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MINIMAL REQUIREMENTS FOR MINKOWSKI'S THEOREM IN THE PLANE II

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Let K be a closed convex set in the Euclidean plane, with area $A(K)$, which contains in its interior only one point 0 of the integer lattice. If K has other than one or three chords through 0 of one of the following types, it is shown that $A(K) \leq 4$, while if K has three of one type, $A(K) \leq 4.5$. The types of chords considered are chords which partition K into two regions of equal area, chords which lie midway between parallel supporting lines of K, and chords such that K is invariant under reflection in them. The results are generalised to any lattice in the plane.

1. Introduction

Let Λ be a lattice in the plane having determinant $\det(\Lambda)$. We say that the set K is *admissible* if it is a closed convex set in the plane with 0 the only point of Λ in its interior. We define a *chord of symmetry* of K to be a chord through 0 bisected by 0, and a *chord of areal symmetry* to be a chord through 0 which splits K into two regions of equal area. A chord through 0 equidistant from two parallel lines of support of K we call a *midchord of symmetry*, while we say that if the set K is invariant under reflection in a chord through 0, then that chord is a *chord of reflective symmetry*. Finally, a *chord of perimeter symmetry* passes through 0 and partitions the boundary of K into two arcs of equal length. We denote the number of each of these types of chords by

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$s(K)$, $a(K)$, $m(K)$, $r(K)$ and $p(K)$ respectively. We say that an integer valued function (possibly infinite) $f(K)$, defined on the set of admissible sets, is an *M-function*, when

(i) $f(K) > 1$ and $f(K) \neq 3$ imply that $A(K) \leq 4 \det(\Lambda)$, and

(ii) $f(K) = 3$ implies that $A(K) \leq 4.5 \det(\Lambda)$.

In an earlier note [1], we showed that $s(K)$ is an *M-function*.

We show

THEOREM 1. *The function $a(K)$ is an M-function.*

THEOREM 2. *The function $m(K)$ is an M-function.*

THEOREM 3. *The function $r(K)$ is an M-function, and $r(K) = 3$ only if adjacent chords of reflective symmetry form angles of $\pi/3$.*

We show that Theorems 1 and 2 give the best possible bounds on $A(K)$ for all Λ and all values of $a(K)$ and $m(K)$. Theorem 3 gives the best possible bound on $A(K)$ for at least one lattice when $r(K) = 2, 3, 4$ or 6. Under a linear transformation of K and Λ, the values of $A(K)/\det(\Lambda)$, $a(K)$ and $m(K)$ are invariant. It suffices to show Theorems 1 and 2 are best possible with respect to any one lattice.

In the hexagonal lattice generated by $a = (2, 0)$ and $b = (1, \sqrt{3})$, the equilateral triangle T, with vertices $v_1 = -a + 2b$, $v_2 = -b + 2a$ and $v_3 = -b - a$ has area $4.5 \det(\Lambda)$. Each chord through the origin 0 and a vertex of T is simultaneously a chord of areal symmetry, a midchord of symmetry, a chord of reflective symmetry and a chord of perimeter symmetry. Hence the bound of $4.5 \det(\Lambda)$ can be attained in each of the above theorems. In the same lattice, the regular hexagon H with vertices $\pm 2/3 v_1$, $\pm 2/3 v_2$, $\pm 2/3 v_3$, has area $4 \det(\Lambda)$, and $r(H) = 6$, $a(H) = m(H) = \infty$. We obtain two further critical examples for Theorem 3 in the integer lattice Λ_0. Let U be the square with vertices $(\pm 1, \pm 1)$, and let R be the rectangle with vertices $\pm(3/2, -1/2)$, $\pm(1/2, -3/2)$. Only the four chords of U parallel to and midway between the coordinate axes are chords of reflective symmetry for U, while only the chords midway between the axes are chords of reflective symmetry for R. As $A(U) = A(R) = 4 = 4 \det(\Lambda_0)$, we have shown Theorem 3 to be best possible in four instances.

The argument that Theorem 1 is best possible when $a(K)$ is finite is a little more complicated. Let int U denote the interior of U, and let $C \subseteq \text{int } U$ be a closed, convex, proper $2n$-gon $(n \geq 2)$ which is centrally symmetric about 0. Further, let the edges of C be labelled $e_1, e_2, \ldots, e_{2n}$ is a clockwise direction, and let e_1 be parallel to the x-axis and bisected by the positive y-axis. For given small $\eta > 0$, we may assume that $A(C) \geq A(U) - \eta$. We now modify to produce a set C' for which $C \subseteq C' \subseteq U$ and $a(C') = n$.

First suppose that n is even. To edge e_1, add a small scalene triangle τ for which the area lying in the halfplane $x \leq 0$ exceeds the area lying in $x \geq 0$ by $\varepsilon/2$ $(\varepsilon > 0)$. To edge e_{n+1}, add the mirror image of τ in the x-axis. To the edges

$$e_2, e_4, \ldots, e_n, e_{n+3}, e_{n+5}, \ldots, e_{2n-1}$$

add a small region of area ε. (See Figure 1 for case $n = 4$.) By

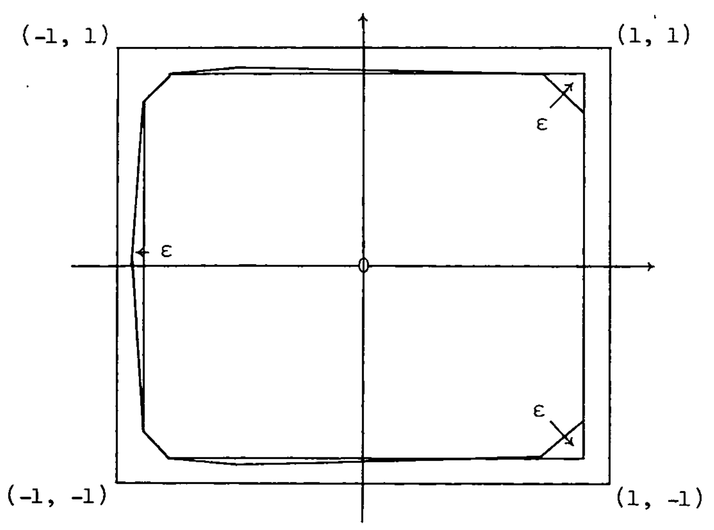

Figure 1

choosing ε sufficiently small we can ensure that the resulting set C' is still convex, $C' \subseteq U$, and $A(U) - \eta < A(C') < 4 = A(U)$. Also, C' has precisely n chords of areal symmetry, namely, the chord along the

y-axis, and the $n/2 + (n/2 - 1)$ chords which bisect the added regions on edges other than e_1 and e_{n+1}.

For odd n, we need not isolate a pair of edges e_1 and e_{n+1} for special treatment as in the above case for even n. Instead, add to each of the edges $e_2, e_4, \ldots, e_{2n}$ a region of area $\varepsilon > 0$, sufficiently small to ensure that the resulting set $C' \subset U$ is still convex. The chords of areal symmetry of C' are precisely the n chords passing through 0 which bisect the area of one of these added regions.

Finally, we argue that Theorem 2 is best possible when $m(K)$ is finite. Let $C \subseteq \text{int } U$ be specified as above. We now modify C to produce a set C'' for which $C \subseteq C'' \subseteq U$ and $m(C'') = n$. To each of the edges $e_1, \ldots, e_n$ we add a region of small positive area, so that the resulting set C'' is convex, $C'' \subseteq \text{int } U$, and so that each vertex of C is a boundary point of C''. We choose the added regions, so that C'' has a unique supporting line at each of the $n - 1$ vertices incident with $e_2, \ldots, e_{n-1}$, and a unique pair of parallel supporting lines at the pair of opposite vertices incident with e_1 and e_n. This latter pair of parallel supporting lines, together with the $n - 1$ unique pairs of parallel supporting lines at the other pairs of opposite vertices, lie parallel to n midchords of symmetry through 0.

We show that there are no other midchords of symmetry. In each pair of parallel supporting lines of C'', one line must meet one of the edges $e_{n+1}, \ldots, e_{2n}$, and hence be incident with a vertex of C''. Therefore, any pair of parallel supporting lines equidistant from 0 must meet C'' at a pair of opposite vertices of C'', and so must be one of the n unique such pairs counted above. As $A(C'') > A(C) \geq A(U) - \eta = 4 \det(\Lambda) - \eta$, the bound of Theorem 2 cannot be improved.

The example chosen above suggests perhaps that $A(K) = 4 \det(\Lambda)$ is impossible when $m(K)$ is finite and not one or three. As a counter-example, the trapezium Z with vertices $(-1, 2)$, $(2, -1)$, $(-1, 0)$ and $(0, -1)$ is admissible with respect to Λ_0, but $A(Z) = 4 \det(\Lambda_0)$, and $m(Z) = 2$.

2. Proof of Theorem 1

Scott [2] and [3] has shown that an admissible set K has area $A(K) \le 4 \det(\Lambda)$ whenever

(1) a chord of symmetry of K is also a chord of areal symmetry of K, or

(2) K has parallel supporting lines at the endpoints of a chord of symmetry (that is, K has an *extremal* chord of symmetry, [1]).

Let $A_R(\theta)$ be the area of that part of an admissible set K to the right of a directed chord $c(\theta) = P'O\vec{P}$ of K, which makes an angle $\angle POX = \theta$ $(0 \le \theta \le \pi)$ with the positive x-axis. Similarly, let $A_L(\theta)$ be the area of that part of K to the left of $c(\theta)$. We show that $A_R(\theta)$ is a differentiable function of θ. The areas of the two almost triangular regions of K which lie between the chords $c(\theta)$ and $c(\theta+d\theta)$ are well approximated by $\frac{1}{2}(OP)^2 d\theta$ and $\frac{1}{2}(OP')^2 d\theta$. Then

$$\frac{d}{d\theta}\left(A_R(\theta)\right) = \lim_{d\theta\to 0} \frac{\frac{1}{2}(OP)^2 d\theta - \frac{1}{2}(OP')^2 d\theta}{d\theta} = \frac{1}{2}\left[(OP)^2 - (OP')^2\right] .$$

The function $d_A(\theta) = A_R(\theta) - A_L(\theta) = 2A_R(\theta) - A(K)$ is therefore differentiable with respect to θ, and has derivative

$$d_A'(\theta) = 2A_R'(\theta) = (OP)^2 - (OP')^2 .$$

Hence we can identify a zero of $d_A(\theta)$ with a chord of areal symmetry of K, and a zero of its derivative $d_A'(\theta)$ with a chord of symmetry of K.

We now show that simplifying assumptions about the function $d_A(\theta)$ can be made. Suppose that at some angle θ_0, $d_A'(\theta_0) = 0$ but $d_A(\theta_0)$ is not an extremum. Without loss of generality assume that $d_A'(\theta) \ge 0$ for θ in a neighbourhood $N(\theta_0, \varepsilon)$ about θ_0. Hence for $\theta \in N(\theta_0, \varepsilon)$, $d(\theta) = P'O\,P$ has the property that the segment OP is no shorter than OP'. Therefore, a supporting line to K at the endpoint P_0 of

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$c(\theta_0) = P_0'OP_0$, together with a parallel line through P_0' form a pair of parallel supporting lines at the endpoints of $c(\theta_0)$, a chord of symmetry of K . Result (2) states that $A(K) \leq 4$ in this case.

If $c(\theta_1)$ is both a chord of areal symmetry of K and a chord of symmetry of K , result (1) gives that $A(K) \leq 4$. A particular case of this occurs when $a(K)$ is infinite, as the infinite set of zeros of $d_A(\theta)$ must have an accumulation point $\theta_1 \in [0, \pi)$. By continuity θ_1 is also a zero of $d_A(\theta)$, and indeed must also be a zero of its derivative $d_A'(\theta)$.

We may therefore assume that $d_A'(\theta) = 0$ only at extrema of $d_A'(\theta)$, and only when $d_A(\theta) \neq 0$. By applying Rolle's theorem to $d_A(\theta)$, we deduce that $d_A'(\theta)$ has an odd, or infinite, number of zeros between each pair of successive zeros of $d_A(\theta)$. The number of pairs of successive zeros of $d_A(\theta)$ equals $a(K)$, for we identify the last and first zero on $[0, \pi)$ as such a pair. The value of $s(K)$ is therefore either infinite, or $a(K) + 2t$, where t is a nonnegative integer. Theorem 1 now follows as a simple corollary of the result proved in [1], that $s(K)$ is an *M-function*.

3. Proof of Theorem 2

Let K be an admissible set having $m(K)$ midchords of symmetry. Hence K is a subset of the $2(m(K))$-gon , P , centrally symmetric about O , formed from those pairs of parallel supports of K parallel to the midchords. When $m(K)$ is infinite, we simply regard some of the sides of this polygon P to have lengths equal to or approaching zero. Since the sides of P are all lines of support of K , K contains a point on each side of P . Let S_1 and S_2 be two opposite sides of P , containing points R_1 and R_2 of K respectively. Let R_3 and R_4 be points such that R_1R_3 and R_2R_4 are chords of K through O . The midpoint of the chord R_1OR_3 lies no further from S_1 than S_2 , since $R_3 \in K$ and S_2

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bounds K, and so the distance $|R_1 0| \geq |0R_3|$. Similarly the midpoint of $R_1 0R_4$ lies no further from S_2 than S_1, and so $|R_4 0| \leq |0R_2|$. Indeed, if equality holds in either of the above expressions, S_1 and S_2 are parallel supports of K at the endpoints of a chord of symmetry of K, and we can deduce that $A(K) \leq 4\det(\Lambda)$ by Scott's result (2). Otherwise, by the continuity of the boundary of K, there must exist a chord of K, $R_5 0R_6$ between the chords $R_1 0R_3$ and $R_4 0R_2$, such that $|R_5 0| = |0R_6|$. Hence we may assume K has at least one chord of symmetry associated exclusively with each of its midchords, and so $s(K) \geq m(K)$. Theorem 1 now follows from the result [1], that $s(K)$ is an *M-function*, with the exception of the case $m(K) = 2$ and $s(K) = 3$. We demonstrate that this situation can never arise, by showing that if an admissible set K has $s(K) = 3$ and $A(K) > 4\det(\Lambda)$, then $m(K) > 2$.

Suppose K is an admissible set, with $s(K) = 3$ and $A(K) > 4\det(\Lambda)$. Since $A(K) > 4\det(\Lambda)$, we may assume without loss of generality that the supporting lines at the endpoints of the three chords of symmetry meet as in Figure 2. This same assumption is justified in [1], in the proof of Theorem 2, from which we adopt the notation $P_i 0P_i'$, T_i, $i \in \{1, 2, 3\}$ for the chords of symmetry and the intersections of the supporting lines at their endpoints. The chord midway between the two parallel supporting lines of K, parallel to T_1P_1, in Figure 2, lies to the same side of 0 as P_1', since no support parallel to T_1P_1 passes through P_1'. Similarly the chord midway between two parallel supporting lines of K, parallel to T_1P_1' lies to the same side of 0 as P_1. Hence, as the signed distance between 0 and a chord of K midway between parallel supports of K at angle θ with the x-axis is a continuous function of θ, there is a midchord of symmetry of K, parallel to T_1X, where X lies on the chord $P_1 0P_1'$. Similarly, midchords of symmetry of K are generated parallel to T_2Y and T_3Z, where $Y \in P_2 0P_2'$ and $Z \in P_3 0P_3'$. That the directions of T_1X, T_2Y, and T_3Z are distinct is easily confirmed from the configuration shown in Figure 2. Hence $m(K) \geq 3$, and the proof is complete.

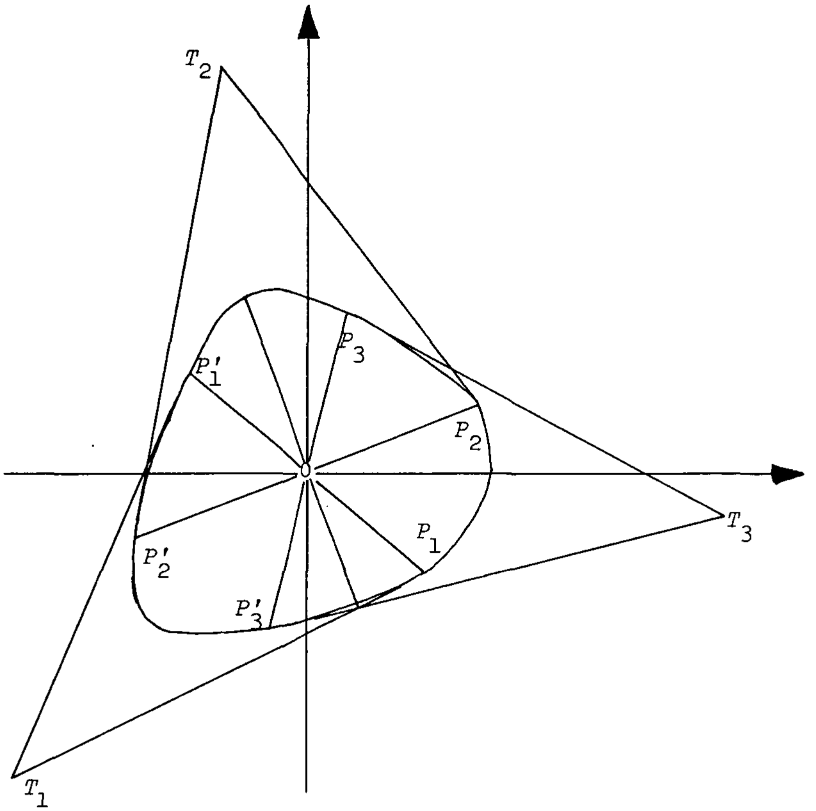

Figure 2

4. Proof of Theorem 3

Every chord of reflective symmetry is also clearly a chord of areal symmetry of K . Hence, by Theorem 1, when $r(K) > 3$, or when $r(K)$ is infinite, we deduce that $A(K) \leq 4 \det(\Lambda)$.

If $r(K) = 2$, the two chords of reflective symmetry must form an angle of $\pi/2$, for otherwise the reflection of one in the other forms a third distinct chord of reflective symmetry. In this case any further chords of areal symmetry give rise to an even number of chords of areal symmetry, by reflection in the two perpendicular chords of reflective symmetry. Hence the number of chords of areal symmetry is even or infinite, and so Theorem 1 implies that $A(K) \leq 4 \det(\Lambda)$.

If $r(K) = 3$, the chords must form angles of $\pi/3$, else their images in each other form further chords of reflective symmetry. By Theorem 1 we thus complete the proof of Theorem 3.

5.

The following conjecture has eluded all my attempts to prove it, or to find a counterexample.

CONJECTURE. *$p(K)$ is an M-function.*

References

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