup_{\partial G} u$.*

Corollary 8.3 *Suppose $|b| \in L^p(G)$, $|\widehat{b}| \in L^q(G)$, $c \in L^{r/2}(G)$ for $p,q,r > d \geq 2$, or $p,q,r \geq 2d = 2$ and $m(G) < \infty$. Let $u \in H_0^1(G)_{\mathrm{loc}} \cap C(\overline{G})$ be an upper bounded $\mathcal{E}_G$-subharmonic function on G. Then $\sup_{\overline{G}} u \leq \sup_{\partial G} u^+$. In particular, $\sup_{\overline{G}} u = \sup_{\partial G} u$ if $u \geq 0$ on ∂G. Furthermore, if $\nu = 0$ on G, then $\sup_{\overline{G}} u = \sup_{\partial G} u$.*

Recall that $\mathbf{M}^G = (\Omega, X_t, \zeta_G, P_x^G)$ is associated with $(\mathcal{E}, H_0^1(G))$. To prove Theorem 8.1, we need the following lemma.

Lemma 8.2 *Suppose Assumption* 8.1. *Let $u \in H_0^1(G)$ be a non-negative $\mathcal{E}$-subharmonic function on G. Then u vanishes on G. In particular, if $m(G) < \infty$, then $P_x^G(\zeta_G < \infty) = 1$ m-a.e. $x \in G$. More generally, if an open subset O of G satisfies $m(O) < \infty$, then $P_x^G(\tau_O < \infty) = 1$ m-a.e. $x \in G$.*

Proof We set $M := \|u\|_{L^\infty(G)}$. It suffices to show $M = 0$. Take $\varepsilon > 0$. Set $\varphi := u/(M+\varepsilon+u)$, $w := \log(\frac{M+\varepsilon}{M+\varepsilon-u})$ and $u_\ell := (u-\ell)^+$ for $\ell > 0$. Then we can confirm $\varphi, w, u_\ell \in H_0^1(G)^+$ in view of [33, Ch. I, Proposition 4.11]. Since u is $\mathcal{E}$-subharmonic on G,

$$0 \geq \mathcal{E}(u, u_\ell) = \mathcal{E}(u_\ell, u_\ell) + \ell\Big(\int_G \langle \widehat{b}, \nabla u_\ell\rangle\, dm + \int_G c u_\ell\, dm\Big).$$

The last term is non-negative by $c - \operatorname{div}\widehat{b} \geq 0$ on G. Then we have $\mathcal{E}(u_\ell, u_\ell) \leq 0$, which implies $\frac{\lambda_G}{2}(1-\delta_0)\|\nabla u_\ell\|_2^2 \leq \gamma_0\|u_\ell\|_2^2$. From this and Nash inequality

$$\|f\|_2^{2(1+2/d)} \leq C_d^2\|\nabla f\|_2^2\|f\|_1^{4/d}, \quad \forall f \in H_0^1(G),$$

we have

$$\begin{aligned} \|u_\ell\|_2 &\leq C_d^{d/2}\Big(\frac{2\gamma_0}{\lambda_G(1-\delta_0)}\Big)^{d/4}\|u_\ell\|_1 \\ &\leq C_d^{d/2}\Big(\frac{2\gamma_0}{\lambda_G(1-\delta_0)}\Big)^{d/4} A(\ell)^{1/2}\|u_\ell\|_2. \end{aligned} \tag{8.1}$$

Then $\|u_\ell\|_2 > 0$ implies $A(\ell) \geq (\frac{\lambda_G(1-\delta_0)}{2C_d^2\gamma_0})^{d/2}$, where $A(\ell) := m(u > \ell)$ and $C_d := (2+d)^{1+2/d}(m(B_1(0))/4\pi)^{2/d}$. Meanwhile,

$$A(\ell)\Big(\log\frac{M+\varepsilon}{M+\varepsilon-\ell}\Big)^2 \leq \|w\|_2^2.$$

Let $\ell(\eta) := (1-\eta)(M+\varepsilon)$ for $\eta \in\,]0,1[$. Then $A(\ell(\eta)) \leq \|w\|_2^2(\log\frac{1}{\eta})^{-2}$. Combining this and the above argument, for sufficiently small $\eta > 0$, we have a contradiction. Hence for small $\eta > 0$, $\|u_{\ell(\eta)}\|_2 = 0$, that is, $M \leq (1-\eta)(M+\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we have $M = 0$.

Next we show the last assertion. We set $u(x) := P_x^G(\zeta_G = \infty)$ for $x \in G$. It is easy to show $u(x) = p_t u(x)$, where $p_t f(x) := E_x^G[f(X_t)]$. Since $m(G) < \infty$, $u \in L^2(G)$, consequently $u = p_t u \in H_0^1(G)$. Then u is $\mathcal{E}$-subharmonic on G by Lemma 2.2. The proof of the rest is quite similar, and we omit it. ∎

Proof of Theorem 8.1 By Lemma 8.2, Assumption 1.1(ii) is satisfied. So we can apply Theorem 1.1. Though the framework of Theorem 1.1 is slightly different from the present one ($\mathbf{M}^G$ is not given as a part process on G), the method of the proof of Theorem 1.1 remains valid. The quasi-left-continuity of $\mathbf{M}^G$ up to ∞ plays the same role as in the proof of Theorem 1.1. ∎

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From now on, we give a complete extension of Stampacchia's weak maximum principle under Assumption 8.1. It is well known that $C^\infty(G) \cap H^1(G)$ is dense in $H^1(G)$ with respect to $\|\cdot\|_{H^1(G)}$-norm (see [11, Ch. V, Theorem 3.2]). By $H^1(G) \subset H^1_0(G)_{\rm loc}$, each element of $H^1(G)$ has an $\mathcal{E}$-quasi-continuous m-version on G. More strongly, each $u \in H^1(G)$ admits an $\mathcal{E}$-quasi-continuous function $\widetilde{u}$ on $\overline{G}$ which is an m-version of u on G by the denseness of $C^\infty(G) \cap H^1(G)$ in $H^1(G)$. We then introduce a disorder $\preceq$ for $u, v \in H^1(G)$ and the maximum in the sense of $H^1(G)$:

Definition 8.3 **([11, Ch. VI, Definition 3.1])** Let A be a subset of $\overline{G}$ and $u \in H^1(G)$. Fix a real number $k \in \mathbb{R}$. We say that u is *greater than* (resp. *less than*) k on A in the sense of $H^1(G)$, and denote $u \succeq k$ (resp. $u \preceq k$) on A if there exists a sequence $\{u_j\}$ in $C^\infty(G) \cap H^1(G)$ such that $u_j \to u$ in $H^1(G)$ and for each $j \in \mathbb{N}$ there exists an open neighborhood U_j of A in $\mathbb{R}^d$ in the following form such that $u_j > k$ (resp. $u_j < k$) on $U_j \cap G$:

$$U_j = U(\varepsilon, \delta) := \{x \in \mathbb{R}^d \mid d(x, \overline{A}) < \varepsilon, |x| > \delta\}.$$

For $u, v \in H^1(G)$, $u \preceq v$ if and only if $v - u \succeq 0$. For a subset A of $\overline{G}$, we set

$$\max_A f := \inf\{k \in \mathbb{R} \mid f \preceq k \text{ on } A\}.$$

We define $\min_A f$ similarly. For $A \subset G$ and $u, v \in H^1(G)$, $u \preceq v$ on A if and only if $u \le v$ m-a.e. on A, then $\max_A$ coincides with the m-essentially supremum on A (see [11, Theorem 3.2]).

Remark 8.2 Our disorder $\preceq$ is slightly different from what is defined in [42], because the 1-order Sobolev space in [42] is defined as the completion of $C^1(\overline{G})$ with respect to $\|\cdot\|_{H^1(G)}$-norm. However, the strategy of the proof below remains valid even if we adopt this definition.

Lemma 8.3 *Fix $\ell \in \mathbb{R}$ and take $u \in H^1(G)$. Suppose $u \preceq \ell$ on ∂G. Then $(u-\ell)^+ \in H^1_0(G)$.*

Proof Since $(u-\ell)^+$ is a normal contraction of $u \in H^1(G)$, we have $(u-\ell)^+ \in H^1(G)$ and

$$\|(u-\ell)^+\|_{H^1(G)} \le \|u\|_{H^1(G)}. \tag{8.2}$$

Suppose that $u \preceq \ell$ on ∂G. Let $u_j \in C^\infty(G) \cap H^1(G)$ and U_j be the functions and neighborhood of ∂G such that $u_j < \ell$ on $U_j \cap G$. Then we see that $(u_j - \ell)^+ \in H^1(G)$ such that $(u_j - \ell)^+$ vanishes on a neighborhood of ∂G in G. By (8.2), $\{(u_j - \ell)^+\}$ is a bounded sequence in $H^1_0(G)$ and converges to $(u-\ell)^+$ in $L^2(G)$ as $j \to \infty$. Hence we have $(u-\ell)^+ \in H^1_0(G)$ in view of the Banach–Saks theorem. ■

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Then we have the following.

Theorem 8.2 *Suppose Assumption* 8.1. *Let $u \in H^1(G)$ be an $\mathcal{E}$-subharmonic function on G. Then* $\max_G u \le \max_{\partial G} u^+$. *In particular,* $\max_G u = \max_{\partial G} u$ *if* $u \succeq 0$ *on* ∂G. *If further* $\nu = 0$ *on G, then* $\max_G u = \max_{\partial G} u$.

Remark 8.3 If $\mathcal{E}$ satisfies the coercivity, that is, there exists $c > 0$ such that $\mathcal{E}(f, f) \ge c\|f\|^2_{H^1(G)}$ for all $f \in H^1(G)$, then the same result without assuming $m(G) < \infty$ is proved by [11, Theorem 4.4]. But the argument of the reduction to unbounded G in their proof seems to be unnecessary. Actually, in view of Lemma 8.3, their proof remains valid without assuming $m(G) < \infty$.

Proof of Theorem 8.2 Note $H^1(G) \subset H^1_0(G)_{\mathrm{loc}}$ and every $u \in H^1(G)$ has an $\mathcal{E}$-quasi-continuous version. We first show that any $\mathcal{E}$-subharmonic function $u \in H^1(G)$ is m-essentially upper bounded under the condition $\max_{\partial G} u^+ < \infty$. For any $\ell > \max_{\partial G} u^+$, we see $u_\ell := (u - \ell)^+ \in H^1_0(G)$ by Lemma 8.3. We have that the same calculation as in the proof of Lemma 8.2 holds for $u \in H^1(G)$. Then we can conclude $A(\ell) = 0$ for sufficiently large ℓ by the argument after (8.1). The m-essentially upper boundedness of u under $\max_{\partial G} u^+ < \infty$ is now proved. Take $\ell > \max_{\partial G} u^+$ again. We can conclude the $\mathcal{E}$-subharmonicity of $(u - \ell)^+ \in H^1_0(G)$ as in the proof of Theorem 1.1. Then $(u - \ell)^+ = 0$ by Lemma 8.2, which implies $u \le \ell$ m-a.e. on G. Thus we obtain the desired result. Next suppose the $\mathcal{E}$-harmonicity of 1 on G. Taking $\ell > \max_{\partial G} u$, we can conclude again the $\mathcal{E}$-subharmonicity of $(u - \ell)^+ \in H^1_0(G)$ and $(u - \ell)^+ = 0$ as noted above. Then $u \le \ell$ m-a.e. on G. Therefore we arrive at the desired result. ■

Remark 8.4 In [42], the weak maximum principle for the subsolution $u \in \widehat{H}^1(G)$ of $Lu = 0$ is only proved under the condition that $b, \widehat{b} \in L^d(G \to \mathbb{R}^d)$ and $c \in L^{d/2}(G)$ for $d \ge 3$, and $\nu(dx) \ge c_0 dx$ on G for some constant $c_0 > 0$ [42, Théorème 3.8], or $c_0 = 0$ with the coercivity of $\mathcal{E}$ on $H^1_0(G)$, that is, $\mathcal{E}(u, u) \ge c\|u\|^2_{H^1_0(G)}$, $u \in H^1_0(G)$ for some constant $c > 0$ [42, Théorème 3.6]. Chen and Wu [7] get rid of the conditions $c_0 > 0$ and the coercivity of $\mathcal{E}$ in the weak maximum principle. But they still assume that G is bounded and the sum of norms of $b, \widehat{b}, c$ is bounded by the half of upper bound of a, and they do not give the assertion for the case that ν vanishes.

To establish the strong maximum principle in our context, we further need a stochastic argument.

Let $(T^a_t)_{t>0}$ be the $L^2(\mathbb{R}^d)$-semigroup associated with $(\mathcal{E}^a, H^1(\mathbb{R}^d))$. Then T^a_t admits a symmetric jointly continuous heat kernel $p^a_t(x, y)$ on $]0, \infty[\times \mathbb{R}^d \times \mathbb{R}^d$ such that $P^a_t f(x) := \int_{\mathbb{R}^d} p^a_t(x, y) f(x)\, dy$ is an m-version of $T^a_t f$ for $f \in L^2(\mathbb{R}^d)$ and $p^a_t(x, y)$ satisfies the Aronson's estimates [43]: there exists an $M := M(\lambda_G, \Lambda_G, d) \in$

$[1, \infty[$ such that for all $x, y \in \mathbb{R}^d, t > 0$

$$\frac{1}{Mt^{d/2}} e^{-M|x-y|^2/t} \leq p_t^a(x, y) \leq \frac{M}{t^{d/2}} e^{-|x-y|^2/Mt}.$$

It should be noted that the ball doubling condition and the strong Poincaré inequality hold for $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ and the pseudo-distance/intrinsic metric derived from $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ is a complete metric compatible with the endowed topology. Hence the parabolic Harnack inequality holds for the local solution of the parabolic equation $(L^a - \frac{\partial}{\partial t})u = 0$ [36, 45]). Here L^a is the $L^2(\mathbb{R}^d)$-generator of $(\mathcal{E}^a, H^1(\mathbb{R}^d))$. In the same way as [46], $\{P_t^a\}$ is a strong Feller semigroup, that is, $P_t^a f$ is bounded continuous for bounded measurable f. Further $\{P_t^a\}$ is a Feller semigroup in view of the upper Gaussian estimate and the estimation [46, Corollary 7.3] of the local Hölder continuity of the local solution of the above parabolic equation (see also [18, Example 4.5.2]). Let $\mathbf{M}^a := (\Omega, X_t, P_x^a)_{x \in \mathbb{R}^d}$ be the Hunt process constructed by the Feller semigroup $\{P_t^a\}$. Then $\mathbf{M}^a$ is a doubly Feller diffusion process [8]. Note that $\mathbf{M}^a$ is conservative [18, Example 5.7.1].

By the same argument in [9, Proposition 1.20, Theorem 2.4] and the strong Feller property of $\mathbf{M}^a$, the part process $\mathbf{M}_G^a$ on G admits a symmetric jointly continuous kernel $p_t^{a,G}(x, y)$ on $]0, \infty[\times G \times G$ defined by

$$p_t^{a,G}(x, y) := p_t^a(x, y) - E_x^a[p_{t-\tau_G}^a(X_{\tau_G}, y) I_{\{\tau_G < t\}}]. \tag{8.3}$$

We consider the Fukushima decomposition in the strict sense for coordinate functions $e^i(x) := x^i$ $(i = 1, 2, \dots, d)$, $x = (x^1, x^2, \dots, x^d)$:

$$X_t - X_0 = M_t + N_t, \quad P_x^a\text{-a.s. for all } x \in \mathbb{R}^d.$$

Here $M_t := (M_t^1, M_t^2, \dots, M_t^d)$, $N_t := (N_t^1, N_t^2, \dots, N_t^d)$, M_t^i is a local CAF in the strict sense and an MAF locally of finite energy, and N_t^i is a local CAF in the strict sense and a CAF locally of zero energy [17, Theorem 2]. Under $\mathbf{M}^a$, we consider the following multiplicative functional L_t:

$$L_t (= L_t(b_0)) := \exp\Big[\int_0^t (a^{-1}b_0)^*(X_s) dM_s - \frac{1}{2} \int_0^t (b_0 a^{-1} b_0^*)(X_s) ds \Big].$$

L_t is a P_x^a-supermartingale and a local P_x^a-martingale for all $x \in \mathbb{R}^d \setminus N_{b_0}$, where N_{b_0} is the exceptional set for the PCAF $\int_0^t |b_0|^2(X_s)\, ds$ [18]. If $|b_0| \in L^r(\mathbb{R}^d)$ with $r > d \geq 2$ or $r/2 \geq d = 1$, then L_t is an exponential P_x^a-martingale for all $x \in \mathbb{R}^d$. More generally, $|b_0|^2 \in S_K$ implies the P_x^a-martingale property of L_t for any $x \in \mathbb{R}^d$.

Let $S_1(G)$ be the totality of smooth measures in the strict sense on G with respect to $(\mathcal{E}^a, H_0^1(G))$ (see the definition of S_1 in [18]). Since $S_D^{\mathrm{loc}}(G) \subset L_{\mathrm{loc}}^1(G)$, $f \in S_D^{\mathrm{loc}}(G)$ implies that $I_K|f|m \in S_{00}(G)$ for any compact set K in G, consequently $|f|m \in S_1(G)$. Hence we have the following.

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Proposition 8.1 *Suppose that $c \in S_D^{\mathrm{loc}}(G)$ and the distributional derivative $\partial_i \widehat{b}^i$ of $\widehat{b}^i$ is a measurable function in $S_D^{\mathrm{loc}}(G)$ for $i = 1, 2, \dots, d$. Then $\nu \in S_1(G)$. In particular, if c, $\partial_i \widehat{b}^i \in L^r_{\mathrm{loc}}(G)$ with $2r > d \geq 2$ or $r \geq d = 1$ for each $i = 1, 2, \dots, d$ (especially, if $c \in C(G)$, $\widehat{b} \in C^1(G \to \mathbb{R}^d)$), then $\nu \in S_1(G)$.*

In what follows, we assume $|b_0|^2 m, \nu \in S_1(G)$. Further we set

$$p_t^G f(x) := E_x^a\left[f(X_t) L_t e^{-A_t^{[\nu],G}} : t < \tau_G \right], \quad x \in G.$$

Here $A_t^{[\nu],G}$ is the PCAF with respect to $\mathbf{M}_G^a$ admitting no exceptional set corresponding to ν under the Revuz's characterization on G: for any $t > 0$ and non-negative Borel functions f, h on G,

$$E^a_{I_G hm}\left[\int_0^t f(X_s)\, dA_s^{[\nu],G} \right] = \int_0^t \langle f\nu, p_s^{a,G} h \rangle \, ds.$$

We omit the detailed definition of the PCAF of $\mathbf{M}_G^a$, but it should be defined on the path space over $\mathbf{M}^a$. The construction of $A_t^{[\nu],G}$ under P_x^a with $x \in G$ is quite similar to that of a PCAF admitting no exceptions over $\mathbf{M}^a$.

The following theorem is a special case of the results in [14].

Theorem 8.3 *The following hold.*

(i) *p_t^G extends to a strongly continuous semigroup P_t^G on $L^2(G)$.*

(ii) *$(P_t^G)_{t>0}$ coincides with $(T_t^G)_{t>0}$. Here T_t^G is the $L^2(G)$-semigroup corresponding to $(\mathcal{E}, H_0^1(G))$.*

It should be noted that the multiplicative functional $L_t I_{\{t<\tau_G\}}$ is defined without exceptional set under the condition $|b_0|^2 m \in S_1(G)$. The subprocess constructed on G by $e^{-\gamma t} p_t^G$ is a right process satisfying the absolute continuity condition with respect to m, because $f = 0$ m-a.e. and $P_x^a(L_t I_{\{t<\tau_G\}} < \infty) = 1$ for all $x \in G$ together imply

$$E_x^a\left[f(X_t) L_t e^{-A_t^{[\nu],G}} : t < \tau_G \right] = 0 \quad \text{for all } x \in G.$$

On the other hand, if G is connected, then $(\mathcal{E}, H_0^1(G))$ is irreducible in view of Corollary 5.1. Hence the strong maximum principle for $\mathcal{E}$-subharmonic finely continuous functions holds under $(\mathcal{E}, H_0^1(G))$ as follows.

Theorem 8.4 *Assume that G is connected. Suppose $|b_0|^2 m$, $\nu \in S_1(G)$. Let $u \in H_0^1(G)_{\mathrm{loc}} \cap C(G)$ be an $\mathcal{E}$-subharmonic function on G. If u attains its maximum at some $x_0 \in G$, then we have $u^+ \equiv u^+(x_0)$. If further $\nu = 0$ on G, then $u \equiv u(x_0)$.*

The following subexample due to K. Kurata (private communication) is not covered by Theorem 8.4.

Subexample 8.3 **(*cf.* [25, Remark 1.2])** Let f_ε be as in Subexample 8.2. We assume $\Lambda_{\mathbb{R}^d} = \lambda_{\mathbb{R}^d} = 1$, $b = \widehat{b} = 0$ and $c := (d-2)f_\varepsilon + 2f_1$ for $d \geq 3$, hence $c \in L_+^{d/2}(\mathbb{R}^d) \cap L_+^1(\mathbb{R}^d)$. Then $(\mathcal{E}, H^1(\mathbb{R}^d))$ (in particular, $(\mathcal{E}, H_0^1(G))$ for any open set G) enjoys the strong maximum principle by Theorem 8.4 if $\varepsilon > 0$, because $f_\varepsilon \in S_K$ in this case. But $\nu := cm \notin S_1$ if $\varepsilon = 0$. Indeed, in this case $v(x) := -1/\log|x|$, $x \neq 0$, $v(0) := 0$ is harmonic on $B_{1/\sqrt{e}}(0)$ with respect to $\frac{1}{2}\Delta - c$ and v takes the minimum 0 at origin. Note that $v \not\equiv 0$ on $B_{1/\sqrt{e}}(0)$. According to the stochastic proof of our strong maximum principle, $\nu = cm \in S_1$ means $v \equiv 0$ on $B_{1/\sqrt{e}}(0)$. So our strong maximum principle does not work for $\varepsilon = 0$. However, $u := -v(\leq 0)$ satisfies the strong maximum principle in the sense that $u^+ \equiv u^+(0) = 0$, which is not covered by Theorem 8.4.

Remark 8.5 In [42], a strong maximum principle is shown in the case that $b \in L^d(G \to \mathbb{R}^d)$, $\widehat{b} \in L^q(G \to \mathbb{R}^d)$, $c \in L^{q-d/2}(G)$ with $q > d \geq 3$, and $c = \operatorname{div}\widehat{b} = 0$ for $\mathcal{E}$-harmonic functions by way of the local Hölder continuity of the $\mathcal{E}$-harmonic function and the elliptic Harnack inequality for it. Our strong maximum principle does not cover the Stampacchia's result, however, if $b \in L^p(G \to \mathbb{R}^d)$, $\widehat{b} \in L^q(G \to \mathbb{R}^d)$, $c \in L^r(G)$ with $p, q, 2r > d \geq 2$ or $p/2, q/2, r \geq d = 1$, $c \geq 0$, $\operatorname{div}\widehat{b} = 0$, then we can confirm $|b|^2, |\widehat{b}|^2, c \in S_K \subset S_H$ with small $\delta(|b_0|^2) < \lambda_G/2$ by setting $b = \widehat{b} \equiv 0$, $c \equiv 0$ on $\mathbb{R}^d \setminus G$ and $|b_0|^2 m$, $\nu = cm \in S_1(G)$. This case is not included in [42].

Example 8.2 **(Doubly Feller symmetric diffusion process)** Let (X, d) be a locally compact separable metric space and m a positive Radon measure with full topological support. Consider a regular strongly local symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$. Let G be an open subset of X. We assume that $(\mathcal{E}, \mathcal{F})$ is associated with a doubly Feller m-symmetric diffusion process $\mathbf{M}$ admitting a jointly continuous heat kernel $p_t(x, y)$ with respect to m. Then the part process $\mathbf{M}_G$ on G is a strong Feller diffusion by Chung [8] which admits a jointly continuous heat kernel $p_t^G(x, y)$, $x, y, \in G$ defined in a similar way as in (8.3) [9, Theorem 2.4]. In this context, if G is a connected open set, then the part space $(\mathcal{E}_G, \mathcal{F}_G)$ of $(\mathcal{E}, \mathcal{F})$ on $L^2(G; m)$ is an irreducible regular strongly local Dirichlet form by [47]. Therefore we have the following strong maximum principle (see also [27]).

Theorem 8.5 *Suppose that G is connected. Let $u \in (\dot{\mathcal{F}}_G)_{\mathrm{loc}} \cap C_f(G)$ be an $\mathcal{E}_G$-subharmonic function on G. If u attains its maximum at $x_0 \in G$, then we have $u \equiv u(x_0)$.*

Let us show some sufficient conditions for our assumptions. Let

$$B_r(x) := \{y \in X \mid d(x, y) < r\}$$

be the open ball with radius $r > 0$ and center $x \in X$. We assume the existence of heat kernel $p_t(x, y)$ associated with $(\mathcal{E}, \mathcal{F})$ satisfying the Li–Yau Gaussian type esti-

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mates (HK): there exist positive constants C_1, C_2, R_{HK} such that for any $t \in]0, R_{HK}[$, $x, y \in X$

$$\frac{C_1}{m(B_{\sqrt{t}}(x))} e^{-C_1 \frac{d(x,y)^2}{t}} \leq p_t(x, y) \leq \frac{C_2}{m(B_{\sqrt{t}}(x))} e^{-C_2 \frac{d(x,y)^2}{t}}.$$

This condition is equivalent to the parabolic Harnack inequality (PHI): there exist positive constants C_{PH}, R_{PH} such that for all open balls $B_r(x)$ with $r \in]0, R_{PH}[$ and for any positive (local) solution of "$(\partial_t + L)w = 0$" in the cylinder set $]s-r^2, s] \times B_r(x)$, we have $\sup_{Q_-} \widetilde{u} \leq C_{PH} \inf_{Q_+} \widetilde{u}$, where $Q_- :=]s - (3/4)r^2, s - (1/2)r^2[\times B_{2r}(x)$ and $Q_- :=]s - (1/4)r^2, s[\times B_{2r}(x)$. We omit the detailed definition of the (local) solution of "$(\partial_t + L)w = 0$" (see [3, 30]). Though the statement on the equivalence between (HK) and (PHI) in [3] is only restricted to the case $R_{HK} = R_{PH} = \infty$, it holds in this generality. The local Hölder continuity estimate of the (local) solution of "$(\partial_t + L)w = 0$" also holds in this generality (see [45, (3.5)] or [38, Theorems 5.4.7, 5.5.1]).

Then there exists a doubly Feller diffusion process $\mathbf{M} = (\Omega, X_t, P_x)_{x \in X}$ such that for Borel $u \in L^2(X; m)$, $P_t u(x) := \int_X p_t(x, y)u(y)m(dy)$ is an m-version of $T_t u$ and

$$P_t u(x) = E_x[u(X_t)] \quad \text{for all } x \in X.$$

The strong Feller property, that is, $P_t u \in C_b(X)$ for $u \in L^\infty(X; m)$, $t > 0$ is essentially proved in [46] by use of (PHI). The Feller property of $P_t u$ follows from the upper Gaussian estimate as above. At this stage, the above upper Gaussian estimate implies $P_t u \in C_\infty(X)$ for $u \in C_\infty(X)$ and the Hölder continuity estimate of the (local) solution of $(\partial_t + L)w = 0$ yields $P_t u(x) \to u(x)$ as $t \to 0$ for $u \in C_\infty(X)$ and $x \in X$. We remark that our assumptions are satisfied in the case that X is a complete smooth Riemannian manifold with lower Ricci curvature bound and m is the volume element derived from the Riemannian metric, and $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form determined by the Laplace–Beltrami operator (see [38, Theorems 5.5.1, 5.5.3, 5.6.3–5.6.6]).

Let X be an Alexandrov space of curvature bounded from below and of finite Hausdorff dimension with Hausdorff measure m and $(\mathcal{E}, \mathcal{F})$ is the canonical Dirichlet form on it [30, 31]. Then the corresponding diffusion process $\mathbf{M}$ on X can be constructed as a strong Feller process in view of [46]. But it is unclear that $\mathbf{M}$ has the Feller property. However, we can directly prove the strong Feller property of $\mathbf{M}_G$ [29]. Hence we have the same assertion as in Theorem 8.5 over Alexandrov spaces.

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