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A NOTE ON PROJECTIONS IN ÉTALE GROUPOID ALGEBRAS AND DIAGONAL-PRESERVING HOMOMORPHISMS

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Abstract

Carlsen ['$*$-isomorphism of Leavitt path algebras over $\mathbb{Z}$', *Adv. Math.* **324** (2018), 326–335] showed that any $*$-homomorphism between Leavitt path algebras over $\mathbb{Z}$ is automatically diagonal preserving and hence induces an isomorphism of boundary path groupoids. His result works over conjugation-closed subrings of $\mathbb{C}$ enjoying certain properties. In this paper, we characterise the rings considered by Carlsen as precisely those rings for which every $*$-homomorphism of algebras of Hausdorff ample groupoids is automatically diagonal preserving. Moreover, the more general groupoid result has a simpler proof.

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1. Introduction

The paper [5] caused quite a stir at the time because it showed that the Cuntz splice does not preserve $*$-isomorphism type of Leavitt path algebras over $\mathbb{Z}$ (or certain more general subrings of $\mathbb{C}$ closed under complex conjugation). The Cuntz splice preserves isomorphism type of graph C^*-algebras and it is a major open question whether it preserves isomorphism type of complex Leavitt path algebras. The result of [5] covered a fairly general class of graphs. The key idea was to show that projections in Leavitt path algebras over such rings are quite restricted, and hence any $*$-isomorphism is forced to be diagonal preserving. This was extended to arbitrary Leavitt path algebras by Carlsen in [2]. A series of results by various authors [1, 3, 7] shows that diagonal-preserving isomorphisms of algebras of Hausdorff ample groupoids forces, under mild hypotheses, an isomorphism of the corresponding groupoids. Hence, over rings in which $*$-isomorphisms are automatically diagonal preserving, the groupoid is entirely encoded by its $*$-algebra.

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The proofs in [2, 5] unfortunately work with Leavitt path algebras as given by generators and relations, rather than as groupoid algebras, rendering them quite technical. Here we give a very simple proof that for the kinds of rings considered in [2, 5] (and only for those rings), every $*$-homomorphism of algebras of Hausdorff ample groupoids is automatically diagonal preserving. We give several equivalent characterisations and prove some basic properties of such rings.

2. The main result

We follow the analytic conventions for groupoids; in particular, we identify objects and identity arrows. An *ample* groupoid $\mathscr{G}$ is a topological groupoid whose unit space $\mathscr{G}^{(0)}$ is locally compact, Hausdorff and totally disconnected, and whose range map $\boldsymbol{r}\colon \mathscr{G} \to \mathscr{G}^{(0)}$ is a local homeomorphism. In this paper, all ample groupoids will be assumed Hausdorff. In this case, the unit space $\mathscr{G}^{(0)}$ is a clopen subspace of $\mathscr{G}$. An open subset $U \subseteq \mathscr{G}$ is a (local) *bisection* if $\boldsymbol{d}|_U, \boldsymbol{r}|_U$ are injective. The compact bisections form a basis for the topology on $\mathscr{G}$.

If R is a commutative ring with unit, the algebra $R\mathscr{G}$ of $\mathscr{G}$ [6] consists of the compactly supported locally constant functions $f\colon \mathscr{G} \to R$ under convolution,

$$f * g(\gamma) = \sum_{\boldsymbol{r}(\alpha)=\boldsymbol{r}(\gamma)} f(\alpha)g(\alpha^{-1}\gamma).$$

The complex algebra $\mathbb{C}\mathscr{G}$ is a $*$-algebra with $f^*(\gamma) = \overline{f(\gamma^{-1})}$. From now on, R will always be a subring of $\mathbb{C}$ closed under complex conjugation. The algebra $R\mathscr{G}$ is then a $*$-subalgebra of $\mathbb{C}\mathscr{G}$, and so we can talk about things like projections and unitaries in $R\mathscr{G}$. For instance, if U is a compact open subset of $\mathscr{G}^{(0)}$, then the indicator function 1_U is a projection. By a $*$-algebra homomorphism $\varphi\colon R\mathscr{G} \to R\mathscr{H}$, we mean an R-algebra homomorphism such that $\varphi(f^*) = \varphi(f)^*$ for all f.

Let us denote by $D(R\mathscr{G})$ the subalgebra of $R\mathscr{G}$ consisting of functions supported on $\mathscr{G}^{(0)}$. Then $D(R\mathscr{G})$ is a commutative $*$-subalgebra of $R\mathscr{G}$, often referred to as the *diagonal subalgebra*. Note that $D(R\mathscr{G})$ is spanned over R by the projections 1_U with $U \subseteq \mathscr{G}^{(0)}$ compact open.

A homomorphism $\varphi\colon R\mathscr{G} \to R\mathscr{H}$ of groupoid algebras is *diagonal preserving* if $\varphi(D(R\mathscr{G})) \subseteq D(R\mathscr{H})$. We say that an isomorphism φ is a diagonal-preserving isomorphism if φ and φ^{-1} are diagonal preserving or, equivalently, φ is an isomorphism taking $D(R\mathscr{G})$ onto $D(R\mathscr{H})$.

An element $n \in R\mathscr{G}$ is a *normaliser* of $D(R\mathscr{G})$ if there is an element n' with $nn'n = n$, $n'nn' = n'$ and $nD(R\mathscr{G})n' \cup n'D(R\mathscr{G})n \subseteq D(R\mathscr{G})$. It is easy to see that any $f \in R\mathscr{G}$ whose support is a bisection is a normaliser. In [7], $\mathscr{G}$ was defined to satisfy the local bisection hypothesis with respect to R if the only normalisers are those which support a bisection. For example, a group G satisfies the local bisection hypothesis with respect to R if and only if the group ring RG has only trivial units. It was shown in [7] that if there is a dense set of units $x \in \mathscr{G}^{(0)}$ such that the group ring over R of the isotropy group at x of the interior of the isotropy bundle of $\mathscr{G}$ has only trivial units, then

$\mathscr{G}$ satisfies the local bisection hypothesis. This includes all boundary path groupoids of graphs and higher rank graphs. The main theorem of [7] admits the following theorem as a special case.

THEOREM 2.1. *Let R be an integral domain and let $\mathscr{G}, \mathscr{G}'$ be Hausdorff ample groupoids such that $\mathscr{G}$ satisfies the local bisection hypothesis. Then the following are equivalent.*

(1) *There is an isomorphism $\varphi\colon \mathscr{G} \to \mathscr{G}'$.*
(2) *There is a diagonal-preserving isomorphism $\Phi\colon R\mathscr{G} \to R\mathscr{G}'$ of R-algebras.*

The papers [2, 5] investigated the case when every $*$-algebra homomorphism must automatically be diagonal preserving in the setting of Leavitt path algebras. We consider here the general case.

For $n \geq 1$, let $\mathcal{R}_n$ be the discrete groupoid associated to the universal equivalence relation on $\{1, \dots, n\}$. So $\mathcal{R}_n$ has these n objects and a unique isomorphism (i, j) from j to i. Multiplication follows the rule

$$(i,j)(k,\ell) = \begin{cases} (i,\ell) & \text{if } j = k, \\ \text{undefined} & \text{else,} \end{cases}$$

and the inversion is given by $(i,j)^{-1} = (j,i)$. It is not difficult to see that $R\mathcal{R}_n$ is $*$-isomorphic to $M_n(R)$ via $f \mapsto [f((i,j))]$ with inverse $A \mapsto f_A$ where $f_A((i,j)) = A_{ij}$. The diagonal subalgebra $D(R\mathcal{R}_n)$ consists of those functions supported on the diagonal and is sent via the above isomorphism onto the subalgebra $D_n(R)$ of diagonal matrices. This explains the nomenclature.

The following theorem greatly generalises and expands on [2, 5], where only Leavitt path algebras were considered.

THEOREM 2.2. *Let R be a subring of $\mathbb{C}$ closed under conjugation. Then the following are equivalent.*

(1) *For every $n \geq 1$, if $v \in R^n$ is a unit vector, then v has exactly one nonzero entry, that is, if $1 = \sum_{i=1}^n |r_i|^2$, then only one $r_i \neq 0$.*
(2) *If $r_1 = \sum_{i=1}^n |r_i|^2$ with $r_1, \dots, r_n \in R$, then $r_2 = \cdots = r_n = 0$.*
(3) *If $\mathscr{G}$ is a Hausdorff ample groupoid, then each projection in $R\mathscr{G}$ belongs to the diagonal subalgebra $D(R\mathscr{G})$.*
(4) *Every $*$-homomorphism $R\mathscr{G} \to R\mathscr{H}$ of algebras of Hausdorff ample groupoids is diagonal preserving.*
(5) *Every $*$-isomorphism $R\mathscr{G} \to R\mathscr{H}$ of algebras of Hausdorff ample groupoids is diagonal preserving.*
(6) *For every $n \geq 1$, every unitary matrix in $GL_n(R)$ is monomial (that is, has exactly one nonzero entry in every row and column).*

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PROOF. The first implication is in [2], but we repeat the proof for the reader's convenience. If $r_1 = \sum_{i=1}^{n} |r_i|^2$, then $r_1 \geq 0$. Therefore,

$$|1 - r_1|^2 + \sum_{i=2}^{n} |r_i|^2 + \sum_{i=1}^{n} |r_i|^2 = 1 - 2r_1 + |r_1|^2 + \sum_{i=2}^{2} |r_i|^2 + \sum_{i=1}^{n} |r_i|^2 = 1.$$

If any $r_i \neq 0$, then $r_1 > 0$, and so we must have $r_2 = \cdots = r_n = 0$ by item (1). Assume now item (2) and let $f \in R\mathscr{G}$ be a projection. Then $f = ff^*$. Let $x \in \mathscr{G}^{(0)}$. Then

$$f(x) = \sum_{\boldsymbol{r}(\gamma)=x} f(\gamma)f^*(\gamma^{-1}) = \sum_{\boldsymbol{r}(\gamma)=x} |f(\gamma)|^2.$$

Thus, by item (2), if $\boldsymbol{r}(\gamma) = x$ and $\gamma \neq x$, then $f(\gamma) = 0$. Therefore, f is supported on $\mathscr{G}^{(0)}$ and hence $f \in D(R\mathscr{G})$ as $\mathscr{G}$ is Hausdorff. It is immediate that item (3) implies item (4) as $D(R\mathscr{G})$ is spanned over R by projections and each projection in $R\mathscr{H}$ is diagonal by item (3). Trivially, item (4) implies item (5). Recalling that $R\mathcal{R}_n \cong M_n(R)$ via a $*$-isomorphism taking $D(R\mathcal{R}_n)$ onto $D_n(R)$, for item (5) implies item (6), it suffices to observe that conjugation by a unitary matrix is a $*$-automorphism, and the normaliser in $GL_n(R)$ of $D_n(R)$ is the group of monomial matrices. Finally, suppose that item (6) holds and that $v \in R^n$ is a unit vector (which we view as a column vector). Then vv^* is a projection, and so $U = I - 2vv^*$ is a self-adjoint unitary matrix. Suppose that $v_i \neq 0 \neq v_j$ with $i \neq j$. Then $U_{ij} = -2v_i\overline{v_j} \neq 0$, $U_{ii} = 1 - 2|v_i|^2$ and $U_{jj} = 1 - 2|v_j|^2$. Thus, if U is monomial, then we must have $|v_i|^2 = |v_j|^2 = 1/2$. However, then

$$\begin{bmatrix} v_1 & -\overline{v}_2 \\ v_2 & \overline{v}_1 \end{bmatrix}$$

is unitary and not monomial, contradicting item (6). Thus, v has a single nonzero entry. This completes the proof. □

We remark that it was claimed without proof in [2] that there are rings satisfying item (2) but not item (1). In fact, it is easy to see directly that item (2) implies item (1) since if $1 = \sum_{i=1}^{n} |r_i|^2$, where without loss of generality $r_1 \neq 0$, then $|r_1|^2 = \sum_{i=1}^{n} |r_1 r_i|^2$, and so $r_i = 0$ for $i \geq 2$ by item (2).

Those rings satisfying the equivalent conditions of Theorem 2.2 were called *kind* in [2] and were said to have a unique partition of the unit in [5]. We prefer 'kind'. Rings with the property that $c_1^2 + \cdots + c_n^2 = 1$ implies $c_i = \pm 1$ for a unique i, and $c_j = 0$ otherwise, were studied in [4] under the name L-rings. Many of the observations in that paper about L-rings apply to kind rings. Note that a subring of $\mathbb{R}$ is kind if and only if it is an L-ring. Any L-ring which is a subring of the complex numbers closed under complex conjugation must be kind. There are, however, kind rings that are not L-rings. For instance, $\mathbb{Z}[i]$ is kind (see Proposition 2.3), but $2^2 + i^2 + i^2 + i^2 = 1$, so it is not an L-ring.

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PROPOSITION 2.3. *Let R be a subring of $\mathbb{C}$ closed under complex conjugation and let F be the field of fractions of R.*

(1) *If R is kind, then $1/n \notin R$ for all $n \geq 2$.*
(2) *If $|r| < 1$ implies $r = 0$ for all $r \in R$, then R is kind. In particular, $\mathbb{Z}$ is kind and if $a \in [1, \infty)$, then $\mathbb{Z}[ia]$ is kind.*
(3) *A directed union of kind rings is kind.*
(4) *If R is kind and $B \subseteq \mathbb{R}$ is algebraically independent over F, then $R[B]$ is kind. In particular, $\mathbb{Z}[\pi]$ and $\mathbb{Z}[e]$ are kind.*
(5) *If R is kind and $n \in \mathbb{Z}$ with $\sqrt{n} \notin F$, then $R[\sqrt{n}]$ is kind. In particular, if R is integrally closed in F and $\sqrt{n} \notin R$, then $R[\sqrt{n}]$ is kind.*

PROOF. If $1/n \in R$, then $(1/n, 1/n, \ldots, 1/n) \in R^{n^2}$ is a unit vector, and so R is not kind. This proves item (1). The second item is clear since if $1 = |r_1|^2 + \cdots + |r_n|^2$, then $|r_i|^2 \geq 1$ for at most one value of i. Item (3) is clear since a unit vector has finitely many entries. For item (4), first note that $R[B]$ is closed under complex conjugation since $B \subseteq \mathbb{R}$. We may assume by item (3) that B is finite and then, by induction, it suffices to handle the case $R[a]$ where $a \in \mathbb{R}$ is transcendental over F. Suppose that $(f_1(a), \ldots, f_n(a)) \in R[a]^n$ is a unit vector with $f_i \in R[x]$. Then since $a \in \mathbb{R}$, we have $|f_i(a)|^2 = (f_i\overline{f}_i)(a)$, and $f_i\overline{f}_i$ is a polynomial over R with real coefficients and with leading coefficient the square of the absolute value of the leading coefficient of f_i. Therefore, if some f_i is a nonconstant polynomial, then $g(x) = f_1\overline{f}_1 + \cdots + f_n\overline{f}_n - 1$ is a nonzero polynomial of degree twice the maximum degree of the f_i with $g(a) = 0$. This is a contradiction since a is transcendental over F. Thus, $f_1, \ldots, f_n$ are constant polynomials, and so $(f_1(a), \ldots, f_n(a)) \in R^n$ and hence has exactly one nonzero entry since K is kind. Finally, if $\sqrt{n} \notin F$ and $(a_1 + b_1\sqrt{n}, \ldots, a_m + b_m\sqrt{n}) \in R[\sqrt{n}]^m$ is a unit vector, then

$$1 = \sum_{i=1}^{m}(|a_i|^2 + |b_i|^2|n|) + \sqrt{n}\sum_{i=1}^{m}(\pm a_i\overline{b}_i + \overline{a}_i b_i).$$

Since $\sqrt{n} \notin F$, we must have $\sum_{i=1}^{m}(\pm a_i\overline{b}_i + \overline{a}_i b_i) = 0$. Since $|n|$ is a positive integer, we deduce using the fact that R is kind that there is a unique i with either $a_i \neq 0$ or $b_i \neq 0$. We conclude that $R[\sqrt{n}]$ is kind. □

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