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p-adic and Motivic Measure on Artin n-stacks

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Abstract. We define a notion of p-adic measure on Artin n-stacks that are of strongly finite type over the ring of p-adic integers. p-adic measure on schemes can be evaluated by counting points on the reduction of the scheme modulo p^n. We show that an analogous construction works in the case of Artin stacks as well if we count the points using the counting measure defined by Toën. As a consequence, we obtain the result that the Poincaré and Serre series of such stacks are rational functions, thus extending Denef's result for varieties. Finally, using motivic integration we show that as p varies, the rationality of the Serre series of an Artin stack defined over the integers is uniform with respect to p.

1 Introduction

Let X be a scheme of finite type and of pure dimension d over $\mathbb{Z}_p$. One can define a measure on the space $X(\mathbb{Z}_p)$, called the p-adic measure, which we denote by μ_d. Roughly speaking, this is defined by choosing bi-analytic isometries of open subsets of the smooth part of $X(\mathbb{Z}_p)$ with balls in $\mathbb{Z}_p^d$ and pulling back the normalized Haar measure on $\mathbb{Z}_p^d$ (see [19]). However, there is another way to define this measure. For each n, let $\tau_n\colon X(\mathbb{Z}_p) \to X(\mathbb{Z}/p^n\mathbb{Z})$ be the "reduction modulo p^n" map. Let A be a sub-analytic or definable (in the language of valued fields) subset of $X(\mathbb{Z}_p)$. Then it can be proved (see [17]) that

$$\mu_d(A) = \lim_{n\to\infty} p^{-nd}|\tau_n(A)|, \tag{1.1}$$

where $|\cdot|$ denotes the cardinality of a set. In other words, the p-adic measure on $X(\mathbb{Z}_p)$ can be obtained from the counting measure on $X(\mathbb{Z}/p^n\mathbb{Z})$ by a limiting process. As an application of p-adic measure, it can be proved that the power series

$$P_X(T) := \sum_{n=0}^{\infty} |\tau_n(X(\mathbb{Z}_p))|T^n, \qquad \widetilde{P}_X(T) := \sum_{n=0}^{\infty} |X(\mathbb{Z}/p^n\mathbb{Z})|T^n$$

(the Serre series and the Poincaré series) are rational functions of T (see [8]).

If X is a scheme of finite type and of pure dimension d over $\mathbb{Z}$, one can consider the schemes $X_p := X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}_p)$ for various primes p. Then motivic integration allows us to compare the p-adic measures on X_p as p varies. Roughly speaking, if we consider a formula ϕ in the language of valued fields and interpret it on the various X_p, we obtain a family of subsets $A_p \subset X(\mathbb{Z}_p)$ for almost all p. The motivic

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measure $\mu(\phi)$ of the formula ϕ lies in a certain localization of the Grothendieck ring of formulas in the language of rings with coefficients in $\mathbb{Z}$. Then, for almost all p, the measure $\mu_d(A_p)$ can be obtained from $\mu(\phi)$ by a process of specialization that amounts to "counting the $\mathbb{F}_p$-valued points satisfying $\mu(\phi)$" (see [6, Section 9] for a precise discussion). As a result, for almost all p, evaluating the p-adic measure of a definable subset for any fixed p boils down to counting the $\mathbb{F}_p$-valued points satisfying a set of formulas that is independent of p. Thus, one is able to strengthen the above-mentioned result regarding the rationality of the power series P_{X_p} and $\widetilde{P}_{X_p}$. Indeed, one is able to obtain rational functions $P_X(T)$ and $\widetilde{P}_X(T)$ in T, with coefficients in the above-mentioned localization of the Grothendieck ring, which specialize to the power series $P_{X_p}(T)$ and $\widetilde{P}_{X_p}(T)$ respectively for almost all p.

We would like to generalize these results to Artin stacks that are strongly of finite type over $\mathbb{Z}$. First we would like to define a p-adic measure on an Artin stack X that is strongly of finite type over $\mathbb{Z}_p$. We do this by first defining a counting measure for the $(\mathbb{Z}/p^n\mathbb{Z})$-valued points of X. Then we use this counting measure in the analogue of equation (1.1) and show that the limit on the right-hand side of equation exists in the case of stacks as well, thus obtaining a notion of p-adic measure (see Theorem 4.5). As a consequence of the proof, we will see in Theorem 4.6 that the power series $P_X(T)$ and $\widetilde{P}_X(T)$ are rational functions of T. Finally, when X is an Artin stack that is strongly of finite type over $\mathbb{Z}$, we vary p and use motivic integration to show that the rationality of the power series P_{X_p} is uniform with respect to p (see Theorem 6.2).

Convention 1.1 We will use the following conventions and notation.

(i) For any scheme T, Aff/T will denote the (big) étale site of affine schemes over T.

(ii) For any scheme T, $(\mathrm{Aff}/T)^\sim$ will denote the model category of simplicial presheaves on T with the *local projective model structure* (see [21]). The homotopy category $Ho((\mathrm{Aff}/T)^\sim)$ will be referred to as the category of stacks over T and denoted by $St(T)$. We will depend on the works of Toën ([20–22]) for all the terminology and basic results regarding this category.

(iii) We will usually be concerned with Artin stacks that are strongly of finite type over the base scheme. For the sake of brevity, we will say that X is an sft-Artin stack over S if X is an Artin stack, strongly of finite type over S (see [22] for an explanation of this notion).

(iv) When we speak of sft-Artin stacks over a discrete valuation ring A, we will always intend it to be *flat* over A.

2 Counting Points on Artin Stacks

In this section, we consider the problem of defining a meaningful way to count the $(\mathbb{Z}/p^n\mathbb{Z})$-valued points of a sft-Artin stack over $\mathbb{Z}_p$. In other words, for such a stack X, we wish to define a counting measure on the set $\pi_0(X(\mathbb{Z}/p^n\mathbb{Z}))$.

If X is an sft-Artin stack over a finite field $\mathbb{F}_q$, there is already a notion of counting for the $\mathbb{F}_q$-valued points on X that was proposed by Toën in [20]. One defines

$$\#X(\mathbb{F}_q) := \sum_{x \in \pi_0(X(\mathbb{F}_q))} \prod_{i>0} |\pi_i(X(\mathbb{F}_q), x)|^{(-1)^i}. \tag{2.1}$$

It was proved in [20, Proposition 3.5] that the sets $\pi_0(X(\mathbb{F}_q))$ and $\pi_i(X(\mathbb{F}_q), x)$ are finite. (We will obtain a different proof for this in the following section; see Remark 3.6.) The product is finite, because the groups $\pi_i(X(\mathbb{F}_q), x)$ are trivial for sufficiently large i. Thus the right-hand side is well defined. This definition is justified by the fact that this counting measure factors through the Grothendieck ring of sft-Artin stacks over $\mathbb{F}_q$.

The above formula suggests that if one has an sft-Artin stack X over $\mathbb{Z}_p$, one may wish to simply define

$$\#X(\mathbb{Z}/p^n\mathbb{Z}) := \sum_{x \in \pi_0(X(\mathbb{Z}/p^n\mathbb{Z}))} \prod_{i>0} |\pi_i(X(\mathbb{Z}/p^n, Z), x)|^{(-1)^i} \tag{2.2}$$

though we would have to verify that the right-hand side of this equation is finite. This is not merely an ad hoc definition. One can functorially construct an Artin stack $Gr_n(X)$ of strongly finite type over $\mathbb{F}_p$ such that we have a weak equivalence $X(\mathbb{Z}/p^n\mathbb{Z}) \simeq Gr_n(X)(\mathbb{F}_p)$. Thus, applying formula (2.1) to $Gr_n(X)$ yields formula (2.2) and also proves that the right-hand side is finite.

More generally, let R be a complete discrete valuation ring with residue field k. Let ω be a uniformizing parameter of R and let $R_n := R/\langle \omega^{n+1} \rangle$ for $n \geq 0$. Then, given a sft-Artin stack X over R, we will construct sft-Artin stacks $Gr_n(X)$ over k such that $X(R) \simeq Gr_n(X)(k)$.

We begin by recalling some material from [11]. Suppose S is an Artin local ring with residue field K such that there exists a bijection of the elements of S with the set K^n for some n and such that the addition and multiplication maps are given by polynomials with coefficients in K. Then there exists a ring variety $\mathcal{S}$ over K such that the underlying scheme of $\mathcal{S}$ is isomorphic to $\mathbb{A}^n_K$ and such that $\mathcal{S}(K) = S$. We can use the ring variety $\mathcal{S}$ to define a functor $\mathrm{Aff}/\mathrm{Spec}(K) \to \mathrm{Aff}/\mathrm{Spec}(S)$ given by $U \mapsto \mathrm{Spec}(\mathcal{S}(A))$.

Convention 2.1 For any ring scheme $\mathcal{A}$ over a base scheme T, we denote by $\widetilde{\mathcal{A}}\colon \mathrm{Aff}/T \to \mathrm{Aff}/\mathrm{Spec}(\mathcal{A}(T))$ the functor $U \mapsto \mathrm{Spec}(\mathcal{A}(U))$.

It is proved in [11], that if X is a scheme of finite type over S, then the presheaf on $\mathrm{Aff}/\mathrm{Spec}(K)$ defined given by $U \mapsto X(\widetilde{\mathcal{S}}(U))$ is represented by a scheme of finite type over K. In the case when S is of the form $S = R_n$ for some $n \geq 0$, we wish to extend this result to sft-Artin stacks over R_n.

Convention 2.2 Let C and D be Grothendieck sites and let $\sigma\colon C \to D$ be a functor. Then we use $(\sigma_!, \sigma_*)$ to denote the adjunction $C^\sim \rightleftarrows D^\sim$, where $\sigma_*(F)$ is defined by $\sigma_*(F)(c) = F(\sigma(c))$. (Of course, this is not always a Quillen adjunction.)

Now for each $n \geq 0$, let $\mathcal{R}_n$ denote the ring variety over k that is constructed from R_n in the manner described above. We wish to examine the functors $(\widetilde{\mathcal{R}_n})_*$ for each $n \geq 0$. We will prove that the adjunction $((\widetilde{\mathcal{R}_n})_!, (\widetilde{\mathcal{R}_n})_*)$ is a Quillen adjunction and that the right derived functor of $(\widetilde{\mathcal{R}_n})_*$ (which will be the desired functor Gr_n) takes sft-Artin stacks over R_n to sft-Artin stacks over k.

The behaviour of the functor $\widetilde{\mathcal{R}_n}$ depends on the characteristics of R and k. We will only need the case in which $\mathrm{char}(R) \neq \mathit{char}(k)$. The case $\mathrm{char}(R) = \mathrm{char}(k)$ is only mentioned in this section for the sake of completeness. Also, this case is much easier to handle, since we know that R is isomorphic to the power series ring $k[[t]]$. It is then easily verified that for $U = \mathrm{Spec}(A)$ in $\mathrm{Aff}/\mathrm{Spec}(k)$, we have $\widetilde{\mathcal{R}_n}(U) = U \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R_n)$. Thus, $(\widetilde{\mathcal{R}_n})_*$ is simply Weil restriction with respect to the morphism $\mathrm{Spec}(R_n) \to \mathrm{Spec}(k)$ and we can define $Gr_n(X)$ to be the stack $\underline{Hom}(\mathrm{Spec}(R_n), X)$ where $\underline{Hom}$ denotes the internal Hom in $St(k)$ (see [21, Section 3.6]). However, we do not have this option when $\mathrm{char}(R) \neq \mathrm{char}(k)$ and thus, for the sake of a unified presentation, we simply follow the argument described in the preceding paragraph for both cases and prove that $((\widetilde{\mathcal{R}_n})_!, (\widetilde{\mathcal{R}_n})_*)$ is a Quillen adjunction. For the equal characteristic case, this is easy to see (and this was presented explicitly in [1]). However, in the unequal characteristic case, a little more work is required as we will see in Propositions 2.3 and 2.6.

Before we prove Proposition 2.3, we will need to understand the structure of the ring varieties $\mathcal{R}_n$ in the case $\mathrm{char}(R) = 0$, $\mathrm{char}(k) = p \neq 0$. In this case, R is obtained as a totally ramified extension of the ring $W(k)$ of Witt vectors with coefficients in k. Let us recall some basic facts about Witt vectors (see [13] or [18]). For any $\mathbb{F}_p$-algebra A, $W(A)$ is actually the set of A-rational points of the ring scheme of Witt vectors, denoted by W. The underlying scheme of W is $\mathbb{A}^{\mathbb{N}} = \mathrm{Spec}\, \mathbb{F}_p[Z_1, Z_2, \dots]$. (Strictly speaking, this is only the fibre of the Witt scheme at the prime p. However, since we are only going to be working with $\mathbb{F}_p$-algebras, this will suffice for our purposes.) The addition and multiplication are given by polynomials with coefficients in $\mathbb{F}_p$. As it turns out, these polynomials, when restricted to the first n coordinates, define a ring scheme structure on $\mathbb{A}^n$ that is denoted by W_n and called the scheme of Witt vectors of length n. For $n > m$, the projection on the first m coordinates defines a "truncation" morphism $W_n \to W_m$ that is a ring scheme homomorphism. Similarly, we have morphisms $W \to W_n$ and W is the projective limit of the system defined by the W_n for $n \geq 0$ along with the truncation morphisms.

The scheme W has two well-known automorphisms: the Verschiebung or "shifting" operator V and the Frobenius operator F. Via the isomorphism $W \cong \mathbb{A}^{\mathbb{N}}$, for any k-algebra A these automorphisms are given on $W(A)$ by the formulas

$$V((a_0, a_1, \dots)) = (0, a_0, a_1, \dots)$$

and

$$F((a_0, a_1, \dots)) = (a_0^p, a_1^p, \dots)$$

where $a_i \in A$, for all i. In other words, F is just induced by the Frobenius operator on A (which we will denote by the same symbol F). (Note that this description of the Frobenius operator only applies when we are working with algebras over $\mathbb{F}_p$.

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For more general rings, the description is via the "ghost components" of the Witt vectors.)

We will require the following easily verifiable facts about these operators.

(a) For any $a, b \in W(A)$, we have $aV(b) = V(F(a)b)$.

(b) Iterations of V induce a filtration of W that is consistent with the ring structure. In other words $V^nW(A) \cdot V^mW(A) \subset V^{n+m}W(A)$. Note that $V^nW(A)$ is the kernel of the truncation map $W(A) \to W_n(A)$.

Note that we have similar operators V and F on $W_n(A)$ for any n and that these maps commute with the truncation map. (It is also an easily verifiable fact that VF and FV are equal to multiplication by p.)

Let $\mathrm{gr}_V W(A)$ denote the graded ring of $W(A)$ with respect to the filtration induced by V. Then it follows from statement (a) that $\mathrm{gr}_V^n W(A)$ is isomorphic to the A-module F^n_*A obtained by considering A as an A-module by scalar restriction via the homomorphism $F^n\colon A \to A$.

Now let us consider the rings R and R_n for $n \geq 0$. We know that R is obtained from $W(k)$ by attaching the root of an Eisenstein polynomial of degree e and coefficients in $W(k)$, where e is the absolute ramification index of R over $W(R_0)$ (*i.e.*, $\mathrm{ord}_R(p) = e$). Thus, R is a free module with e generators over the ring $W(R_0)$. From this, it is easy to see that there exists a ring scheme $\mathcal{R}$ that is a module scheme over W of the form W^e and such that $R = \mathcal{R}(k)$. The rings R_n are Artin local rings, and so by the above discussion we see that there exists ring schemes $\mathcal{R}_n$ of finite type over k such that such that $\mathcal{R}_n(k) = R_n$.[1] We also have ring homomorphisms $\mathcal{R}_n \to \mathcal{R}_m$ for $n > m$ obtained from the surjections $R_n \to R_m$, and $\mathcal{R}$ is the limit of this projective system of schemes.

We wish to prove that the functor $\mathcal{R}_n$ preserves étale morphisms of schemes. This has been proved for the case $R = W(k)$ in [13]. We prove this result in the general case using the same method with some modifications.

For any integer $m \geq 1$, let $\mathcal{R}^m$ (resp. $\mathcal{R}^m_n$) denote the kernel of the morphism $\mathcal{R} \to \mathcal{R}_{m-1}$ (resp. $\mathcal{R}_n \to \mathcal{R}_{m-1}$). Let $\mathcal{R}^0$ (resp. $\mathcal{R}^0_n$) be the scheme $\mathcal{R}$ (resp. $\mathcal{R}_n$). Then $\{\mathcal{R}^m\}_{m\geq 0}$ (resp. $\{\mathcal{R}^m_n\}_{m\geq 0}$) is a decreasing filtration of closed ideal subschemes on $\mathcal{R}$ (resp. $\mathcal{R}_n$).

The ideals $\mathcal{R}^m$ and $\mathcal{R}^m_n$ can be easily described if we choose good presentations of $\mathcal{R}$ and $\mathcal{R}_n$ as W-modules. We know that $R = W(R_0)[\omega]$ (see [18, Chapter 1, Prop. 18]). Thus we can choose $1, \omega, \dots, \omega^{e-1}$ as generators for R and R_n as $W(k)$-modules. A typical element of R is of the form

$$x = \sum_{i=0}^{e-1} (a_{0i}, a_{1i}, \dots) \cdot \omega^i$$

where $(a_{0i}, a_{1i}, \dots) \in W(k)$. Clearly,

$$\mathrm{ord}_R((a_{0i}, a_{1i}, \dots) \cdot \omega^i) = i + ke,$$

where k is the least integer such that $a_{ki} \neq 0$. Thus

$$\mathrm{ord}_R(x) = \min_i \{\mathrm{ord}_R(a_{0i}, a_{1i}, \dots) \cdot \omega^i)\}.$$

[1]It is claimed in [2] that $\mathcal{R}_n(A) = R_n \otimes_{W(k)} W(A)$ where A is *any* algebra over k. However, in reality, this will not hold unless A is perfect.

Suppose that $m+1 = q\cdot e + r$ for $0 \le r \le e-1$. Then it is easy to see from the above calculation of $\operatorname{ord}_R(x)$ that $\mathcal{R}^m = V^{m_0}W \oplus \cdots \oplus V^{m_{(e-1)}}W$, where $m_i = q + \epsilon_i$ where $\epsilon_i = 1$ if $0 \le i < r$ and $= 0$ otherwise. (The description of $\mathcal{R}_n^m$ is similar.) From this description, it is easy to see that the filtration is consistent with the ring structure and that the ideals $\mathcal{R}_n^1$ are nilpotent for all n.

Let $\operatorname{gr}\mathcal{R}(A)$ (resp. $\operatorname{gr}\mathcal{R}_n(A)$) denote the graded ring of $\mathcal{R}(A)$ (resp. $\mathcal{R}_n(A)$) with respect to this filtration. It follows that for $m < n$,

$$\operatorname{gr}^m \mathcal{R}(A) = \operatorname{gr}^m \mathcal{R}_n(A) = \begin{cases} F_*^q A & \text{if } r \neq 0, \\ F_*^{q+1} A & \text{if } r = 0, \end{cases}$$

where q and r are as in the preceding paragraph.

Now we prove that the functor $\widetilde{\mathcal{R}}_n$ takes étale maps into étale maps. (In the special case $\mathcal{R} = W$, this has been proved in [13]. An extension of the result in this special case, involving general Witt vectors, was proved in [3]). Our argument for general $\mathcal{R}$ is an adaptation of the proof in [13], but we present it in detail for the sake of completeness.

Proposition 2.3 *Let $X \to Y$ be an étale morphism of schemes over k. Then $\widetilde{\mathcal{R}}_n(X) \to \widetilde{\mathcal{R}}_n(Y)$ is étale, and the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widetilde{\mathcal{R}}_n(X) \arrow[d] \\ Y \arrow[r] & \widetilde{\mathcal{R}}_n(Y) \end{tikzcd}$$

is cartesian.

Proof In the case $\operatorname{char}(R) = \ char(k)$, the statement is obvious. Thus we now focus on the case $\operatorname{char}(R) = 0$, $\operatorname{char}(k) = p \neq 0$. Without any loss of generality, we can assume that $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(A)$ and that the morphism $X \to Y$ is given by a k-algebra homomorphism $A \to B$. First we show that $\mathcal{R}_n(A) \to \mathcal{R}_n(B)$ is flat. For this we use a modification of the flatness criterion for filtered modules in [4, Chap. III, § 5, Thm. 1]. This criterion is stated there for I-adic filtrations but the arguments are easily adapted to this case as follows.

Lemma 2.4 *Let A be a ring with a given decreasing filtration $\{A_i\}_{i=0}^\infty$. Let M be an A-module with a filtration $\{M_i\}_{i=0}^\infty$ that is compatible with the filtration of A. Suppose the following conditions hold:*

(i) *there exists an integer k such that $A_i = 0$ and $M_i = 0$ for all $i > k$;*
(ii) *M/M_1 is a flat A/A_1-module;*
(iii) $\operatorname{gr}^n A \otimes_{\operatorname{gr}^0 A} \operatorname{gr}^0 M \to \operatorname{gr}^n M$ *is an isomorphism.*

Then M is a flat A-module.

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Proof First we prove that $A_n \otimes_A M \to M_n$ is a surjection for all n. Indeed, this is trivially true for $n > k$. For general n, we consider the diagram

$$\begin{tikzcd} & A_{n+1}M \arrow[r] \arrow[d] & A_nM \arrow[r] \arrow[d] & A_nM/A_{n+1}M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_{n+1} \arrow[r] & M_n \arrow[r] & \operatorname{gr}^n M \arrow[r] & 0. \end{tikzcd}$$

It is clear from the diagram that it will suffice to prove that the right vertical map is surjective (since we can then apply decreasing induction on n). The image of $A_nM/A_{n+1}M$ in $\operatorname{gr}^n M$ is the same as the image of

$$A_n/A_{n+1} \otimes_A M \to \operatorname{gr}^n M.$$

But A_n/A_{n+1} is annihilated by A_1, so

$$A_n/A_{n+1} \otimes_A M \cong A_n/A_{n+1} \otimes_{A/A_1} M/(A_1M).$$

Since $A_1M \subset M_1$, we have a surjection

$$A_n/A_{n+1} \otimes_{A/A_1} M/(A_1M) \to A_n/A_{n+1} \otimes_{A/A_1} M/M_1.$$

But $A_n/A_{n+1} \otimes_{A/A_1} M/M_1 \to \operatorname{gr}^n M$ is an isomorphism by hypothesis. Thus, we see that $A_n \otimes_A M \to M_n$ is a surjection. In other words, $A_nM = M_n$ for all n.

Now we claim that $A_n \otimes_A M \to M_n$ is an injection as well. Indeed, consider the diagram

$$\begin{tikzcd} & A_{n+1} \otimes_A M \arrow[r] \arrow[d] & A_n \otimes_A M \arrow[r] \arrow[d] & (A_n/A_{n+1}) \otimes_A M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_{n+1} \arrow[r] & M_n \arrow[r] & \operatorname{gr}^n M \arrow[r] & 0. \end{tikzcd}$$

Again, using decreasing induction on n, we see that it is enough to show that the right vertical map is an isomorphism. For this, we observe that

$$(A_n/A_{n+1}) \otimes_A M \cong (A_n/A_{n+1}) \otimes_{A/A_1} M/A_1M \cong (A_n/A_{n+1}) \otimes_{A/A_1} M/M_1.$$

Thus the right vertical map is an isomorphism by hypothesis. Thus, $A_n \otimes_A M \to M_n$ is an isomorphism for all n.

Now we can forget about the given filtrations on A and M and apply [4, Chap. III, §5, Thm. 1] for the A_1-adic filtrations. In other words, the facts that $M/M_1 = M/A_1M$ is a flat A/A_1 module and $A_1 \otimes M \to A_1M$ is a bijection imply that M is a flat A-module. ∎

We continue the proof of Proposition 2.3. By [7, XIV, §1, Prop. 2], the relative Frobenius map $F_{\operatorname{Spec}(B)/\operatorname{Spec}(A)}$ is surjective and radicial. Since it is also étale, it is an isomorphism. Thus $F_*B \cong B \otimes_A F_*A$, and by iteration $F_*^qB \cong B \otimes_A F_*^qA$ for any positive integer q. Thus, by our earlier arguments, $\operatorname{gr}^n \mathcal{R}_n(B) \cong B \otimes_A \operatorname{gr}^n \mathcal{R}_n(A)$. By Lemma 2.4, we see that $\mathcal{R}_n(B)$ is flat over $\mathcal{R}_n(A)$. Also, from the proof of the previous

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lemma, we see that $\mathcal{R}_n^1(B)$ is generated by the image of $\mathcal{R}_n^1(A)$. Thus

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widetilde{\mathcal{R}_n}(X) \arrow[d] \\ Y \arrow[r] & \widetilde{\mathcal{R}_n}(Y) \end{tikzcd}$$

is cartesian and the vertical arrows are flat morphisms. Now the result follows from Lemma 2.5 ∎

Lemma 2.5 *Suppose $A \to B$ is a flat ring homomorphism. Let I be a nilpotent ideal in A and suppose that $A/I \to B/IB$ is étale. Then $A \to B$ is étale.*

Proof Since I is a nilpotent ideal, one can easily prove that the fact that $A/I \to B/IB$ is of finite type implies that $A \to B$ is of finite type. Indeed if the homomorphism $p\colon A/I[X_1, \dots, X_r] \to B/IB$ is a surjection, we define a homomorphism $q\colon A[X_1, \dots, X_r] \to B$ to be an arbitrary lift of this surjection. Then for any element b, there exists $f(X) \in A[X_1, \dots, X_r]$ such that $q(f(X)) - b \in IB$. Thus $q(f(X) - b = i_1 b_1 + \cdots + i_s b_s$. Now choose $f_i(X) \in A[X_1, \dots, X_r]$ for $1 \leq i \leq s$ such that $q(f_i(X)) - b_i \in IB$. Then if $g(X) = f(X) - \sum_i f_i(X)$, we see that $q(g(X)) - b \in I^2 B$. Continuing in this manner and using the fact that I is nilpotent, we see that q is surjective.

Thus now we merely need to prove that $A \to B$ is unramified. But this is immediate, since $A/I \to B/IB$ is unramified. (A morphism of schemes is unramified if and only if its geometric fibres are unramified). ∎

Proposition 2.6 *The functor $\widetilde{\mathcal{R}_n}\colon \mathrm{Aff}\,/\,\mathrm{Spec}(k) \to \mathrm{Aff}\,/\,\mathrm{Spec}(R_n)$ satisfies*

$$\widetilde{\mathcal{R}_n}(X \times_Z Y) \cong \widetilde{\mathcal{R}_n}(X) \times_{\widetilde{\mathcal{R}_n}(Z)} \widetilde{\mathcal{R}_n} Y$$

if $X \to Z$ is étale.

Proof In the diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \widetilde{\mathcal{R}_n}(X) \arrow[d] \\ Y \arrow[r] & Z \arrow[r] & \widetilde{\mathcal{R}_n}(Z), \end{tikzcd}$$

the left and right squares are cartesian, and so the outer square is cartesian. In the diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & \widetilde{\mathcal{R}_n}(X) \times_{\widetilde{\mathcal{R}_n}(Z)} \widetilde{\mathcal{R}_n}(Y) \arrow[r] \arrow[d] & \widetilde{\mathcal{R}_n}(X) \arrow[d] \\ Y \arrow[r] & \widetilde{\mathcal{R}_n}(Y) \arrow[r] & \widetilde{\mathcal{R}_n}(Z), \end{tikzcd}$$

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the right and outer squares are cartesian. Thus the left square is cartesian. But since Y is a closed subscheme of $\widetilde{\mathcal{R}}_n(Y)$ defined by a nilpotent ideal, there is a unique étale morphism $T \to \widetilde{\mathcal{R}}_n(Y)$ such that $T \times_{\widetilde{\mathcal{R}}_n(Y)} Y \cong X \times_Z Y$. Since both

$$\widetilde{\mathcal{R}}_n(X \times_Z Y) \longrightarrow \widetilde{\mathcal{R}}_n(Y) \quad \text{and} \quad \widetilde{\mathcal{R}}_n(X) \times_{\widetilde{\mathcal{R}}_n(Z)} \widetilde{\mathcal{R}}_n(Y) \longrightarrow \widetilde{\mathcal{R}}_n(Y)$$

satisfy this property, they must be equal. ■

Proposition 2.7 *$((\widetilde{\mathcal{R}}_n)_!, (\widetilde{\mathcal{R}}_n)_*)$ is a Quillen adjunction.*

Proof By [10, Cor. 8.3] it suffices to check that $\widetilde{\mathcal{R}}_n$ preserves étale covers and that it commutes with limits of finite diagrams of étale maps. This follows from Propositions 2.3 and 2.6. ■

Definition 2.8 For any $n \geq 0$, the n-th Greenberg functor for R is defined to be the right-derived functor $\mathbb{R}(\widetilde{\mathcal{R}}_n)_*: St(\mathrm{Spec}(R_n)) \to St(\mathrm{Spec}(R_0))$ and is denoted by Gr_n. If X is a stack over R, we abuse notation and write $Gr_n(X)$ instead of $Gr_n^R(X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R_n))$.

Convention 2.9 Actually, it would be appropriate to include a reference to R in the notation for the n-th Greenberg functor. However, we will avoid this to prevent the notation from becoming too cumbersome. This will not lead to any confusion.

Proposition 2.10 *Let $n \geq 0$ be an integer. The functor Gr_n has the following properties:*

(i) *Gr_n preserves homotopy fibre products;*
(ii) *Gr_n takes schemes of finite type over R_n to schemes of finite type over k;*
(iii) *Gr_n takes smooth (étale, unramified) morphisms between schemes of finite type over R_n to smooth (resp. étale, unramified) morphisms between schemes of finite type over k;*
(iv) *Gr_n preserves epimorphisms of stacks;*
(v) *Gr_n takes sft-Artin stacks over R_n to sft-Artin stacks over k.*

Proof (i) is obvious, since $(\widetilde{\mathcal{R}}_n)_*$ is a right Quillen functor. (ii) is proved in [11].

(iii) is stated in [2] without a proof. A proof is included here for the sake of completeness. Suppose $\mathrm{char}(R) = 0$ and $\mathrm{char}(k) = p \neq 0$.

For any positive integer m, let $x = (x_0, \dots, x_{m-1})$ and $y = (y_0, \dots, y_{m-1})$ denote generic elements of W_m. We claim that all the monomials appearing in the polynomials defining the product xy involve both the x_i and y_i to non-zero degree. Indeed, suppose $xy = (p_0, \dots, p_{m-1})$, where for every i, p_i is a polynomial in x and y. Suppose that for some j a monomial of the form the monomial x_j^α, $\alpha \in \mathbb{N}$, appears in one of the polynomials $p_0, p_1, \dots, p_{m-1}$. Then we see that if $x = (0, \dots, 1, \dots 0)$ (*i.e.*, 1 in the j-th place and 0 elsewhere) and $y = (0, \dots, 0)$ (0 in all places), then $xy \neq 0$, which is a contradiction. This proves our claim.

Thus we see that for any ring A, if $a = (a_0, \dots, a_{n-1})$ and $(b_0, \dots, b_{n-1})$ are elements of $W_n(A)$ such that the a_i are in an ideal I and the b_i are in an ideal J, then the coordinates of ab are in the ideal IJ. By examining the multiplication rule on $\mathcal{R}_n$, we see that this argument continues to hold with $\mathcal{R}_n$ in place of W_n. Also note that the set of elements having all coordinates in an ideal I is precisely the kernel of $\mathcal{R}_n(A) \to \mathcal{R}_n(A/I)$. Thus we see that if I is a nilpotent ideal in A, then the kernel of $\mathcal{R}_(A) \to \mathcal{R}_n(A/I)$ is a nilpotent ideal in $\mathcal{R}(A)$. This proves that Gr_n preserves the property of being formally smooth, étale, or unramified. This proves (iii) in the unequal characteristic case. The argument in the case $\mathrm{char}(R) = \mathrm{char}(k)$ is similar but simpler, and so we omit it.

To prove (iv), it suffices to see that for any affine scheme U over k, any étale cover of $\widetilde{\mathcal{R}_n}(U)$ can be refined by a cover of the form $\{\widetilde{\mathcal{R}_n}(U_i) \to \mathcal{R}_n(U\}_i$. But this is obvious since U is a closed subscheme of $\mathcal{R}(U)$ defined by a nilpotent ideal.

(v) now follows immediately, since the notion of an Artin stack is defined in terms of affine schemes, smoothness, and homotopy fibre products. ∎

Let $m \geq n$ be non-negative integers and let U be an affine scheme over k. The ring-scheme homomorphism $\mathcal{R}_m \to \mathcal{R}_n$ induces a morphism $e_n^m \colon \widetilde{\mathcal{R}_n}(U) \to \widetilde{\mathcal{R}_m}(U)$. These morphisms induce the "truncation morphisms" as follows.

Definition 2.11 Let X be a stack over R. The truncation morphism $\tau^m_{n,X} \colon Gr_m(X) \to Gr_n(X)$ is the one that maps

$$x_m \in Gr_m(X)(U) \colon \widetilde{\mathcal{R}_m}(U) \longrightarrow X$$

to

$$x_n \in Gr_n(X)(U) \colon \widetilde{\mathcal{R}_n}(U) \xrightarrow{e_n^m} \widetilde{\mathcal{R}_m}(U) \xrightarrow{x_m} X$$

for any affine scheme U over k.

Also, for every n and for any affine scheme U let $\tau_{n,X}(U)$ denote the function

$$\pi_0\big(X(\mathcal{R}(U))\big) \to \pi_0\big(X(\mathcal{R}_n(U))\big) \equiv \pi_0\big(Gr_n(X)(U)\big).$$

If there is no risk of confusion, we will write τ_n^m instead of $\tau^m_{n,X}$. We will also write τ_n instead of $\tau_{n,X}(U)$ (*i.e.*, we will omit the reference to both U and X).

3 Lifting R-valued Points to an Atlas

Let X be an Artin stack and let $p \colon U \to X$ be a smooth atlas, with U being an affine scheme. Then if K is a field and $x \colon \mathrm{Spec}(K) \to X$ is a morphism, we may not always be able to find a lift $u \colon \mathrm{Spec}(K) \to X$. However, given x, p can be chosen appropriately so that a lift does exist, as we will prove in Lemma 3.1. If X is an sft-Artin stack over a noetherian base scheme, then p can be chosen to be independent of x.

Lemma 3.1 is proved in [14, Thm. II.6.4] for algebraic spaces, and the argument is generalized in [15, Chapter 6] for Artin 1-stacks. The proof for Artin n-stacks is obtained by a further extension of this argument.

Let $X \to Y$ be a morphism of stacks. Then $(X/Y)^d$ denotes the d-fold fibre product

$$\underbrace{X \times^h_Y X \times^h_Y \times \cdots \times^h_Y X}_{d \text{ times}}.$$

Let $\mathscr{S}_d$ denote the symmetric group on d letters. Then $\mathscr{S}_d$ acts on $(X/Y)^d$ by permuting the factors and the quotient stack is denoted by $\Sigma_d(X/Y)$. We observe that this construction is well behaved with respect to base changes of the form $Y' \to Y$.

Lemma 3.1 *Let S be an arbitrary base scheme and let X be an Artin stack over S. Let K be a field and suppose we have a morphism* $x\colon \operatorname{Spec}(K) \to X$*. Then there exists a diagram*

$$\begin{tikzcd} & V \arrow[d, "\phi"] \\ \operatorname{Spec}(K) \arrow[ur, "v"] \arrow[r, "x"'] & X \end{tikzcd}$$

that commutes up to homotopy, where V is an affine scheme and ϕ is smooth.

Proof Choose a smooth morphism $U \to X$, where U is an affine scheme such that the stack $U_x := U \times^h_{X,x} \operatorname{Spec}(K)$ is non-empty. Since U_x is non-empty, there exists a K-morphism $x'\colon \operatorname{Spec}(K') \to U_x$, where K' is a separable finite field extension of K. Let $d = [K' {:} K]$. Let L be a Galois extension of K containing K', so that $\operatorname{Spec}(K') \times_{\operatorname{Spec}(K)} \operatorname{Spec}(L) = \coprod_J \operatorname{Spec}(L)$ where J is a set of representatives for the cosets of $\operatorname{Gal}(L/K')$ in $\operatorname{Gal}(L/K)$. Note that the action of $\operatorname{Gal}(L/K)$ on $\operatorname{Spec}(K') \times_{\operatorname{Spec}(K)} \operatorname{Spec}(L)$ merely permutes the components of $\coprod_J \operatorname{Spec}(L)$.

Clearly, x' induces a morphism $\operatorname{Spec}(L) \times \{1, \dots, d\} \to U_x \times^h_{\operatorname{Spec}(K)} \operatorname{Spec}(L)$, or equivalently, a morphism $\operatorname{Spec}(L) \to (U_x/\operatorname{Spec}(K))^d \times^h_{\operatorname{Spec}(K)} \operatorname{Spec}(L)$. On composing with the projection, we get a morphism $\operatorname{Spec}(L) \to (U_x/\operatorname{Spec}(K))^d$. On composing with the quotient map for the $\mathscr{S}_d$ action, we get a morphism $\operatorname{Spec}(L) \to \Sigma_d(U_x/\operatorname{Spec}(K))$. As we noted above, the action of $\operatorname{Gal}(L/K)$ permutes the components of $\operatorname{Spec}(K') \times_{\operatorname{Spec}(K)} \operatorname{Spec}(L) = \coprod_J \operatorname{Spec}(L)$, and thus we see that the L-valued point $\operatorname{Spec}(L) \to \Sigma_d(T_1/T)$ that we have obtained is invariant under the action of $\operatorname{Gal}(L/F)$, *i.e.*, it is an F-valued point. This gives us an F-valued point of $\Sigma_d(U/X)$.

Since $U \to X$ is smooth, so is $(U/X)^d \to X$. The quotient morphism $(U/X)^d \to \Sigma_d(U/X)$ is obviously a smooth covering map, and thus $\Sigma_d(U/X) \to X$ is smooth. If X is an n-stack, it is clear that $(U/X)^d$ is an $(n-1)$-stack. Let $s\colon Z \to (U/X)^d$ be a geometric point (*i.e.*, Z is the spectrum of a separably closed field) of $(U/X)^d$ and let t be its image in $\Sigma_d(U/X)$. Then we have the long exact sequence of homotopy groups

$$\cdots \to \pi_i\big(F_t(Z), s\big) \to \pi_i\big((U/X)^d(Z), s\big) \to \pi_i\big(\Sigma_d(U/X))(Z), t\big) \\ \to \pi_{i-1}\big(F_t(Z), s\big) \to \cdots,$$

where F_t is the fibre of $(U/X)^d \to \Sigma_d(U/X)$ at t. Since F_t is isomorphic to $Z \times \mathscr{S}_d$, we see immediately that $\pi_i((U/X)^d(Z), s) \cong \pi_i(\Sigma_d(U/X)(Z), t)$ for $i > 1$. Thus if X is an n-stack for $n \geq 2$, then $\Sigma_d(U/X)$ is an $(n-1)$-stack. Now replace X by $\Sigma_d(U/X)$ and repeat the procedure until we come to the case $n = 1$. If X is an Artin 1-stack, then $(U/X)^d$ is an algebraic space, and thus $\Sigma_d(U/X)$ is the quotient of an algebraic space under the action of a finite group.

Now the required result follows immediately from [15, Thm. 6.1]; we briefly reproduce the argument. Choose the usual embedding of $\mathscr{S}_d$ into the group scheme $GL_{n,S}$. Let V' be the quotient of the action of $\mathscr{S}_d$ on $(U/X)^d \times_S GL_{n,S}$. Then it can be checked that V' is an algebraic space and that V' is a $GL_{n,S}$-torsor over $\Sigma_d(U/X)$. Thus for any morphism $T \to \Sigma_d(U/X)$ from a semi-local scheme into $\Sigma_d(U/X)$, there exists a lift $T \to V'$. In particular, there exists a morphism $v'\colon \operatorname{Spec}(K) \to V'$ lifting x.

Finally, now we apply [14, Thm. II.6.4] to construct an étale map $V \to V'$ such that V is an affine scheme and such that there exists a $v\colon \operatorname{Spec}(K) \to V$ lifting v'. ■

We note that in the above lemma, the morphism $V \to X$ is specifically chosen for the given morphism $x\colon \operatorname{Spec}(K) \to X$. However, for sft-Artin stacks over a noetherian base, we are able to strengthen this result.

Definition 3.2 Let $p\colon X \to Y$ be a morphism of Artin stacks.

(i) We say that p is of *f-class n* if for any field K and any morphism $y\colon \operatorname{Spec}(K) \to Y$, there exists a finite field extension L of K with $[L{:}K] \leq n$ such that there exists a morphism $x\colon \operatorname{Spec}(L) \to X$ such that the square

$$\begin{array}{ccc} \operatorname{Spec}(L) & \xrightarrow{x} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(K) & \xrightarrow{y} & Y \end{array}$$

commutes up to homotopy.

(ii) We say that p is *f-surjective* if it is of f-class 0. In other words, for any field K, the morphism $\pi_0(X(K)) \to \pi_0(Y(K))$ is surjective.

Lemma 3.3 *Let $p\colon X \to Y$ be a morphism of sft-Artin stacks over a noetherian base scheme S. Then there exists a integer n such that p is of f-class n.*

Proof We break the proof down into three cases.

Case 1. X and Y are affine schemes: Suppose $Y = \operatorname{Spec}(R)$ for some ring R and $X = \operatorname{Spec}(R[X_1, \dots, X_r]/I)$, where $I = \langle h_1, \dots, h_s\rangle$ is some finitely generated ideal of the noetherian ring $R[X_1, \dots, X_r]$. Now if y is a point of Y given by a ring homomorphism $R \to K$ for some field K, then the fibre $X \times_{Y,y} \operatorname{Spec}(K)$of p over Y is a scheme of finite type over K whose underlying set is the set of solutions of the images of the polynomials h_i in the ring $K[X_1, \dots, X_r]$. Clearly, there exists a number n depending only on the degrees of the polynomials h_i such that there exists a field L with $[L{:}K]$ and a K-morphism $\operatorname{Spec}(L) \to X \times_{Y,y} \operatorname{Spec}(K)$. This proves the result when X and Y are affine schemes.

Case 2. Y is an affine scheme and p is arbitrary: Note that if we have morphisms $F \to G \to H$ of Artin stacks over S and if the statement of the lemma is true for the morphisms $F \to H$ then it is also true for the morphism $G \to H$. Now let $U \to X$ be a smooth atlas of X with U being an affine scheme of finite type over k. The result follows from Case 1.

Case 3. The general case: Let $k \geq -1$ be an integer such that Y is k-geometric and p is k-representable (see [20] for this terminology). We prove the result by induction on k. The case $k = -1$ is covered in Case 1. Suppose the result is true for $k \leq m-1$.

Now suppose Y is m-geometric and Y is m-representable. Let $V \to Y$ be a smooth atlas where V is an affine scheme. By the observation in Case 2, it is enough to prove the result for the composition of the morphisms $V \times_Y X \to X \to Y$. But this is also the composition of the morphisms $V \times_Y X \to V \to Y$. The statement of the lemmma holds for $V \times_Y X \to V$ by Case 2. The morphism $V \to Y$ is $(m-1)$-representable, and thus the statement of the lemma holds for this morphism by the induction hypothesis. This completes the proof. ■

Lemma 3.4 *Let X be an sft-Artin stack over a noetherian base scheme S. Then there exists an affine scheme V and a smooth covering map $V \to X$ which is f-surjective.*

Proof The proof is based on the arguments in the proof of Lemma 3.1. Thus we will refer to that proof for the details.

Suppose X is an n-stack. Choose any smooth covering map $U \to X$. By Lemma 3.3, there exists an m such that $U \to X$ is of f-class m. Then by the argument in the proof of Lemma 3.1, the map $\Sigma_{m!}(U/X)$ is f-surjective. Then $\Sigma_{m!}(U/X)$ is an $(n-1)$-stack. Proceeding as in the proof of Lemma 3.1, we get a smooth covering map $X' \to X$ that is f-surjective and X' is an algebraic space. Thus it remains to construct an f-surjective smooth covering map $V \to X'$, where V is an affine scheme.

Let x be any point of X' and let K be the residue field of X' at x. Let $\overline{\{x\}}$ denote the closure of x in X' and let $Z \subset \overline{\{x\}}$ be an open dense subscheme of $\overline{\{x\}}$. Using Lemma 3.1 (or, more honestly, [14, Thm. II.6.4]), there exists a smooth (or even étale) morphism $V_x \to X'$ such that $x\colon \operatorname{Spec}(K) \to X$ can be lifted to V_x. Then it follows that there is a dense open subscheme $U_x \subset Z$ such that the immersion $U_x \to X'$ has a lift $U_x \to V_x$. In particular, $V_x \times_{X'} U_x \to U_x$ is f-surjective.

Now, we apply this construction to all the generic points of the top-dimensional components of X'. This gives us a dense open subscheme $U_0 \subset X'$ and a smooth map $V_0 \to X'$, the image of which contains U_0 and such that $V_0 \times_{X'} U_0 \to U_0$ is f-surjective. Then we apply this argument to all the generic points of the top-dimensional components of $X' \backslash U_0$. Proceeding in this manner and using the fact that X' is a noetherian space, we get the required result. ■

Corollary 3.5 *Let X be an sft-Artin stack over a complete discrete valuation ring A. Let α be a uniformizing parameter in A. Then X has an atlas $U \to X$, where U is an affine scheme of finite type over A such that the maps $U(A) \to \pi_0(X(A))$ and $U(A/\alpha^{n+1}) \to \pi_0(X(A/\alpha^{n+1}))$ are surjective for all $n \geq 0$.*

Proof Let K denote the residue field of A. Using Lemma 3.4, there exists an f-surjective smooth atlas $U \to X$ such that U is an affine scheme.

We prove that $U(A) \to \pi_0(X(A))$ is surjective. Indeed, pick any morphism $t\colon \operatorname{Spec}(A) \to X$. It suffices to show that the smooth stack $U_t : U \times^h_{X,t} \operatorname{Spec}(A)$ has an A-valued point. By construction, U_t has a K-valued point u. We construct an atlas $V_t \to U_t$ such that V_t is a (smooth) affine scheme over A and such that u lifts

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to V_t. Now we already know by Hensel's lemma that this lift can be extended to an A-valued point of V_t that gives an R-valued point of U_t. Thus $U(A) \to \pi_0(X(A))$ is surjective. The proof for the maps $U(A/\alpha^{n+1}) \to \pi_0(X(A/\alpha^{n+1}))$ is similar. ■

Remark 3.6 If, in Lemma 3.5, K is a finite field, the result follows from Lemma 3.1 itself, since then $\pi_0(X(K))$ is known to be a finite set by [20, Prop. 3.5]. On the other hand, once we have Lemma 3.4, we get an alternative proof of the fact that $\pi_0(X(K))$ is finite. Indeed, choose a smooth atlas $U \to X$, where U is an affine scheme. Then by Lemma 3.3, it follows that there exists a finite algebraic extension L of K such that any K-valued point of X lifts to an L-valued point of U. But the number of L-valued points of U is known to be finite.

4 p-adic Measure on Artin Stacks

We recall the notation from Section 1 that R is a complete discrete valuation ring with a finite residue field k of cardinality $q = p^r$, ω is a uniformizing parameter in R and $R_n := R/\langle \omega^{n+1} \rangle$ for each $n \geq 0$. In this section, we examine the numbers $\#X(R_n)$ for $n \geq 0$. We also define a p-adic measure on $\pi_0(X(R))$. For this, we view $\pi_0(X(R))$ as a locally compact topological space by the quotient topology given by the map $U(R) \to X(R)$, where $U \to X$ is an f-surjective smooth atlas with U being an affine scheme. It is easily seen that this topology is independent of the choice of the atlas $U \to R$, since any two such atlases $U_1 \to X$ and $U_2 \to X$ have a common refinement (for example, choose a f-surjective smooth covering map $U_3 \to U_1 \times^h_X U_2$). The p-adic measure will be a Borel measure on this space.

For an stf-Artin stack X over k, we note that the counting formula (2.1) defines a measure $\pi_0(X(k))$. For any subset $A \subset \pi_0(X(k))$, we denote this measure by $\#A$. To be precise, we write

$$\#A := \sum_{x \in A} \prod_{i>0} |\pi_i(X(k), x)|^{(-1)^i}. \tag{4.1}$$

In the following discussion, $|\cdot|$ will continue to denote the cardinality of a set, even if it is a subset of $\pi_0(X(k))$.

Lemma 4.1 *Let $p\colon F \to G$ be a morphism of sft-Artin stacks over k. Let $y \in \pi_0(G(k))$. Let $F_y = F \times^h_G \operatorname{Spec}(k)$. Then*

$$\#p^{-1}(y) = \big(\#F_y(k)\big) \cdot \big(\#\{y\}\big)$$

where $p^{-1}(y) = \{x \in \pi_0(F(k)) | p(x) = y\}$.

Proof Let $i_y\colon F_y \to F$ be projection morphism. Let $x \in p^{-1}(y)$. Let $x' \in F(k)$ such that $i_y \circ x' \cong x$. Then by the long exact sequence of homotopy groups corresponding to the fibration sequence $F_y(k) \to F(k) \to G(k)$, we have

$$|i_y^{-1}(x)| \cdot \prod_{i=1}^{\infty} \Big(\frac{|\pi_i(F_y(k), x')| \cdot |\pi_i(G(k), y)|}{|\pi_i(F(k), x)|} \Big)^{(-1)^i} = 1.$$

Thus $(\#\{x'\})\cdot(\#\{y\}) = |i_y^{-1}(x)|^{-1}\cdot(\#\{x\})$. Summing up over all $x'\in i_y^{-1}(x)$, we get

$$\left(\#i_y^{-1}(x)\right)\cdot\left(\#\{y\}\right) = \#\{x\}.$$

Summing up over all $x\in p^{-1}(y)$, we get

$$\left(\#F_y(k)\right)\cdot\left(\#\{y\}\right) = \#p^{-1}(y). \qquad \blacksquare$$

Proposition 4.2 *Let X be a smooth sft-Artin stack over R with* $\dim(X/R) = d$. *Let $n\geq 0$ be an integer and let $x\in\pi_0(Gr_n(X)(k))$. Then $\#(\tau_n^{n+1})^{-1}(x) = q^d\cdot\#\{x\}$.*

Proof Suppose X be an sft-Artin stack that is m-geometric. We prove the result by induction on m. When $m=-1$, *i.e.*, when X is an affine scheme, the result is well known.

Suppose the result has been proved for stacks that are m'-geometric for $m'\leq m$. Let $f\colon U\to X$ be a f-surjective smooth atlas such that U is an affine scheme with $\dim(U/R) = e$. Let $x'\in(\tau_n^{n+1})^{-1}(x)$ and let $\widetilde{x}$ be an element of $(\tau_{n+1})^{-1}(x')$ (it is easy to see that $\widetilde{x}$ exists because X is smooth). Let $F = U\times^h_{X,\widetilde{x}}\operatorname{Spec}(R)$. Then F is a smooth sft-Artin stack over R that is $(m-1)$-geometric.

We have

$$\#Gr_n(f)^{-1}(x) = \left(\#\pi_0(Gr_n(F)(k))\right)\cdot\left(\#\{x\}\right)$$

and, similarly,

$$\#Gr_{n+1}(f)^{-1}(x') = \left(\#\pi_0(Gr_{n+1}(F)(k))\right)\cdot\left(\#\{x'\}\right).$$

Then by the induction hypothesis, we have

$$\#\pi_0(Gr_{n+1}(F)(k)) = q^{(e-d)}\cdot\#\pi_0(Gr_n(F)(k)).$$

Thus we have

$$\#Gr_{n+1}(f)^{-1}(x') = q^{(e-d)}\cdot\#Gr_n(p)^{-1}(x)\frac{\#\{x'\}}{\#\{x\}}.$$

Letting x' vary over the set $(\tau_n^{n+1})^{-1}(x)$ and summing up, we get

$$\#Gr_{n+1}(f)^{-1}\left((\tau_n^{n+1})^{-1}(x)\right)| = q^{(e-d)}\cdot\#Gr_n(f)^{-1}(x)\frac{\#(\tau_n^{n+1})^{-1}(x)}{\#\{x\}},$$

but

$$Gr_{n+1}(f)^{-1}\left((\tau_n^{n+1})^{-1}(x)\right) = \left(\tau_n^{n+1}\right)^{-1}\left(Gr_n(f)^{-1}(x)\right).$$

Thus, since we know the result to be true for $m=-1$, we have

$$\left|Gr_{n+1}(f)^{-1}((\tau_n^{n+1})^{-1}(x))\right| = q^e\cdot\left|Gr_n(f)^{-1}(x)\right|$$

which completes the proof. $\blacksquare$

Now let X be an arbitrary sft-Artin stack over R with $\dim(X/R) = d$. Let $f\colon U\to X$ be an f-surjective smooth atlas where U is an affine scheme with $\dim(U/R) = e$. For any $s\in\pi_0(X(k))$, let $\overline{U}_s := U\times^h_{X,s}\operatorname{Spec}(k)$ and let $m_s := \#U_s(k)$. Note that $m_s\neq 0$ for all s.

Let A be any subset of $\pi_0(Gr_n(X)(k))$. Let A_s denote the set $A \cap (\tau_0^n)^{-1}(s)$ so that $A = \coprod_{s\in\pi_0(X(k))} A_s$. By Lemma 4.1 and Proposition 4.2, we have

$$|Gr_n(f)^{-1}(A_s)| = q^{n(e-d)} \cdot m_s \cdot \#A_s.$$

Thus,

$$\begin{aligned} q^{-nd} \cdot \#A &= q^{-nd} \cdot \Big(\sum_{s\in\pi_0(X(k))} \#A_s \Big) \\ &= q^{-nd} \cdot \Big(\sum_{s\in\pi_0(X(k))} \frac{|Gr_n(f)^{-1}(A_s)|}{m_s \cdot q^{-n(e-d)}} \Big) \\ &= \sum_{s\in\pi_0(X(k))} \Big(\frac{1}{m_s}\Big) \cdot \Big(\frac{|Gr_n(f)^{-1}(A_s)|}{q^{-ne}} \Big). \end{aligned}$$

Similarly, if $B \subset \pi_0(X(R))$, we define $B_s := B \cap \tau_0^{-1}(s)$ for every $s \in \pi_0(X(k))$. Then it is clear that $\tau_n(B_s) = \tau_n(B)_s$. Thus, we get

$$q^{-nd} \cdot \#(\tau_n(B)) = \sum_{s\in\pi_0(X(k))} \Big(\frac{1}{m_s}\Big) \cdot \Big(\frac{|Gr_n(f)^{-1}(\tau_n(B_s))|}{q^{-ne}} \Big). \tag{4.2}$$

This leads us to define p-adic measure as follows.

Definition 4.3 With the above notation, note that a set $B \subset \pi_0(X(R))$ is a Borel subset if and only if $f^{-1}(B) \subset U(R)$ is a Borel subset. We define the p-adic measure of such a set by

$$\mu_d^f(B) := \sum_{s\ inS} \Big(\frac{1}{m_s}\Big) \cdot \mu_e\big(f^{-1}(B_s)\big).$$

Lemma 4.4 *With the above notation, μ_d^f is independent of the choice of f.*

Proof This almost follows from the definition. Indeed, the limit of the left-hand side of equation (4.2) as $n \to \infty$, if it exists, clearly does not depend on f. This limit exists if the limit of the right-hand side of equation (4.2) exists as $n \to \infty$. This is clearly so if B is an open subset (for the topology described at the beginning of this section). Indeed, in this case, for any s, the set B_s is a sub-analytic set. Thus, by equation (1.1), the right-hand side converges.This shows that the result is true when B is open.

If B is a Borel subset, so is $f^{-1}(B)$. Since Borel subsets are outer regular for the p-adic measure on $U(R)$, we see that μ_d^f is independent of f on an arbitrary Borel subset of $\pi_0(X(R))$. ∎

The arguments above have given us the following result, which we restate explicitly.

Theorem 4.5 *Let X be an sft-Artin stack over R with $\dim(X/R) = d$. Then the sequences*

$$\{q^{-nd}\#\pi_0(X(R_n))\}_{n=0}^{\infty} \quad \textit{and} \quad \{q^{-nd}\#\tau_n(\pi_0(X(R)))\}_{n=0}^{\infty}$$

both converge to $\mu_d(\pi_0(X(R)))$.

Finally, we look at the power series $\widetilde{P}_X(T)$ and $P_X(T)$.

Theorem 4.6 *Let X be an sft-Artin stack over R. Then the power series $P_X(T)$ and $\widetilde{P}_X(T)$ are rational functions of T.*

Proof With the notation as above, writing $A_n := \pi_0(X(R_n)) = \pi_0(Gr_n(X)(k))$, we have the equalities

$$\begin{aligned}\widetilde{P}_X(T) &= \sum_{s\in\pi_0(X(k))} \sum_{n=0}^{\infty} \#(A_n)_s T^n \\ &= \sum_{s\in\pi_0(X(k))} \frac{1}{m_s \cdot p^{n(e-d)}} \sum_{n=0}^{\infty} |Gr_n(f)^{-1}((A_n)_s)| T^n.\end{aligned}$$

Now it follows from [8, Thm. 4.1] that $\widetilde{P}_X(T)$ is a rational function of T. The proof for $P_X(T)$ is similar. ■

Remark 4.7 It has been proved in [12] that zeta functions arising from definable equivalence relations are rational. François Loeser had posed the question of whether a notion p-adic measure on stacks may lead to an alternative method for proving such results. (This question was also raised by the referee.) The construction in this paper is probably not suitable for exploring such a connection. Indeed, if X is an Artin stack and $U \to X$ is an atlas, we do obtain an equivalence relation on the points of U, but not every definable equivalence relation occurs in this manner. It may be worth exploring whether the ideas in this paper can be adapted to define a notion of p-adic measure on more general stack-like objects that may be more suitable for addressing these issues.

While the above argument is adequate to establish the rationality of $P_X(T)$, in order to prove a "uniform rationality" theorem, it is useful to view the power series $P_X(T)$ a little differently. We recall the definition of the singular locus of a stack.

Definition 4.8 Let X be an sft-Artin stack over an affine scheme S. Then the singular locus X_{sing} is a closed substack of X defined as follows:

(i) If X is an affine scheme of dimension d over S, then X_{sing} is the closed subscheme of X defined by the d-th Fitting ideal of $\Omega_{X/S}$ (the module of relative differentials of X over S.

(ii) In general, let $f\colon U \to X$ be a smooth atlas with U being an affine scheme of finite type over S. Then X_{sing} is the closed substack of X which is the image of $U_{\text{sing}} \to X$.

In (ii) above, it is easy to check that the image stack of $U_{\text{sing}} \to X$ is a closed substack of X and that the definition of X_{sing} does not depend on the choice of the atlas f.

Definition 4.9 Let X be an sft-Artin stack over R and let X_{sing} denote its singular locus over R. Then the power series $Q_X(T)$ is defined as

$$Q_X(T) := P_X(T) - P_{X_{\text{sing}}}(T).$$

The reason that the power series $Q_X(T)$ is useful is that the problem of proving the rationality of $P_X(T)$ is equivalent to that of proving the rationality of $Q_X(T)$. Indeed, suppose we know that the power series $Q_X(T)$ is a rational function for any sft-Artin stack over R. Then we can prove the rationality of $P_X(T)$ by noetherian induction on the closed substacks of X. Indeed, if $P_{X_{\text{sing}}}(T)$ and $Q_X(T)$ are both known to be rational functions, it follows that $P_X(T)$ is a rational function. The advantage here is that the coefficients of $Q_X(T)$ have a simple description in terms of p-adic measure.

Lemma 4.10 *Let X be an sft-Artin stack over R with* $\dim(X/R) = d$. *Then* $Q_X(T) = \sum_{n=0}^{\infty} q^{nd}\mu_d(M_n)T^n$, *where M_n is the subset of $\pi_0(X(R))$ given by*

$$M_n = \pi_0\big(X(R)\big)\setminus\tau_{n,X}^{-1}\big(X_{\text{sing}}(R_n)\big).$$

Proof Let $S_n := \tau_n(\pi_0(X(R)))\setminus\tau_n(\pi_0(X_{\text{sing}}(R))) \subset \pi_0(X(R_n))$ such that the coefficient of T^n in $Q_X(T)$ is $\#S_n$. We wish to prove that $\mu_d(M_n) = \#S_n/q^{nd}$. Note that $M_n = \tau_n^{-1}(S_n)$.

First suppose that X is an affine scheme. Then it is known that for any n, $x \in S_n$ implies that $\#[(\tau_n^{n+1})^{-1}(x) \cap \tau_n(\pi_0(X(R)))] = q^d$ (for example, see the argument in [16, Lemma 9.1]).

The case of a general sft-Artin stack follows from equation (4.2) and our definition of p-adic measure. ■

5 Motivic Measure on Stacks

In this section k will denote a field of characteristic zero. Let Field_k denote the category of field extensions of k. We will now associate a "motivic measure" to definable subassignments on Artin stacks over $k[[t]]$. In doing so, we will assume familiarity with the theory of motivic integration as presented in [5]. For the sake of completeness, we recall some of the definitions and notation from that work. We are essentially reproducing the summary from [6, Section 2] while incorporating the changes that are necessary for our setting.

For any field extension K of k, we consider the power series ring $K[[t]]$. This is a discrete valuation ring with valuation $\text{ord}\colon K[[t]]\setminus\{0\} \to \mathbb{Z}$. Let $\overline{\text{ac}}\colon K[[t]] \to K$ be the "angular component" map, *i.e.*,

$$\overline{\text{ac}}(x) = \begin{cases} xt^{-\operatorname{ord}(x)} \mod t & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

In order to work with these power series rings, we use the language of Denef–Pas, which we denote by $\mathcal{L}_{\text{LD,P}}$. This is a 3-sorted language

$$\mathcal{L}_{\text{LD,P}} := (\mathbf{L}_{\text{Val}}, \mathbf{L}_{\text{Res}}, \mathbf{L}_{\text{Ord}}, \text{ord}, \overline{\text{ac}})$$

with three sorts, Val, Res, and Ord corresponding to the valuation ring, the residue field and the order group. $\mathbf{L}_{\mathrm{Val}}$ and $\mathbf{L}_{\mathrm{Res}}$ are equal to the language of rings $\{+,-,\cdot,0,1\}$, while the language $\mathbf{L}_{\mathrm{Ord}}$ is the Presburger language

$$\{+,-,0,1,\leq\}\cup\{\equiv_n |m\in\mathbb{N},n>1\},$$

where $\equiv_n$ is interpreted as congruence modulo n.

If $\mathcal{C}$ is a category and $F\colon \mathcal{C}\to \mathrm{Sets}$ is a functor, a *subassignment* h of F is a rule that assigns a subset $h(C)\subset F(C)$ for each object C of $\mathcal{C}$. For subassignments, the usual set-theoretic notions such as $\cup$, $\cap$, $\subset$, etc., are defined objectwise. In particular the subassignments of a fixed functor form a Boolean algebra.

We use this notion with $\mathcal{C}=\mathrm{Field}_k$. Consider a triple $(\mathcal{X},X,r)$, where $\mathcal{X}$ is an Artin stack over $k[[t]]$, X is an Artin stack over k, and $r\geq 0$ is an integer. Consider the functor

$$(\mathcal{X}\times X\times\mathbb{Z}^r)(K):=\pi_0\big(\mathcal{X}(K[[t]])\times\pi_0(X(K)\big)\times\mathbb{Z}^r.$$

When any of the elements in the triple $(\mathcal{X},X,r)$ are trivial (*i.e.*, $\mathcal{X}=\mathrm{Spec}(k[[t]])$, $X=\mathrm{Spec}(k)$, or $r=0$) we may abuse notation and simply omit to write them if there is no risk of confusion. When $\mathcal{X}=\mathbb{A}^n_{k[[t]]}$ and $X=\mathbb{A}^m_k$, the above functor is denoted by $h[m,n,r]$.

Remark 5.1 Note that we are considering $K[[t]]$-valued points when we define a subassignment, while in [5, 6] one considers $K((t))$-valued points. However, when one is working with a separated scheme X, $X(K[[t]])$ maps injectively into $X(K((t)))$ and thus we can apply the results regarding motivic measure on such schemes without any problems. We will use these results only for affine schemes.

We will now define what it means for a subassignment of such a functor to be definable in the language of Denef–Pas. Given such a triple $(\mathcal{X},X,r)$, we choose f-surjective smooth atlases $\mathcal{U}\to\mathcal{X}$ and $U\to X$ such that $\mathcal{U}$ and U are affine schemes over $k[[t]]$ and k respectively. Then consider the triple $(\mathcal{U},U,r)$. There is an obvious morphism of functors $f\colon \mathcal{U}\times U\times\mathbb{Z}^r\to\mathcal{X}\times X\times\mathbb{Z}^r$. We say that a subassignment of $\mathcal{X}\times X\times\mathbb{Z}^r$ is *definable* if and only if its preimage in $\mathcal{U}\times U\times\mathbb{Z}^r$ is definable by a formula in the language $\mathcal{L}_{\mathrm{LD,P}}$ with Val coefficients in $k[[t]]$ and Res coefficients in k. It is easy to see that this notion is independent of the choice of $\mathcal{U}\to\mathcal{X}$ and $U\to X$.

Suppose $(\mathcal{X},X,r)$ and $(\mathcal{Y},Y,s)$ are two triples with $\mathcal{X}$ and $\mathcal{Y}$ (resp. X and Y) being sft-Artin stacks over $k[[t]]$ (resp. k). Suppose we are given morphisms of stacks $\mathcal{X}\to\mathcal{Y}$ and $X\to Y$ along with a linear map $\mathbb{Z}^r\to\mathbb{Z}^s$. Then this induces a morphism of the functors $\mathcal{X}\times X\times\mathbb{Z}^r\to\mathcal{Y}\times Y\times\mathbb{Z}^s$. We call such a morphism a *geometric morphism*.

Now, we consider the category $\mathcal{D}$ whose objects are pairs $(S,(\mathcal{X},X,r))$, where $(\mathcal{X},X,r)$ is a triple as above and S is a definable subassignment of $\mathcal{X}\times X\times r$. A morphism $(S,(\mathcal{X},X,r))\to(T,(\mathcal{Y},Y,s))$ is a geometric morphism $\mathcal{X}\times X\times\mathbb{Z}^r\to\mathcal{Y}\times Y\times\mathbb{Z}^s$ that maps S into T. We say that morphism is an geometric equivalence if it induces a weak equivalence $S(K)\to T(K)$ for every field extension K of k and let us denote the class of geometric equivalences by $\mathcal{W}$. (Here we view $S(K)\subset\pi_0(\mathcal{X}(K[[t]])\times\pi_0(X(K))\times\mathbb{Z}^r$ not just as an ordinary set, but as a set whose objects are homotopy types.) Now consider the category $\mathcal{W}^{-1}\mathcal{D}$ obtained from $\mathcal{D}$ by

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localizing with respect to $\mathcal{W}$. It is easy to see that $\mathcal{W}$ is a left-multiplicative system of morphisms, thus the localization makes sense. Indeed, any morphism $S \to T$ in the localization is given by an equivalence $W \to S$ and a geometric morphism $W \to T$. We refer to the morphisms in $\mathcal{W}^{-1}\mathcal{D}$ as *definable morphisms*. (Note that unlike the case of definable subassignments of varieties, we cannot simply say that a morphism $f\colon S \to T$ is definable if its graph Γ_f is a definable subassignment of $S \times T$, since $\Gamma_f \to S \times T$ is not a monomorphism in general.)

In the above construction, if we restrict ourselves to the full subcategory of objects of the type $(S, (\mathbb{A}^n_{k[[t]]}, \mathbb{A}^m_k, r))$, where $n, m \geq 0$, we denote the resulting category by Def_k. (This is exactly the category Def_k defined in [5] except for the slight difference explained in Remark 5.1.) More generally, for any subassignment S, we denote by Def_S the category of subassignments contained in $S \times \mathbb{A}^n_{k[[t]]} \times \mathbb{A}^m_k \mathbb{Z}^r$, where $n, m \geq 0$. We denote by RDef_S the subcategory of Def_S consisting of subassignments of $S \times \mathbb{A}^m_k$ for $m \geq 0$. We denote by $K_0(\mathrm{RDef}_S)$ the corresponding Grothendieck ring (see [5, Section 5]).

Let A denote the ring $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}, \{(1 - \mathbb{L}^{-o})^{-1}\}_{i>0}]$. Let $\mathcal{P}(S)$ denote the ring of functions from the set of points of S into A generated by constant functions, definable functions from S into $\mathbb{Z}$, and functions of the form $\mathbb{L}^{\beta}$ with $\beta\colon S \to \mathbb{Z}$ being a definable morphism. We denote by $\mathcal{P}^0(S)$ the subring of $\mathcal{P}(S)$ generated by the characteristic functions of definable subassignments contained in S and the constant function $\mathbb{L}$. There is a natural ring homomorphism $\mathcal{P}^0(S) \to K_0(\mathrm{RDef}_S)$ sending $\mathbb{L}$ to the class of $S \times \mathbb{A}^1_k$ and sending the characteristic function $\mathbb{1}_T$ of a subassignment $T \subset S$ to the class of T itself (viewed as an element of RDef_S). Then the ring of constructible functions on S is defined by

$$\mathcal{C}(S) := K_0(RDef_S) \otimes_{\mathcal{P}^0(S)} \mathcal{P}(S).$$

Let $\mathcal{X}$ be an sft-Artin stack over $k[[t]]$ and let S be a definable subassignment contained in $\mathcal{X}$ (viewed as a definable subassignment itself). Then the Zariski closure W of S in $\mathcal{X}$ is the intersection of all closed substacks $\mathcal{Y} \subset \mathcal{X}$ such that $S \subset \mathcal{Y}$. We set $\dim(S) = \dim(\mathcal{Y}/\operatorname{Spec}(k[[t]])$. More generally, if S is a definable subassignment of a functor of the type $\mathcal{X} \times X \times \mathbb{Z}^r$ where $\mathcal{X}$ and X are sft-Artin stacks over $k[[t]]$ and k-respectively, then we define the dimension of S to be the dimension of its projection to $\mathcal{X}$. For every integer d, we denote by $\mathcal{C}^{\leq d}(S)$ the ideal of $\mathcal{C}(S)$ generated by the characteristic functions of subassignments $Z \subset S$ with $\dim(Z) \leq d$. This defines a filtration of $\mathcal{C}(S)$ and we denote the associated graded group by $C(S) := \bigoplus_d C^d(S)$ (the group of constructible motivic Functions—note the capital 'F'), where $C^d(S) := \mathcal{C}^{\leq d}(S)/\mathcal{C}^{\leq d-1}(S)$. Note that d can be negative but the set of d such that $\mathcal{C}^{\leq d}(S) \neq \{0\}$ is bounded below. For any $\phi \in \mathcal{C}(S)$, its image in $C(S)$ will be denoted by $[\phi]$.

Now suppose S is in Def_k and Z is in Def_S. Then one can define a subgroup $I_S C(Z)$ of $C(Z)$ together with pushforward morphisms

$$f_!\colon I_S C(Z) \to C(Y)$$

for every morphism $f\colon Z \to Y$ in Def_S (see [5, §10]). When S is simply the final object of Def_k, and f is the map $Z \to S$, then we write $f_!$ as μ and call it the motivic measure.

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More generally, if Λ is an object of Def_k, the above construction can be performed relative to Λ. Using relative dimension instead of dimension, we can obtain relative analogues $C(Z \to \Lambda)$ for a definable morphism $f\colon Z \to \Lambda$ in Def_Λ. In particular, we obtain a morphism

$$\mu_\Lambda\colon I_\Lambda C(Z \to \Lambda) \to \mathcal{C}(\Lambda) = I_\Lambda C(\Lambda \to \Lambda).$$

Note that $f_!$ and μ_Λ are usually distinct. However, when Λ is a subassignment of $\mathbb{A}_k^m$, the groups $I_\Lambda C(Z \to \Lambda)$ and $IC(Z)$ are identical and the morphisms μ_Λ and $f_!$ from $IC(Z)$ to $\mathcal{C}(\Lambda) = C(\Lambda)$ are the same. (See [5, Remark 14.2.3 and Theorem 10.1.1, Part (5)] for a clarification of this.) This will be made use of in the following arguments, particularly in Lemma 5.2 and in the proof of Theorem 4.6 (particularly where [5, Theorem 14.4.1] is applied).

This completes our review of the basic terminology and results. Now, in the following two lemmas, we will try to adapt the essence of equation (4.2) to motivic integration. For an sft-Artin stack $\mathcal{X}$ over $k[[t]]$, we will denote by $\overline{\mathcal{X}}$ the Artin stack $\mathcal{X} \times_{\mathrm{Spec}(k[[t]])} \mathrm{Spec}(k)$ over k. If $p\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of sft-Artin stacks over $k[[t]]$, let $\overline{p}$ denote the pullback $\overline{\mathcal{X}} \to \overline{\mathcal{Y}}$. Note that the morphism $\rho_{\mathcal{X}}\colon h[\mathcal{X}] \to \overline{\mathcal{X}}$ is definable.

Lemma 5.2 *Let $f\colon \mathcal{U} \to \mathcal{X}$ be an f-surjective smooth covering map of affine schemes over $k[[t]]$. Then $[\phi] \in I_{\overline{\mathcal{X}}}C(\mathcal{X} \to \overline{\mathcal{X}})$ if and only if $[f^*(\phi)] \in I_{\overline{\mathcal{U}}}C(\mathcal{U} \to \overline{\mathcal{U}})$. If these conditions hold, then*

$$\mu_{\overline{\mathcal{U}}}\big([f^*(\phi)]\big) = \overline{f}^*\big(\mu_{\overline{\mathcal{X}}}([\phi])\big).$$

Proof As we recalled (from [5]) in the discussion above, $[\phi] \in I_{\overline{\mathcal{X}}}C(X \to \overline{\mathcal{X}})$ if and only if $[\phi] \in IC(X)$. Thus we simply wish to prove that $[\phi] \in IC(\mathcal{X})$ if and only if $f^*([\phi]) \in IC(\mathcal{U})$.

First we note a consequence of the f-surjectivity of f that we will use below. Since f is f-surjective, so is $\overline{f}$. In particular, every generic point of $\overline{\mathcal{X}}$ can be lifted to $\overline{\mathcal{U}}$. Thus there is an open dense subscheme $X_0 \subset \overline{\mathcal{X}}$ and a $\overline{\mathcal{X}}$-morphism $X_0 \to \overline{\mathcal{U}}$. Repeating the procedure for $\overline{\mathcal{X}} - X_0$, we see that $\overline{f}$ has a definable (in the language of rings) section $s\colon \overline{\mathcal{X}} \to \overline{\mathcal{U}}$.

We will now prove that $[\phi] \in IC(\mathcal{X})$ if and only if $f^*([\phi]) \in IC(\mathcal{U})$. First we observe that due to the [5, Theorem 10.1.1, Part (A3)] (the "projection formula"), for $\alpha \in \mathcal{C}(\mathcal{X})$ and $\beta \in IC(\mathcal{U})$, $\alpha f_!(\beta) \in IC(\mathcal{X})$ if and only if $f^*(\alpha)\beta \in IC(\mathcal{U})$ and that if these conditions are verified, then $f_!(f^*(\alpha)\beta) = \alpha f_!(\beta)$. We apply this with $\beta = [\mathbb{1}_{\mathcal{U}}]$ and $\alpha = [\phi]$. Since f is a smooth morphism, it is easy to see that $f_!([\mathbb{1}_{\mathcal{U}}]) = \rho_{\mathcal{X}}^*(\overline{f}_!(\mathbb{1}_{\overline{\mathcal{U}}}))$. Thus, $f^*([\phi]) = f^*([\phi])\mathbb{1}_{\mathcal{U}} \in IC(\mathcal{U})$ if and only if $\rho_{\mathcal{X}}^*(\overline{f}_!(\mathbb{1}_{\overline{\mathcal{U}}}))[\phi] \in IC(\mathcal{X})$.

Suppose $[\phi] \in IC(\mathcal{X})$. Then we apply the projection formula mentioned above to the morphism $\rho_{\mathcal{X}}$. We see, as a result, that $\rho_{\mathcal{X}}^*(\overline{f}_!(\mathbb{1}_{\overline{\mathcal{U}}}))[\phi] \in IC(\mathcal{X})$ if and only if $\overline{f}_!(\mathbb{1}_{\overline{\mathcal{U}}})(\rho_{\mathcal{X}})_!([\phi]) \in IC(\overline{\mathcal{X}})$. But the latter condition is trivial since $IC(\overline{\mathcal{X}}) = \mathcal{C}(\overline{\mathcal{X}})$. Thus we see that $\rho_{\mathcal{X}}^*(\overline{f}_!(\mathbb{1}_{\overline{\mathcal{U}}}))[\phi] \in IC(\mathcal{X})$.

We now prove the converse; we assume that $\rho_{\mathcal{X}}^*(\overline{f}_!(\mathbb{1}_{\overline{\mathcal{U}}}))[\phi] \in IC(\mathcal{X})$ and show that $[\phi] \in IC(\mathcal{X})$. To prove this, we can assume that $\phi \in \mathcal{C}_+(X)$, the semiring of positive constructible motivic functions on $\mathcal{X}$ (see [5, Section 5] for the definition). Let $\gamma = [\overline{\mathcal{U}}\backslash s(\overline{\mathcal{X}})] \in K_0(RDef_{\overline{\mathcal{X}}})$ so that $\overline{f}_!(\mathbb{1}_{\overline{\mathcal{U}}}) = \gamma + [\overline{\mathcal{X}}]$. Thus we have

$$\rho_{\mathcal{X}}^*\big(\overline{f}_!(\mathbb{1}_{\overline{\mathcal{U}}})\big)\,[\phi] = \gamma[\phi] + [\phi]$$

and $\gamma[\phi] \in C_+(\mathcal{X})$, the semigroup positive constructible motivic Functions on $\mathcal{X}$. Now apply [5, Thm 12.2.1], to conclude that $[\phi] \in IC(\mathcal{X})$.

Finally, we prove the equality in the statement of the lemma. (For this part, it is not necessary to assume that f is f-surjective.) It suffices to choose a suitable cover $\{\mathcal{U}_i\}_i$ of $\mathcal{U}$ by affine open subschemes $\mathcal{U}_i$ and prove the result after replacing $\mathcal{X}$ by $f(\mathcal{U}_i)$ and $\mathcal{U}$ by $\mathcal{U}_i$. By choosing the pieces of the cover to be sufficiently small, we can assume that f factors as

$$\mathcal{U} \xrightarrow{f'} \mathbb{A}_{\mathcal{X}}^{\dim(\mathcal{U}/\mathcal{X})} \longrightarrow X,$$

where f' is étale and the morphism $\mathbb{A}_{\mathcal{X}}^{\dim(\mathcal{U}/\mathcal{X})} \to X$ is the projection. Thus it suffices to prove the result for étale maps and maps of the form $\mathbb{A}_{\mathcal{X}}^{\dim(\mathcal{U}/\mathcal{X})} \to X$. The latter case is trivial. Thus we now consider the case when f is étale. But in this case, it is clear that $g\colon \mathcal{U} \to \overline{\mathcal{U}} \times_{\overline{\mathcal{X}}} \mathcal{X}$ is a definable isomorphism with ordjac(g) $= 0$ (see [5, Section 8 and Thm. 12.1.1]). ∎

This leads us to the following definition.

Definition 5.3 Let $\mathcal{X}$ be an sft-Artin stack over $k[[t]]$ and let $f\colon \mathcal{U} \to \mathcal{X}$ be an f-surjective smooth atlas with $\mathcal{U}$ being an affine scheme of finite type over $k[[t]]$. Then we define

$$IC(\mathcal{X}) = \{[\phi] \in C(\mathcal{X}) | [f^*(\phi)] \in IC(\mathcal{U})\}.$$

It follows from the preceding lemma that $IC(\mathcal{X})$ is independent of the choice of f.

Lemma 5.4 *Let $\mathcal{X}$ be an sft-Artin stack over $k[[t]]$, and let $[\alpha] \in IC(\mathcal{X})$. Let $f\colon \mathcal{U} \to \mathcal{X}$ be a smooth covering map, where $\mathcal{U}$ is an affine scheme of finite type over $k[[t]]$. Let ϕ be the constructible function $\mu_{\overline{\mathcal{U}}}([f^*(\alpha)]) \in \mathcal{C}(\overline{\mathcal{U}})$. Let Z be any affine scheme of finite type over k, let $x\colon Z \to \overline{\mathcal{X}}$ by any morphism, and let $u\colon Z \to \overline{\mathcal{U}}$ be such that $\overline{p} \circ u \cong x$. Then the element*

$$u^*(\phi) \in \mathcal{C}_+(Z)$$

depends only on x and α, i.e., *it is independent of the choice of $\mathcal{U}$ and u.*

Proof We can assume that Z is irreducible. Suppose $f_1\colon \mathcal{U}_1 \to \mathcal{X}$ and $f_2\colon \mathcal{U}_2 \to \mathcal{X}$ are two choices for the atlas as mentioned in the statement of the lemma. Let $\phi_i := \mu_{\overline{\mathcal{U}}}([f_i^*(\alpha)]) \in \mathcal{C}(\overline{\mathcal{U}_i})$. Let $u_i\colon Z \to \overline{\mathcal{U}}_i$ be a lift of x to $\overline{\mathcal{U}_i}$ for $i = 1, 2$. We wish to prove that $u_i^*(\phi_i) \in \mathcal{C}_+(Z)$ is the same element for $i = 1, 2$.

There exists a morphism $v\colon Z \to \overline{\mathcal{U}_1 \times_{\mathcal{X}} \mathcal{U}_2}$ that lifts u_1 and u_2. Let η denote the generic point of Z. Choose a smooth atlas $\mathcal{U}_3 \to \mathcal{U}_1 \times_{\mathcal{X}}^h \mathcal{U}_2$ such that $\mathcal{U}_3$ is an affine scheme of finite type over $k[[t]]$ and such that $v|_\eta$ can be lifted to U_3. Then there is an open subscheme $Z_0 \subset Z$ such that the morphisms $u_i|_{Z'}$ have a common lift to U_3. It will suffice to prove the result with Z_0 in place of Z. Indeed, if we can do this, we can repeat the procedure for $Z_1 = Z\backslash Z_0$. Proceeding in this manner, we can get the required result, since Z is noetherian. Thus we can assume that the morphisms $u_i\colon Z \to U_i$ for $i = 1,2$ have a common lift $u_3\colon Z \to U_3$.

Let $g_i\colon U_3 \to U_i$ for $i = 1,2$ be the obvious maps and let $f_3 := f_1 \circ g_1 = f_2 \circ g_2$. Let $\phi_3 := \mu_{\overline{\mathcal{U}}}([f_3^*(\alpha)]) \in \mathcal{C}(\overline{\mathcal{U}_3})$. Then it is clearly enough to show that $u_1^*(\phi_1) = u_3^*(\phi) = u_2^*(\phi_2)$. In other words, it is enough to prove the lemma assuming that $\mathcal{X}$ is an affine scheme.

Now assume that $\mathcal{X}$ is an affine scheme and that $f\colon \mathcal{U} \to \mathcal{X}$ is a smooth covering map of affine schemes. It will suffice to prove the result when $Z = \overline{\mathcal{U}}$. In other words, we wish to prove that

$$\phi = \overline{f}^*\left(\mu_{\overline{\mathcal{X}}}([\alpha])\right).$$

This is precisely the content of the previous lemma. ∎

Thus, with the notation of the above lemma, we have a rule that, for every affine scheme Z of finite type over k and a morphism $x\colon Z \to \overline{\mathcal{X}}$, assigns a constructible function $\phi_x \in \mathcal{C}(Z)$ in a coherent manner. This leads us to the following definition.

Definition 5.5 Let X be a sft-Artin stack over k. A *constructible pseudo-function on* X is a rule ϕ which for every affine scheme Z of finite type over k and morphism $x\colon Z \to X$ assigns an element $\phi_x \in \mathcal{C}(Z)$ such that if we have a commutative diagram (up to homotopy)

$$\begin{tikzcd} Z_2 \arrow[r, "\alpha"] \arrow[dr, "x_2"'] & Z_1 \arrow[d, "x_1"] \\ & X, \end{tikzcd}$$

then $\alpha^*(\phi_{x_1}) = \phi_{x_2}$.

We abuse notation and write $x^*(\phi)$ instead of ϕ_x, even though ϕ is not a constructible function on X in the usual sense. The constructible functions on X clearly from a ring, which we denote by $\mathcal{C}(X)^{ps}$.

Lemma 5.6 *Suppose X is an sft-Artin stack over k and $\phi \in \mathcal{C}(X)^{ps}$. Let $f\colon U \to X$ be an f-surjective morphism (not necessarily smooth) with U being an affine scheme of finite type over k. Then ϕ is completely determined by $f^*(\phi)$.*

Proof Indeed, suppose Z is an affine scheme of finite type over k and $x \to X$ is a morphism. We wish to compute $x^*(\phi)$. We can assume that Z is irreducible. If η is its generic point, then we can lift $x|_\eta$ to U. Thus there exists an open subscheme $Z_0 \subset Z$ such that $x|_{Z_0}$ can be lifted to U. Then, as in the proof of Lemma 5.4, the fact that Z is noetherian allows us to compute ϕ_x. It is easy to see that the calculation does not depend on the choice of Z_0. ∎

Using Lemmas 5.4 and 5.6, we have the following definition.

Definition 5.7 Let $\mathcal{X}$ is a sft-Artin stack over $k[[t]]$. Let $f\colon \mathcal{U} \to \mathcal{X}$ be an f-surjective atlas with $\mathcal{U}$ being an affine scheme of finite type over $k[[t]]$. We denote by $\mu_{\overline{\mathcal{X}}}\colon IC(\mathcal{X}) \to \mathcal{C}(\overline{\mathcal{X}})^{ps}$ the group homomorphism that maps an element $[\alpha] \in IC(\mathcal{X})$ to the element of $\mathcal{C}(\overline{\mathcal{X}})^{ps}$ determined by the element $\mu_{\overline{\mathcal{U}}}([f^*(\alpha)]) \in \mathcal{C}(\overline{\mathcal{U}})$.

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At this point, we pause for some remarks to clarify some of the choices we have made in this construction.

Remark 5.8 Obviously, the above definition is not quite satisfactory, since we expect $\mu_{\overline{\mathcal{X}}}$ to take values in the ring $\mathcal{C}(\overline{\mathcal{X}})$. Indeed, if we were able to use $\mathcal{C}(\overline{\mathcal{X}})$, then we would be able to use the obvious pushforward from the ring $\mathcal{C}(\overline{\mathcal{X}})$ into the ring $\mathcal{C}(\operatorname{Spec}(k[[t]] \times \operatorname{Spec}(k))^{st}$, which would give us motivic measure in the usual sense. By using the ring $\mathcal{C}(\overline{\mathcal{X}})^{ps}$ instead, we find that our measure takes values in a ring that depends on the stack. The reason for doing this is that we do not know if the obvious ring homomorphism $\mathcal{C}(\overline{\mathcal{X}}) \to \mathcal{C}(\overline{\mathcal{X}})^{ps}$, mapping $\phi \in \mathcal{C}(\overline{\mathcal{X}})$ to $f^*(\phi) \in \mathcal{C}(\overline{\mathcal{U}})$, is an isomorphism. We do know this to be the case when $\mathcal{X}$ is an algebraic space (in which case the fact that this homomorphism is an isomorphism can be proved by choosing a "definable section" for the morphism $\overline{\mathcal{U}} \to \overline{\mathcal{X}}$). But we do not know if this is true in general and hence our usage of the ring $\mathcal{C}(\overline{\mathcal{X}})^{ps}$ is essentially a compromise. We note that when $\mathcal{X}$ is an affine scheme, we are using the symbol $\mu_{\overline{\mathcal{X}}}$ to denote two maps, one with domain $\mathcal{C}(\overline{\mathcal{X}})^{ps}$ and the other with domain $\mathcal{C}(\overline{\mathcal{X}})$. However, these two maps can be identified via $\mathcal{C}(\overline{\mathcal{X}}) \xrightarrow{\sim} \mathcal{C}(\overline{\mathcal{X}})^{ps}$ (which is known to be an isomorphism in this case), and so this is a harmless abuse of notation.

However, this construction is good enough to allow us to obtain the uniformity results relating p-adic integration for various primes p when we are working with an sft-Artin stack defined over the ring of integers in a number field (see Theorems 6.2 and 4.6).

6 Specialization to p-adic Integration

We now return to the problem of applying the theory of motivic integration to p-adic integration on sft-Artin stacks. We briefly recall the process of specialization to p-adic integration using [6] as our reference.

Let k be a number field with $\mathcal{O}$ being its ring of integers. Recall the terminology from [6, Section 9] that the language $\mathcal{L}_{\mathcal{O}}$ is the language $\mathcal{L}_{\mathrm{LD,P}}$ with Val-type constants added for the elements of $\mathcal{O}[[t]]$ and Res-type constants added for the elements of $\mathcal{O}$. Then we may consider definable subassignments and constructible functions that are definable in the language $\mathcal{L}_{\mathcal{O}}$. As before, for a subassignment S definable in $\mathcal{L}_{\mathcal{O}}$, we can define the rings $K_0(\mathrm{RDef}_S, \mathcal{L}_{\mathcal{O}})$ and $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$.

Let K be a p-adic completion of k with valuation ring R_K and residue field k_K. Let ω_K be a uniformizing parameter in R_K. Then one has a $\mathcal{O}$-algebra homomorphism $\lambda_{\mathcal{O},K} \colon \mathcal{O}[[t]] \to K$ defined by

$$\lambda_{\mathcal{O},K}\Big(\sum_{i\geq 0} a_i t^i\Big) = \sum_{i\geq 0} a_i \omega^i.$$

Also, for every α in $\mathcal{O}$, let $\overline{\alpha}$ denote the image of α under the quotient map $\mathcal{O} \to k_K$. In an $\mathcal{L}_{\mathcal{O}}$ formula ϕ, if we interpret every Val-type constant $a \in \mathcal{O}[[t]]$ as $\lambda_{\mathcal{O},K}(a) \in K$, and every Res-type constant $\alpha \in \mathcal{O}$ as $\overline{\alpha} \in k_K$, then ϕ defines a subset ϕ_K of $R_K^m \times k_K^n \times \mathbb{Z}^r$ for some non-negative integers n, m, r. We recall ([6] or [9]) that if two formulas ϕ and ψ define the same subassignment S of $\mathbb{A}^m_{k[[t]]} \times \mathbb{A}^n_k \times \mathbb{Z}^r$, then the subsets ϕ_K and ψ_K of $R_K^m \times k_K^n \times \mathbb{Z}^r$ are equal for almost all choices of K. One abuses notation to denote this set by S_K. Similarly, definable morphisms $f\colon S \to T$ give functions $f_K\colon S_K \to T_K$ for almost all K.

Similarly, a constructible function on S can be interpreted to give a function from S_K into $\mathbb{Q}$ for almost all K. For this we first interpret the elements of $K_0(\mathrm{RDef}_S, \mathcal{L}_{\mathcal{O}})$, and then the elements of $\mathcal{P}(S)$, as functions on S_K.

If $\phi \in K_0(\mathrm{RDef}_S, \mathcal{L}_{\mathcal{O}})$ is such that it is represented by $[\pi\colon W \to S$ with $W \in S \times \mathbb{A}^n_k$. Then W_K is in $S_K \times k^n_K$, and we have the projection map $\pi_K\colon W_K \to S_K$. We define the function ϕ_K on X_K by $\phi_K(x) := |\pi_K^{-1}(x)|$. Then one extends this construction by linearity to the whole of $K_0(\mathrm{RDef}_S, \mathcal{L}_{\mathcal{O}})$.

To interpret the elements of $\mathcal{P}(S)$ over K, we express such an element ϕ in terms of $\mathbb{L}$ and definable functions $\alpha\colon S \to \mathbb{Z}$. Then we interpret $\mathbb{L}$ as $q_K = |k_K|$ and α as a function $\alpha_K\colon S_K \to \mathbb{Q}$, which is well defined for almost all K. Now, we can interpret the elements of $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$ as functions on S_K by tensoring.

Now let X be an sft-Artin stack over $\mathcal{O}$. Let $X_t := X \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(k[[t]])$, which is a sft-Artin stack over $k[[t]]$. We claim that a definable subassignment S on X_t defines a subset S_K of $\pi_0(X(R_K))$ for almost all K. Indeed, we choose a f-surjective smooth atlas $f\colon U \to X$ with U being an affine scheme of finite type over $\mathcal{O}[[t]]$. Let $V \to U \times^h_X U$ be an f-surjective smooth atlas, where V is an affine scheme of finite type over $\mathcal{O}$. Let $g_1, g_2\colon V \to U$ be the maps obtained by composing $V \to U \times^h_X U$ with the two projections $U \times^h_X U \to U$. Then $f^{-1}(S)$ is a definable subassignment on U_t that defines a subset given by a formula ϕ. Let ψ_i be the pullback of the formula ϕ via g_i for $i = 1, 2$. Clearly, ψ_1 and ψ_2 define the same subassignment on V_t. The formula ϕ defines a subset ϕ_K of $U(R_K)$. This is the preimage of a subset of $\pi_0(X(R_K))$ if and only if the subsets $(\psi_1)_K$ and $(\psi_2)_K$ are equal. But we know that this is true for almost all K. Thus we see that a definable subassignment on X_t defines a subset S_K of $\pi_0(X(R_K))$ for almost all K. By similar arguments, one can interpret constructible functions ϕ on S_K to give functions $S_K \to \mathbb{Q}$ for almost all K.

For any affine scheme T of finite type over $\mathcal{O}$, we note that the set $(\overline{T_t})_K$ is simply the set $T(k_K)$ of k_K valued points on T. Indeed, T can be defined as a closed subscheme of $\mathbb{A}^n_{\mathcal{O}}$ cut out by polynomials with coefficients in $\mathcal{O}$. The same polynomials can be used to define T_t as a closed subscheme of $\mathbb{A}^n_{\mathcal{O}[[t]]}$ and $\overline{T_t}$ as a closed subscheme of $\mathbb{A}^n_k$. Interpreting the coefficients of those polynomials as elements in k_K via the map $\alpha \mapsto \overline{\alpha}$, we see that

$$(\overline{T_t})_K = \big(T \times_{\mathrm{Spec}(\mathcal{O}} \mathrm{Spec}(k_K)\big)(k_K) = T(k_K).$$

Let $\phi \in \mathcal{C}(\overline{X_t}, \mathcal{L}_{\mathcal{O}})^{ps}$. Then with U as above, $f^*(\phi) \in \mathcal{C}(\overline{U_t}, \mathcal{L}_{\mathcal{O}})$. Then by the above arguments we can define a function $(f^*(\phi))_K$ on the set $(\overline{U_t})_K = U(k_K)$ for almost all K.

Claim 6.1 *The function $(f^*(\phi))_K$ is constant on the fibres of the map $U(k_K) \to \pi_0(X(k_K))$.*

Proof Using the notation we set up above, we look at the functions $g_1^* \circ f^*(\phi)$ and $g_2^* \circ f^*(\phi)$ on $\overline{V_t}$. By the definition of a constructible pseudo-function, these functions are equal. Thus,

$$(g_1)_K^*(f^*(\phi))_K = (g_1^* \circ f^*(\phi)))_K = (g_2^* \circ f^*(\phi)))_K = (g_2)_K^*(f^*(\phi))_K$$

for almost all K. If u_1 and u_2 are two points of $U(k_K)$ that lie in the same fibre of $U(k_K) \to \pi_0(X(k_K))$, then they have a common lift v to $V(k_K)$. This proves our claim. ∎

Given any $\phi \in \mathcal{C}(\overline{X_t}, \mathcal{L}_{\mathcal{O}})^{ps}$, we now define its evaluation $\gamma_K(\phi) \in \mathbb{Q}$, which is defined for almost all K. To do this, for each $x \in \pi_0(X(k_K))$ we choose an arbitrary u_x in the fibre of $U(k_K) \to \pi_0(X(k_K))$ over x. Then we define

$$\gamma_K(\phi) = \sum_{x \in \pi_0(X(k_K))} \big(f^*(\phi)\big)_K(u_x) \cdot \#\{x\},$$

where # is the counting measure we defined in Section 4.3, equation (4.1). By the claim we proved above, this is independent of the choice of u_x for almost all K.

Theorem 6.2 *Let X be an sft-Artin stack over $\mathcal{O}$ with $\dim(X/\mathcal{O}) = d$. let $\phi \in IC^d(X_t, \mathcal{L}_{\mathcal{O}})$. Then for almost all K, ϕ_K is integrable over $\pi_0(X(R_K))$ and*

$$\gamma_K\big(\mu_{\overline{X_t}}(\phi)\big) = \int_{\pi_0(X(R_K))} \phi_K d\mu_d.$$

Proof Let $f\colon U \to X$ be an f-surjective smooth atlas with U being an affine scheme over $\mathcal{O}$ with $\dim(U/\mathcal{O}) = e$. Then we know that $f_t^*(\phi)$ is in $IC^e(U_t)$. From [6, Thm. 9.1.5], it follows that for any $u \in U(k_K)$,

$$\big(\mu_{\overline{U_t}}(f_t^*(\phi))\big)_K(u) = \int_{\tau_0^{-1}(u)} \phi_K d\mu_d,$$

where τ_0 is the truncation map as defined in Section 4. If $x \in \pi_0(X(k_K))$, $U_x := U \times_{X,x} \operatorname{Spec}(k_K)$, and $f^{-1}(x) := \{u \in U(k_K) | f(u) = x\}$, we know from Lemma 4.1 that $\#f^{-1}(x) = (\#U_x(k_K)) \cdot (\#\{x\})$. Also, by the above claim, $(\mu_{\overline{U_t}}(f_t^*(\phi)))_K(u)$ is constant as u varies through $f^{-1}(x)$. Thus,

$$\begin{aligned}
\gamma_K(\mu_{\overline{X_t}}(\phi)) &= \sum_{x \in \pi_0(X(k_K))} (\mu_{\overline{U_t}}(f_t^*(\phi)))_K(u_x) \cdot \#\{x\} \\
&= \sum_{x \in \pi_0(X(k_K))} \Big(\frac{1}{\#U_x(k_K)}\Big) (\mu_{\overline{U_t}}(f_t^*(\phi)))_K(u_x) \cdot \#f^{-1}(x) \\
&= \sum_{u \in U(k_K)} \Big(\frac{1}{\#U_x(k_K)}\Big) \mu_{\overline{U_t}}(f_t^*(\phi))_K(u) \\
&= \sum_{u \in U(k_K)} \Big(\frac{1}{\#U_x(k_K)}\Big) \int_{\tau_0^{-1}(u)} \phi_K d\mu_d \\
&= \int_{\pi_0(X(R_K))} \phi_K d\mu_d,
\end{aligned}$$

as required. (The last equality is a consequence of Definition 4.3.) ∎

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We define $\mathcal{C}(\overline{X_t}, \mathcal{L}_{\mathcal{O}})^{ps}[[T]]_{\mathrm{rat}}$ to be the subring of $\mathcal{C}(\overline{X_t}, \mathcal{L}_{\mathcal{O}})^{ps}[[T]]$ generated by the polynomial ring $\mathcal{C}(\overline{X_t}, \mathcal{L}_{\mathcal{O}})^{ps}[T]$ and the set $(1 - \mathbb{L}^a T^b)^{-1}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{N}\backslash\{0\}$. For any $R(T) \in \mathcal{C}(\overline{X_t}, \mathcal{L}_{\mathcal{O}})^{ps}[T]$ we can define the element $\gamma_K(R(T)) \in \mathbb{Q}[T]$ for almost all K by applying γ_K to the coefficients of $R(T)$. We also define $\gamma_K((1 - \mathbb{L}^a T^b)^{-1}) := (1 - q_K^a T^b)^{-1}$. Thus, we see that for any element $P \in \mathcal{C}(\overline{X_t}, \mathcal{L}_{\mathcal{O}})^{ps}[[T]]_{\mathrm{rat}}$, we can obtain an element $\gamma_K(P(T))$ for almost all K by simply applying γ_K to its coefficients.

Theorem 6.3 *Let X be an sft-Artin stack over $\mathcal{O}$. Then there exists an element $P_X(T) \in \mathcal{C}(\overline{X_t}, \mathcal{L}_{\mathcal{O}})^{ps}[[T]]_{\mathrm{rat}}$ such that $\gamma_K(P_X(T)) = P_{X_K}(T)$ for almost all K.*

Proof By the comments preceding Lemma 4.10, it suffices to prove the theorem with Q_X in place of P_X. In other words, we wish to show that there exists an element $Q_X(T) \in \mathcal{C}(\overline{X_t}, \mathcal{L}_{\mathcal{O}})^{ps}[[T]]_{\mathrm{rat}}$ such that $\gamma_K(Q_X(T)) = Q_{X_K}(T)$ for almost all K (see Definition 4.9 for the definition of the power series $Q_{X_K}(T)$).

Let $f\colon U \to X$ be an f-surjective smooth atlas with U being an affine scheme over $\mathcal{O}$. Let M_n^U be the definable subassignment on U_t defined by the condition $u \notin U_{\mathrm{sing}}$ mod (t^{n+1}). Let M_n^X be the image of M_n^U in X_t. It is easy to see that $M_n^U = f_t^{-1}(M_n^X)$. We define ϕ_n^X (resp. ϕ_n^U) to be the characteristic function of M_n^X (resp. M_n^U). Then it is easy to see from Lemma 4.10 and Theorem 6.2 that if we define $Q_X(T)$ by the formula

$$Q_X(T) := \sum_{n=0}^{\infty} \mathbb{L}^{nd} \mu_{\overline{X_t}}(\phi_n^X) T^n,$$

then $\gamma_K(Q_X(T)) = Q_{X_K}(T)$ for almost all K. Also, $Q_X(T) \in \mathcal{C}(\overline{X_t}, \mathcal{L}_{\mathcal{O}})^{ps}[[T]]$ is represented by

$$f_t^*(Q_U(T)) = \sum_{n=0}^{\infty} \mathbb{L}^{nd} \mu_{\overline{U_t}}(\phi_n^U) T^n,$$

in $\mathcal{C}(\overline{U_t}, \mathcal{L}_{\mathcal{O}})[[T]]$. By [5, Theorem 14.4.1], it is known that $f_t^*(Q_U(T))$ is contained in $\mathcal{C}(\overline{U_t}, \mathcal{L}_{\mathcal{O}})[[T]]_{\mathrm{rat}}$. This proves the result. ∎

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