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Abstract

In this paper, we extend the notion of Shintani descent to general (possibly disconnected) algebraic groups defined over a finite field $\mathbb{F}_q$. For this, it is essential to treat all the pure inner $\mathbb{F}_q$-rational forms of the algebraic group at the same time. We prove that the notion of almost characters (introduced by Shoji using Shintani descent) is well defined for any neutrally unipotent algebraic group, i.e. an algebraic group whose neutral connected component is a unipotent group. We also prove that these almost characters coincide with the 'trace of Frobenius' functions associated with Frobenius-stable character sheaves on neutrally unipotent groups. In the course of the proof, we also prove that the modular categories that arise from Boyarchenko and Drinfeld's theory of character sheaves on neutrally unipotent groups are in fact positive integral, confirming a conjecture due to Drinfeld.

1. Introduction

Let us fix a prime number p, q a power of p, and let us set $\mathtt{k} = \overline{\mathbb{F}}_q$. Let G be an algebraic group over $\mathtt{k}$ equipped with an $\mathbb{F}_q$-structure defined by a Frobenius $F : G \to G$. Shintani descent on G compares the irreducible representations of the finite groups $G(\mathbb{F}_{q^m})$ and $G(\mathbb{F}_q)$ where m is a positive integer. We begin by recalling the notion of Shintani descent for connected algebraic groups from [Sho92].

1.1 Shintani descent for connected algebraic groups

We refer to [Sho92] for the details of the constructions in this introduction. On the other hand, all these constructions will be carried out in detail in a more general setting in the subsequent sections of this paper.

Suppose that G is connected. Shintani descent for G was studied by Shoji in [Sho92], extending the work [Shi76] of Shintani for general linear groups. Let m be any positive integer and consider the mth power of the Frobenius

$$F^m : G \to G.$$

This is a Frobenius map for an $\mathbb{F}_{q^m}$-rational structure on G. The Frobenius F restricts to a group automorphism $F : G^{F^m} \to G^{F^m}$ of order m and hence acts on the finite set $\mathrm{Irrep}(G^{F^m})$ of irreducible characters of G^{F^m}. Shintani descent gives us a way to compare the two sets $\mathrm{Irrep}(G^{F^m})^F$ and $\mathrm{Irrep}(G^F)$.

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More precisely, consider the action of G^{F^m} on itself by F-twisted conjugation defined by

$$g : h \mapsto ghF(g^{-1}). \tag{1}$$

Let us denote the set of orbits of this action by $G^{F^m}/{\sim_F}$. Let $\mathrm{Fun}(G^{F^m}/{\sim_F})$ denote the space of F-conjugation invariant $\overline{\mathbb{Q}}_\ell$-valued functions on G^{F^m} where $\ell \neq p$ is a prime number.

Remark 1.1. In this paper we will always consider functions and characters with values in $\overline{\mathbb{Q}}_\ell$ since we will use the theory of $\overline{\mathbb{Q}}_\ell$-sheaves and complexes. Once we fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we can talk of 'complex conjugation' on the field $\overline{\mathbb{Q}}_\ell$ and we can define the usual Hermitian inner product on the function spaces $\mathrm{Fun}(G^{F^m}/\sim_F) \subset \mathrm{Fun}(G^{F^m})$. The choice of the isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ does not matter because in this paper we will only need to consider 'complex conjugation' on the subfield $\mathbb{Q}^{ab} \subset \overline{\mathbb{Q}}_\ell$ where 'complex conjugation' can be canonically defined.

Now given $\chi \in \mathrm{Irrep}(G^{F^m})^F$, we can define[1] a function $\widetilde{\chi} \in \mathrm{Fun}(G^{F^m}/{\sim_F})$ (well defined up to scaling by mth roots of unity). The set $\{\widetilde{\chi} \mid \chi \in \mathrm{Irrep}(G^{F^m})^F\}$ forms an orthonormal basis of the Hermitian inner product space $\mathrm{Fun}(G^{F^m}/{\sim_F})$.

Using Lang's theorem, we can define the norm map which is a bijection between the set $G^{F^m}/{\sim_F}$ and the set $G^F/\sim$ of conjugacy classes in G^F:

$$N_m : (G^{F^m}/{\sim_F}) \xrightarrow{\cong} (G^F/{\sim}). \tag{2}$$

Hence we get an isomorphism of function spaces

$$N_m^{-1*} : \mathrm{Fun}(G^{F^m}/{\sim_F}) \xrightarrow{\cong} \mathrm{Fun}(G^F/{\sim}). \tag{3}$$

Definition 1.2. For a positive integer m, Shintani descent is defined to be the map

$$\mathrm{Sh}_m : \mathrm{Irrep}(G^{F^m})^F \hookrightarrow \mathrm{Fun}(G^F/\sim) \tag{4}$$

defined by

$$\mathrm{Irrep}(G^{F^m})^F \ni \chi \mapsto N_m^{-1*}(\widetilde{\chi}) \in \mathrm{Fun}(G^F/\sim). \tag{5}$$

Remark 1.3. The map Sh_m is only well defined up to scaling by mth roots of unity since this is so for the assignment $\mathrm{Irrep}(G^{F^m})^F \ni \chi \mapsto \widetilde{\chi} \in \mathrm{Fun}(G^{F^m}/\sim_F)$.

The image of Sh_m forms an orthonormal basis of the Hermitian space $\mathrm{Fun}(G^F/\sim)$ which we call the mth Shintani basis. The set $\mathrm{Irrep}(G^F)$ of irreducible characters of G^F gives another orthonormal basis.

Remark 1.4. In general these two bases of $\mathrm{Fun}(G^F/\sim)$ are not equal (even for $m = 1$; see § 3.2). In fact, describing the relationship between these two bases is one of the objectives of this paper. However, if $G = \mathrm{GL}_n$ as in the original case studied by Shintani, then these bases are in fact equal and Shintani descent defines an explicit bijection between the sets $\mathrm{Irrep}(\mathrm{GL}_n(\mathbb{F}_{q^m}))^F$ and $\mathrm{Irrep}(\mathrm{GL}_n(\mathbb{F}_q))$.

[1] Extend χ to an irreducible character χ' of the semidirect product $G^{F^m}\langle F\rangle$, where $\langle F\rangle$ is the cyclic group of order m generated by F. Then set $\widetilde{\chi}(g) = \chi'(gF)$ for $g \in G^{F^m}$. Note that there is a choice involved in extending χ to χ', and the different choices differ by scaling by mth roots of unity on the coset $G^{F^m} \cdot F \subset G^{F^m}\langle F\rangle$.

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In [Sho92] Shoji proved that if G is connected and satisfies certain further conditions, then the mth Shintani basis of $\mathrm{Fun}(G^F/\sim)$ is (up to scaling by roots of unity) independent of m if m is *sufficiently divisible*. The elements of this common Shintani basis (when it exists) for sufficiently divisible m are known as the almost characters of G^F. Shoji proved that the almost characters are eigenvectors for the twisting operator defined by setting $m = 1$ in (3):

$$N_1^* : \mathrm{Fun}(G^F/\sim) \to \mathrm{Fun}(G^F/\sim). \tag{6}$$

Moreover, if G is reductive and satisfies some further conditions (e.g. if G has connected center or G is special linear and if p is not too small) then Shoji has proved that the almost characters of G^F coincide with the 'trace of Frobenius' functions associated with F-stable character sheaves on G. Our main result, Theorem 1.7 below, says that analogues of all these results are true for neutrally unipotent groups G, namely groups such that their neutral connected components G° are unipotent algebraic groups.

1.2 Shintani descent in general and pure inner forms

In [Sho92], an analogous descent construction was also carried out for disconnected groups under some additional conditions. In § 3 we will define Shintani descent in complete generality. The main difficulty in treating the disconnected case is the fact that the Lang isogeny for disconnected groups may not be surjective. However, this can be overcome if we consider not just the Frobenius F that we start out with, but also all pure inner forms of F. We will now briefly describe the main results of this paper.

Let G be any (possibly disconnected) algebraic group over $\mathtt{k}$. Let $F : G \to G$ be a Frobenius map defining an $\mathbb{F}_q$-rational structure on G. Consider the F-twisted conjugation action of G on itself as described by (1). Let $H^1(F, G)$ denote the set of F-twisted conjugacy classes in G. By Lang's theorem we have a natural bijection $H^1(F, G) = H^1(F, \Pi_0(G))$. In particular, $H^1(F, G)$ is a finite set and each orbit of the F-twisted conjugation action is a union of some of the connected components of the group G.

For each $g \in G$, we can define a new Frobenius map $gF := \mathrm{ad}(g) \circ F : G \to G$ which we call an inner form of F. The isomorphism class of the $\mathbb{F}_q$-rational form of G defined by the Frobenius gF only depends on the F-twisted conjugacy class of g since we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "gF"] \arrow[d, "ad(h)"'] & G \arrow[d, "ad(h)"] \\ G \arrow[r, "hgF(h^{-1})F"] & G \end{tikzcd}$$

for each $h, g \in G$. We are interested in the representation theory of G^F. However, it is more natural to consider all pure inner forms $G^{\mathrm{ad}(g)\circ F}$ as $g \in G$ runs over a set of representatives of $H^1(F, G)$.

Definition 1.5. We define the set

$$\mathrm{Irrep}(G, F) := \coprod_{\langle g\rangle \in H^1(F,G)} \mathrm{Irrep}(G^{\mathrm{ad}(g)\circ F}) = \coprod_{\langle g\rangle \in H^1(F,G)} \mathrm{Irrep}(G^{gF}) \tag{7}$$

and the commutative $\overline{\mathbb{Q}}_\ell$-algebra (under convolution of functions)

$$\mathrm{Fun}([G], F) := \prod_{\langle g\rangle \in H^1(F,G)} \mathrm{Fun}(G^{gF}/\sim). \tag{8}$$

We equip $\mathrm{Fun}([G],F)$ with the standard Hermitian inner product. By taking the characters of irreducible representations, we can consider the set $\mathrm{Irrep}(G,F)$ as an orthonormal basis of $\mathrm{Fun}([G],F)$.

In § 3.1, for each positive integer m, we will define a Shintani descent map (well defined up to scaling by mth roots of unity)

$$\mathrm{Sh}_m : \mathrm{Irrep}(G,F^m)^F \hookrightarrow \mathrm{Fun}([G],F) \tag{9}$$

generalizing (4). The image of Sh_m inside $\mathrm{Fun}([G],F)$ is an orthonormal basis which we again call the mth Shintani basis (which is well defined up to scaling by mth roots of unity).

Also, just as in the connected case (see (6)), we will construct a (unitary) twisting operator

$$\Theta = N_1^* : \mathrm{Fun}([G],F) \to \mathrm{Fun}([G],F). \tag{10}$$

The map N_1 that we will define in § 3.1 permutes the set of rational conjugacy classes of all the pure inner forms within each geometric conjugacy class.

Remark 1.6. Each irreducible character $\chi \in \mathrm{Irrep}(G,F) \subset \mathrm{Fun}([G],F)$ is supported on exactly one pure inner form G^{gF}. However, if $\chi \in \mathrm{Irrep}(G,F^m)^F$, then $\mathrm{Sh}_m(\chi) \in \mathrm{Fun}([G],F)$ can be non-zero on multiple pure inner forms. Also the map N_1 can map a rational conjugacy class of one inner form to a rational conjugacy class of a different inner form. From these considerations we see that it is essential to consider all pure inner forms G^{gF} at the same time.

We now introduce the category $\mathscr{D}_G(G)$. This is the triangulated braided monoidal category of conjugation equivariant $\overline{\mathbb{Q}}_\ell$-complexes on G. The monoidal structure comes from the convolution with compact support of $\overline{\mathbb{Q}}_\ell$-complexes. For each object $C \in \mathscr{D}_G(G)$, we have functorial automorphisms $\theta_C : C \to C$ known as twists. We refer to [BD14] for more about the structure of the category $\mathscr{D}_G(G)$. We recall from [Des14b, § 2.4.8] that, given an object $C \in \mathscr{D}_G(G)$ and an isomorphism $\psi : F^*C \to C$, we have the associated 'trace of Frobenius' function $T_{C,\psi} \in \mathrm{Fun}([G],F)$.

We can now state our main result. Let G be an algebraic group such that its neutral connected component G° is a unipotent group. In this case, the notion of character sheaves on G has been defined[2] in [BD14]. We let $CS(G)$ denote the set of character sheaves on G. These are (the isomorphism classes of) certain special objects in the category $\mathscr{D}_G(G)$.

Theorem 1.7. *Let G be a neutrally unipotent group over* $\mathtt{k}$ *as above. Let $F : G \to G$ be an $\mathbb{F}_q$-Frobenius. Then we have the following results.*

(i) *There exists a positive integer m_0 such that if m is divisible by m_0, then the mth Shintani basis of* $\mathrm{Fun}([G],F)$ *is (up to scaling by roots of unity) independent of m. We define almost characters to be the elements of this common Shintani basis.*

(i′) *The mth Shintani basis of* $\mathrm{Fun}([G],F)$ *only depends (up to scaling by roots of unity) on the residue of m modulo m_0.*

(ii) *The almost characters as defined above are eigenvectors for the twisting operator Θ (see (10)).*

(iii) *Recall that the Frobenius F induces a permutation of the set $CS(G)$. The mapping which takes an F-stable character sheaf to its associated trace of Frobenius function in* $\mathrm{Fun}([G],F)$ *defines a bijection from the set $CS(G)^F$ to the set of almost characters.*

[2] To be precise, in [BD14] character sheaves are only studied for unipotent groups. However, everything can be readily extended to the neutrally unipotent case.

Remark 1.8. We conjecture that in fact the above results should hold for all algebraic groups. In particular, we expect that there should be an interesting theory of character sheaves on all algebraic groups.

In § 2 we describe some preliminary constructions. In particular, for each $m \in \mathbb{Z}$, we introduce the $\overline{\mathbb{Q}}_\ell$-linear triangulated category $\mathscr{D}_G^{F^m}(G)$ which encodes the representation theory of G^{F^m} and all its pure inner forms, and we define associative convolution products $\mathscr{D}_G^{F^{m_1}}(G) \times \mathscr{D}_G^{F^{m_2}}(G) \longrightarrow \mathscr{D}_G^{F^{m_1+m_2}}(G)$ which satisfy some crossed braiding relations. We construct analogues of the sheaf-function correspondence in certain general settings.

We then use the various constructions from § 2 to define Shintani descent for general algebraic groups in § 3.1. In § 3.2 we study the special case of Shintani descent for $m = 1$, namely the twisting operator introduced in (10). In § 3.3 we prove that the trace functions associated with F-stable simple objects in $\mathscr{D}_G(G)$ are always eigenvectors for the twisting operator Θ from (10).

In § 4 we define the notion of twists in the categories $\mathscr{D}_G^{F^m}(G)$ and study their properties. We prove that these twists are compatible with the traces in $\mathscr{D}_G^{F^m}(G)$ that were defined in [Des14b, § 2.4.5].

In § 5 we express the inner products between elements of the mth Shintani basis and irreducible characters in terms of the traces and convolution structure of the categories $\mathscr{D}_G^{F^m}(G)$. Note that these inner products are the entries of the unitary matrix which relates the mth Shintani basis of $\mathrm{Fun}([G], F)$ to the orthonormal basis of $\mathrm{Fun}([G], F)$ formed by the irreducible characters of all the pure inner forms.

From § 6 onwards we restrict to the case of neutrally unipotent groups. In § 6 we recall some relevant facts from the theory of character sheaves on unipotent groups. In particular, we recall the notion of $\mathbb{L}$-packets and prove that Shintani descent respects the $\mathbb{L}$-packet decompositions. We also state a refined version of Theorem 1.7 which takes into account the $\mathbb{L}$-packet decompositions, and we deduce Theorem 1.7 from this refined version.

In § 7 we study some properties of the modular categories and their module categories that arise from the theory of character sheaves on neutrally unipotent groups. In particular, we will prove (see Theorem 7.5) that the modular categories that arise from this theory are positive integral.

Finally, in § 8 we complete the proof of the refined version of our main theorem.

2. Preliminary constructions

In this section we will describe some preliminary constructions which will help us to define and study Shintani descent for general algebraic groups over $\mathtt{k}$ equipped with a Frobenius map. In this paper, by passing to the perfectizations, we will always assume that all our groups are in fact perfect quasi-algebraic groups over $\mathtt{k}$ even though we may not mention this explicitly. For more about this convention, see [BD14, § 1.9]. With this convention, the Frobenius maps in fact become automorphisms.

2.1 Twisted conjugation actions

Let G be a (perfect quasi-)algebraic group over $\mathtt{k}$ and let $\gamma : G \to G$ be an automorphism. We define the γ-twisted conjugation action of G on itself by $g : h \mapsto gh\gamma(g)^{-1}$. Equivalently, we can think of this action as the action of G by conjugation on the coset $G\gamma$ in the semidirect product $\widetilde{G} := G\langle\gamma\rangle = G \rtimes \mathbb{Z}$ under the natural identification of G with $G\gamma$. We let $\mathscr{D}_G^\gamma(G)$ denote the equivariant derived category for the γ-conjugation action of G on itself.

Now suppose that $\gamma_1, \gamma_2 : G \to G$ are automorphisms of G that commute with each other. Then the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\gamma_1\text{-conjugation}"] \arrow[d, "\gamma_2\times\gamma_2"'] & G \arrow[d, "\gamma_2"] \\ G \times G \arrow[r, "\gamma_1\text{-conjugation}"] & G \end{tikzcd} \tag{11}$$

Hence pullback by γ_2 induces a functor $\gamma_2^* : \mathscr{D}_G^{\gamma_1}(G) \to \mathscr{D}_G^{\gamma_1}(G)$. We let $\mathscr{D}_G^{\gamma_1}(G)^{\gamma_2}$ denote the equivariantization. The objects of $\mathscr{D}_G^{\gamma_1}(G)^{\gamma_2}$ are pairs (M, ψ) where $M \in \mathscr{D}_G^{\gamma_1}(G)$ and $\psi : \gamma_2^*(M) \xrightarrow{\cong} M$ is an isomorphism in $\mathscr{D}_G^{\gamma_1}(G)$.

Let us define

$$R_{\gamma_1,\gamma_2} := \{(g,h) \in G \times G \mid h\gamma_2(g)\gamma_1(h)^{-1} = g\}. \tag{12}$$

We equip R_{γ_1,γ_2} with an action of G by setting ${}^x(g,h) = (xg\gamma_1(x)^{-1}, xh\gamma_2(x)^{-1})$ for $x \in G$ and $(g,h) \in R_{\gamma_1,\gamma_2}$. We let $\mathrm{Fun}_G(R_{\gamma_1,\gamma_2})$ denote the space of $\overline{\mathbb{Q}}_\ell$-valued G-invariant functions on R_{γ_1,γ_2}. In our applications in this paper, one or both of the automorphisms will be Frobenius maps.

Remark 2.1. For $(g,h) \in R_{\gamma_1,\gamma_2}$, we can check that we have ${}^{g^{-1}}(g,h) = (\gamma_1(g), \gamma_1(h))$ and also that ${}^{h^{-1}}(g,h) = (\gamma_2(g), \gamma_2(h))$.

Example 2.2. Let G be connected, let $F : G \to G$ be a Frobenius map and let $\gamma : G \to G$ be an automorphism that commutes with F. Then the quotient set $G\backslash R_{\gamma,F}$ is finite and in fact can be naturally identified with the set $G^F/\sim_\gamma$. Note that in this case, γ induces an automorphism of the finite group G^F. To construct this bijection, we note that since G is connected, the G-orbit of each $(g,h) \in R_{\gamma,F}$ contains an element of the form $(x,1)$ by Lang's theorem. Also we see that $(x,1) \in R_{\gamma,F}$ if and only if $x \in G^F$. Moreover, the γ-twisted conjugacy class of x in G^F is uniquely determined by the G-orbit of (g,h). This defines the desired natural bijection. Hence in this case we have a natural identification of the spaces $\mathrm{Fun}_G(R_{\gamma,F}) \cong \mathrm{Fun}(G^F/\sim_\gamma)$. Some cases of particular interest to us are when $\gamma = \mathrm{id}_G$ or γ is another Frobenius.

Remark 2.3. In this paper we wish to work with possibly disconnected groups. In the general situation it is more natural to work with the space $\mathrm{Fun}_G(R_{\gamma,F})$ than with the space $\mathrm{Fun}(G^F/\sim_\gamma)$ since the former takes into account all the pure inner forms.

Example 2.4. Let G be any algebraic group equipped with a Frobenius F. In this case the G orbits in $R_{\mathrm{id}_G,F}$ are in a natural bijection with the sets of conjugacy classes in all the pure inner forms G^{gF}. Hence we have a natural identification $\mathrm{Fun}([G],F) = \mathrm{Fun}_G(R_{\mathrm{id}_G,F})$. We will use these two notations interchangeably.

2.2 Some auxiliary maps

The following simple lemma is key in defining the Shintani descent maps for general algebraic groups.

LEMMA 2.5. (i) *Swapping the factors defines a G-equivariant isomorphism*

$$\tau : R_{\gamma_1,\gamma_2} \to R_{\gamma_2,\gamma_1}. \tag{13}$$

(ii) *We have G-equivariant twisting isomorphisms*

$$t_1 : R_{\gamma_1,\gamma_2} \to R_{\gamma_1\gamma_2,\gamma_2} \tag{14}$$

defined by $t_1(g,h) = (h\gamma_2(g),h) = (g\gamma_1(h),h)$ *and*

$$t_2 : R_{\gamma_1,\gamma_2} \to R_{\gamma_1,\gamma_1\gamma_2} \tag{15}$$

defined by $t_2(g,h) = (g,h\gamma_2(g)) = (g,g\gamma_1(h))$.

Example 2.6. Suppose that G is connected and that $F_1,F_2 : G \to G$ are commuting Frobenius maps. By Example 2.2, $G\backslash R_{F_1,F_2} = G^{F_2}/\sim_{F_1}$ and $G\backslash R_{F_2,F_1} = G^{F_1}/\sim_{F_2}$ and the swapping map τ induces an identification $G^{F_2}/\sim_{F_1} \cong G^{F_1}/\sim_{F_2}$ which agrees with the identification defined in [Sho92, § 1.1] up to an inversion. This identification was used in [Sho92] to define Shintani descent for connected groups.

Example 2.7. Let G be connected and let $F : G \to G$ be a Frobenius. Then by Example 2.2 we have $G\backslash R_{\mathrm{id}_G,F} = G^F/\sim$. In this case the G-isomorphism $t_2 : R_{\mathrm{id}_G,F} \to R_{\mathrm{id}_G,F}$ induces the twisting map $N_1 : (G^F/\sim) \longrightarrow (G^F/\sim)$ that is studied in [Sho92, § 4] and that we introduced in (6). This twisting map is a map which permutes the rational conjugacy classes of G^F within each geometric conjugacy class.

2.3 The sheaf-function correspondence

Given an object $(M,\psi) \in \mathscr{D}_G^{\gamma_1}(G)^{\gamma_2}$, we will define the associated 'trace function' $T_{M,\psi} \in \mathrm{Fun}_G(R_{\gamma_1,\gamma_2})$. For any $g,h \in G$, the equivariance structure φ_M of $M \in \mathscr{D}_G^{\gamma_1}(G)$ and the γ_2-equivariance structure ψ give us isomorphisms of stalks

$$M_{h\gamma_2(g)\gamma_1(h)^{-1}} \xrightarrow{\varphi_M(h^{-1},h\gamma_2(g)\gamma_1(h)^{-1})} M_{\gamma_2(g)} \xrightarrow{\psi(g)} M_g. \tag{16}$$

If $(g,h) \in R_{\gamma_1,\gamma_2}$, then this composition is an automorphism of the stalk $M_g \in D^b(\mathrm{Vec})$. Hence we can define $T_{M,\psi}(g,h)$ to be the trace of this composition.

Example 2.8. Let G be any algebraic group. Set $\gamma_1 = \mathrm{id}_G$ and γ_2 to be an $\mathbb{F}_q$-Frobenius map $F : G \to G$. In this case, the construction of the trace functions defined here reduces to the notion of the 'trace function' associated with an F-equivariant object of $\mathscr{D}_G(G)$ that was studied in [Des14b, § 2.4.8]. Recall from Example 2.4 that in this case the space $\mathrm{Fun}_G(R_{\mathrm{id}_G,F})$ can be identified with the space $\mathrm{Fun}([G],F)$ of class functions on all pure inner forms of G^F. We refer to [Des14b, § 2.4.8] for details.

Example 2.9. Let G be any algebraic group, but now let γ_1 be a Frobenius $F : G \to G$ and let $\gamma_2 = \mathrm{id}_G$. In this case, our construction takes an object in $\mathscr{D}_G^F(G)^{\mathrm{id}_G}$ and produces a function in $\mathrm{Fun}_G(R_{F,\mathrm{id}_G})$. Let $(M,\mathrm{id}_M) \in \mathscr{D}_G^F(G)^{\mathrm{id}_G}$. This gives us a function $\tau^*T_{M,\mathrm{id}_M} \in \mathrm{Fun}_G(R_{\mathrm{id}_G,F})$ where $\tau : R_{\mathrm{id}_G,F} \to R_{F,\mathrm{id}_G}$ is the swapping map. Then it is easy to check that $\tau^*T_{M,\mathrm{id}_M} = \overline{\chi_M}$, where χ_M is the character of $M \in \mathscr{D}_G^F(G)$ as defined in [Des14b, § 2.4.7].

2.4 Algebraic groups over $\mathbb{F}_q$ equipped with an $\mathbb{F}_q$-automorphism

In this section we study algebraic groups defined over $\mathbb{F}_q$ equipped with an automorphism defined over $\mathbb{F}_q$. Let G be any algebraic group equipped with an $\mathbb{F}_q$-Frobenius $F : G \to G$ and an automorphism $\gamma : G \to G$ commuting with F. Let us now understand the map of $\mathtt{k}$-schemes $R_{\gamma,F} \xrightarrow{p_2} G$. For this it is convenient to identify G with the coset $G\gamma \subset G \rtimes \gamma^{\mathbb{Z}}$. Then the conjugation action of G on $G\gamma$ corresponds to the γ-twisted conjugation on G. For each $h \in G$, we have the Frobenius map $hF : G\gamma \to G\gamma$. Under the identification with G, this modified Frobenius maps $g \mapsto hF(g)\gamma(h)^{-1}$. Note that this modified Frobenius does not respect the group structure of G. In other words, we have the following lemma.

Lemma 2.10. *In the notation above, $(g,h) \in R_{\gamma,F}$ if and only if $g \in (G\gamma)^{hF}$.*

Remark 2.11. Hence we can identify the spaces $\mathrm{Fun}_G(R_{\gamma,F})$ and $\mathrm{Fun}([G\gamma], F)$, the space of class functions on the $\mathbb{F}_q$-points of all pure inner forms of the coset $G\gamma$.

Let us now study the category $\mathscr{D}_G^F(G)^\gamma$ and the space $\mathrm{Fun}_G(R_{F,\gamma})$. There are only finitely many G-orbits in $R_{F,\gamma}$ and hence $\mathrm{Fun}_G(R_{F,\gamma})$ is finite-dimensional. For $(g,h) \in R_{F,\gamma}$ its stabilizer,

$$\mathrm{Stab}_G(g,h) = \{x \in G^{gF} \mid h\gamma(x)h^{-1} = x\} \tag{17}$$

is finite. We define an Hermitian inner product on $\mathrm{Fun}_G(R_{F,\gamma})$ as follows:

$$\langle f_1, f_2\rangle := \sum_{\langle (g,h)\rangle \in G\backslash R_{F,\gamma}} \frac{f_1(g,h)\overline{f_2(g,h)}}{|\mathrm{Stab}_G(g,h)|}. \tag{18}$$

As stated earlier, we need to take into consideration all pure inner forms of F. For each $g \in G$ we have the following commutative diagram.

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[d, "\mathrm{ad}(g)\circ F"'] & G \arrow[d, "\mathrm{ad}(\gamma(g))\circ F"] \\ G \arrow[r, "\gamma"] & G \end{tikzcd} \tag{19}$$

Hence γ induces an isomorphism $\gamma : G^{gF} \xrightarrow{\cong} G^{\gamma(g)F}$ and hence it induces a bijection

$$\gamma^* : \mathrm{Irrep}(G^{\gamma(g)F}) \xrightarrow{\cong} \mathrm{Irrep}(G^{gF}). \tag{20}$$

We recall from [Des14b] that the set $\mathrm{Irrep}(G^{gF})$ is canonically determined by the F-conjugacy class of g in $H^1(F,G)$. Hence we see that γ induces a bijection

$$\gamma^* : \mathrm{Irrep}(G,F) \to \mathrm{Irrep}(G,F). \tag{21}$$

2.5 Irreducible representations fixed by automorphisms

In this section we study the fixed point set $\mathrm{Irrep}(G,F)^\gamma$ of the bijection $\gamma^* : \mathrm{Irrep}(G,F) \to \mathrm{Irrep}(G,F)$ defined above. Let us recall a simple observation from [Des14b, § 2.4.2]. (See § 2.1 for the notation.)

Lemma 2.12. *We have an equivalence of triangulated categories*

$$\mathscr{D}_G^F(G) \cong \bigoplus_{\langle g\rangle \in H^1(F,G)} D^b\mathrm{Rep}(G^{gF}).$$

The category $\mathscr{D}_G^F(G)$ is in fact a semisimple abelian category whose (isomorphism classes of) simple objects are parametrized by $\mathrm{Irrep}(G,F) \times \mathbb{Z}$, where the integers keep track of the degree shift. If $W \in \mathrm{Irrep}(G^{gF}) \subset \mathrm{Irrep}(G,F)$ then we have the associated local system $W_{\mathrm{loc}} \in \mathscr{D}_G^F(G)$ supported on the F-twisted conjugacy class of g.

We note that γ acts on the set $H^1(F,G)$ which parametrizes the pure inner forms and only the irreducible representations of γ-fixed pure inner forms can lie in $\mathrm{Irrep}(G,F)^\gamma$. Hence suppose that $g \in G$ represents a γ-fixed F-twisted conjugacy class, i.e. there exists an $h \in G$ such that $g = h\gamma(g)F(h)^{-1}$, or in other words an h such that $(g,h) \in R_{F,\gamma}$. We have an automorphism

$$h\gamma : G^{gF} \xrightarrow{\gamma} G^{\gamma(g)F} \xrightarrow{\mathrm{ad}(h)} G^{gF}. \tag{22}$$

We note that a different choice of $h \in G$ only changes the automorphism above by an inner automorphism of G^{gF}. Hence the action of $h\gamma$ on $\mathrm{Irrep}(G^{gF})$ is independent of the choice of h. Let $W \in \mathrm{Irrep}(G^{gF}) \subset \mathrm{Irrep}(G,F)$ and consider the local system $W_{\mathrm{loc}} \in \mathscr{D}_G^F(G)$ supported on the F-twisted conjugacy class of g. Then $W \in \mathrm{Irrep}(G,F)^\gamma$ if and only if there exists an intertwiner $\psi : (h\gamma)^*W \xrightarrow{\cong} W$. Then we can define an object $(W_{\mathrm{loc}},\psi) \in \mathscr{D}_G^F(G)^\gamma$. Taking the associated trace function, we get $T_{W,\psi} \in \mathrm{Fun}(G^{gF}/\sim_{h\gamma}) \subset \mathrm{Fun}_G(R_{F,\gamma})$ (see Lemma 2.13). In fact we can choose the intertwiner ψ in such a way that we obtain an irreducible representation of the finite group $G^{gF}\langle h\gamma\rangle$, where $\langle h\gamma\rangle$ is the finite cyclic group generated by the automorphism $h\gamma$ of G^{gF}. For each $W \in \mathrm{Irrep}(G,F)^\gamma$ we fix such an intertwiner ψ_W as above. In this case we have $\|T_{W,\psi_W}\| = 1$ and T_{W,ψ_W} is well defined by W up to scaling by a root of unity. Here $\|\cdot\|$ denotes the Hermitian norm on $\mathrm{Fun}_G(R_{F,\gamma})$.

In other words, we have the following lemma.

LEMMA 2.13. *Let $W \in \mathrm{Irrep}(G,F)$. Then $W \in \mathrm{Irrep}(G,F)^\gamma$ if and only if $\gamma^*W_{\mathrm{loc}} \cong W_{\mathrm{loc}}$. If such a W lies in $\mathrm{Irrep}(G^{gF})^{h\gamma}$, we can choose an intertwiner $\psi_W : (h\gamma)^*W \xrightarrow{\cong} W$ such that we obtain a representation of $G^{gF}\langle h\gamma\rangle$ on W extending the original representation of G^{gF}. Such a ψ_W is well defined up to scaling by roots of unity. Further, we have an orthogonal decomposition*

$$\mathrm{Fun}_G(R_{F,\gamma}) = \bigoplus_{\langle g\rangle \in H^1(F,G) \mid \gamma\langle g\rangle = \langle g\rangle} \mathrm{Fun}(G^{gF}/\sim_{h\gamma}). \tag{23}$$

The set

$$\{T_{W,\psi_W} \mid W \in \mathrm{Irrep}(G^{gF})^{h\gamma}\} \subset \mathrm{Fun}(G^{gF}/\sim_{h\gamma}) \tag{24}$$

is an orthonormal basis, and hence the set

$$\{T_{W,\psi_W} \mid W \in \mathrm{Irrep}(G,F)^{\gamma}\} \subset \mathrm{Fun}_G(R_{F,\gamma}) \tag{25}$$

is an orthonormal basis.

2.6 Irreducible representations fixed by Frobenius

Let us now refine the results of the previous section in the special case which is of interest from the point of view of Shintani descent. Let m be any positive integer and let us consider the set $\mathrm{Irrep}(G,F^m)^F$ which appears on one side of the Shintani descent map. If $W \in \mathrm{Irrep}(G,F^m)^F$ we will place some further restrictions on the choice of the intertwiner ψ_W such that with these restrictions the different choices differ by mth roots of unity. Hence with each $W \in \mathrm{Irrep}(G,F^m)^F$ we will be able to associate a function $T_{W,\psi_W} \in \mathrm{Fun}_G(R_{F^m,F})$ which will be unique up to scaling by mth roots of unity.

Suppose that $W \in \mathrm{Irrep}(G,F^m)^F$, where W is an irreducible representation of some pure inner form G^{gF^m}. By § 2.5 there must be an $h \in G$ such that $(g,h) \in R_{F^m,F}$. We can check that

$$hF(h)\cdots F^{m-1}(h)g^{-1} \in G^{gF^m} \cap G^{hF}.$$

Consider the group automorphism $hF : G^{gF^m} \to G^{gF^m}$ from § 2.5. Then $W \in \mathrm{Irrep}(G^{gF^m})^{hF}$. Let $\widetilde{G^{gF^m}\langle hF\rangle}$ be the quotient of the semidirect product $G^{gF^m} \rtimes (hF)^{\mathbb{Z}}$ modulo the relation

$$G^{gF^m} \ni hF(h) \cdots F^{m-1}(h)g^{-1} = (hF)^m.$$

This gives us an extension

$$0 \to G^{gF^m} \to \widetilde{G^{gF^m}\langle hF\rangle} \to \mathbb{Z}/m\mathbb{Z} \to 0. \tag{26}$$

Let us note that the semidirect product $G^{gF^m} \rtimes (hF)^{\mathbb{Z}}$ as well as the extension above do not depend (up to a canonical isomorphism) on the choice of h since the possible choices all lie in the coset $G^{gF^m}h$. For example, a different choice of h merely amounts to a different splitting of the semidirect product.

Since $W \in \mathrm{Irrep}(G^{gF^m})^{hF}$ we can extend the action to obtain a representation of $\widetilde{G^{gF^m}\langle hF\rangle}$ on W. This extension amounts to choosing a suitably normalized intertwiner $\psi_W : (hF)^*W \to W$ of G^{gF^m}-representations. We thus obtain an object $(W_{\mathrm{loc}}, \psi_W) \in \mathscr{D}_G^{F^m}(G)$ and the corresponding trace function $T_{W,\psi_W} \in \mathrm{Fun}_G(R_{F^m,F})$. It is clear that the different choices of extensions of the representation as above, namely the different choices of normalization of the intertwiner ψ_W, differ by scaling by mth roots of unity. (See also Remark 4.1 for another equivalent characterization of ψ_W.) Hence we have proved that given $W \in \mathrm{Irrep}(G, F^m)^F$, we can obtain the function $T_{W,\psi_W} \in \mathrm{Fun}_G(R_{F^m,F})$ as above, uniquely determined up to scaling by mth roots of unity. By Lemma 2.13, we see that the set $\{T_{W,\psi_W}\}_{W\in\mathrm{Irrep}(G,F^m)^F}$ is an orthonormal basis of $\mathrm{Fun}_G(R_{F^m,F})$.

2.7 Convolution of functions and module structure

Let $F : G \to G$ be a Frobenius map and let $\gamma_1, \gamma_2 : G \longrightarrow G$ be group automorphisms that commute with the Frobenius. In this section we define convolution products (which are associative)

$$\mathrm{Fun}_G(R_{\gamma_1,F}) \times \mathrm{Fun}_G(R_{\gamma_2,F}) \longrightarrow \mathrm{Fun}_G(R_{\gamma_1\gamma_2,F}). \tag{27}$$

If $(g_i, h) \in R_{\gamma_i,F}$ for $i = 1, 2$, then we can check that $(g_1\gamma_1(g_2), h) \in R_{\gamma_1\gamma_2,F}$. For $f_i \in \mathrm{Fun}_G(R_{\gamma_i,F})$ we define

$$f_1 * f_2(g, h) = \sum_{\substack{g_1\gamma_1(g_2)=g \\ (g_i,h)\in R_{\gamma_i,F}}} f_1(g_1, h) f_2(g_2, h) \tag{28}$$

for each $(g, h) \in R_{\gamma_1\gamma_2,F}$. Note that the sum is finite (e.g. using Lemma 2.10) and hence the convolution is well defined. If is easy to check that $f_1 * f_2 \in \mathrm{Fun}_G(R_{\gamma_1\gamma_2,F})$ and also that the convolution is associative.

Remark 2.14. This convolution of functions may perhaps be more transparent using the identification of Remark 2.11.

Remark 2.15. Using the convolution, each function space $\mathrm{Fun}_G(R_{\gamma,F}) = \mathrm{Fun}([G\gamma], F)$ in fact becomes a module over the commutative algebra $\mathrm{Fun}([G], F)$.

2.8 Convolution of complexes and sheaf-function correspondence

Let γ_1, γ_2, F be as in the previous section. Then, using the 'coset multiplication'

$$\mu_{\gamma_1,\gamma_2} : G \times G \to G, \quad \text{defined by } (g_1, g_2) \mapsto g_1\gamma_1(g_2), \tag{29}$$

we can define a convolution (with compact supports)

$$\mathscr{D}_G^{\gamma_1}(G)\times\mathscr{D}_G^{\gamma_2}(G)\longrightarrow\mathscr{D}_G^{\gamma_1\gamma_2}(G). \tag{30}$$

It may be useful to think of this as the standard convolution $\mathscr{D}_G(G\gamma_1)\times\mathscr{D}_G(G\gamma_2)\longrightarrow\mathscr{D}_G(G\gamma_1\gamma_2)$. The map $\gamma_1 : G\longrightarrow G$ induces the equivalence

$$\gamma_1 := \gamma_1^{-1*} = \gamma_1^{*-1} : \mathscr{D}_G^{\gamma_1}(G)\xrightarrow{\cong}\mathscr{D}_G^{\gamma_1\gamma_2\gamma_1^{-1}}(G). \tag{31}$$

Moreover, for $M_i\in\mathscr{D}_G^{\gamma_i}(G)$ we have functorial crossed braiding isomorphisms

$$\beta_{M_1,M_2} : M_1 * M_2\xrightarrow{\cong}\gamma_1(M_2)*M_1. \tag{32}$$

We also have the induced convolution

$$\mathscr{D}_G^{\gamma_1}(G)^F\times\mathscr{D}_G^{\gamma_2}(G)^F\longrightarrow\mathscr{D}_G^{\gamma_1\gamma_2}(G)^F. \tag{33}$$

Now the sheaf-function correspondence is compatible with pullbacks and pushforwards with compact supports (see also [Des14b, § 2.2]). Using this, we obtain the following proposition.

PROPOSITION 2.16. *The convolution of complexes (as defined above) is compatible with the convolution of functions (as defined in § 2.7) under the sheaf-function correspondence. Namely, let $M_i\in\mathscr{D}_G^{\gamma_i}(G)^F$ and let $T_{M_i}\in\mathrm{Fun}_G(R_{\gamma_i,F})$ be the associated trace function. Then*

$$T_{M_1}*T_{M_2}=T_{M_1*M_2}. \tag{34}$$

3. Shintani descent for general algebraic groups

In this section we define Shintani descent using the constructions from § 2.

3.1 Definition of Shintani descent

In this section we apply our previous results and constructions and define Shintani descent in general. Let G be any algebraic group over $\mathtt{k}$. Let $F : G\to G$ be an $\mathbb{F}_q$-Frobenius map. Let m be a positive integer. Note that we have a G-equivariant isomorphism (which we will call the inverse norm map)

$$N_m^{-1} : R_{1,F}\xrightarrow{\cong}R_{F^m,F} \tag{35}$$

given by $R_{1,F}\ni(g,h)\mapsto(ghF(h)\cdots F^{m-1}(h),h)\in R_{F^m,F}$, i.e. it is the composition of the twists (14):

$$R_{1,F}\xrightarrow{t_1}R_{F,F}\xrightarrow{t_1}R_{F^2,F}\xrightarrow{t_1}\cdots R_{F^m,F}. \tag{36}$$

Hence we obtain an isomorphism of Hermitian inner product spaces

$$N_m^{-1*} : \mathrm{Fun}_G(R_{F^m,F})\xrightarrow{\cong}\mathrm{Fun}_G(R_{1,F})=\mathrm{Fun}([G],F). \tag{37}$$

LEMMA 3.1. *For each positive integer m, the map $t_1^* : \mathrm{Fun}_G(R_{F^m,F})\longrightarrow\mathrm{Fun}_G(R_{F^{m-1},F})$ is an isomorphism of $\mathrm{Fun}([G],F)$-modules (see also Remark 2.15). Hence the map N_m^{-1*} defined above is a $\mathrm{Fun}([G],F)$-module isomorphism.*

Proof. It is enough to check that t_1^* preserves the module structure. Moreover, it is easy to check that $f_1 * t_1^*f_2 = t_1^*(f_1 * f_2)$ for $f_1\in\mathrm{Fun}([G],F)$ and $f_2\in\mathrm{Fun}_G(R_{F^m,F})$. □

Given W in $\mathrm{Irrep}(G,F^m)^F$, we have the associated trace function $T_{W,\psi_W} \in \mathrm{Fun}_G(R_{F^m,F})$ as in § 2.6.

Definition 3.2. We define Shintani descent

$$\mathrm{Sh}_m : \mathrm{Irrep}(G,F^m)^F \hookrightarrow \mathrm{Fun}([G],F) = \mathrm{Fun}_G(R_{\mathrm{id}_G,F}) \tag{38}$$

by $\mathrm{Sh}_m(W) = N_m^{-1*}(T_{W,\psi_W})$ for each $W \in \mathrm{Irrep}(G,F^m)^F$. By Lemma 2.13, the image of Shintani descent $\{\mathrm{Sh}_m(W)\}_{W\in \mathrm{Irrep}(G,F^m)^F} \subset \mathrm{Fun}([G],F)$ is an orthonormal basis known as the mth Shintani basis.

Remark 3.3. By § 2.6, for each $W \in \mathrm{Irrep}(G,F^m)^F$, $\mathrm{Sh}_m(W)$ is uniquely determined up to scaling by an mth root of unity.

3.2 The twisting operator

In this section we study a special case of Shintani descent when $m = 1$. In this case Shintani descent is a map

$$\mathrm{Sh}_1 : \mathrm{Irrep}(G,F) = \mathrm{Irrep}(G,F)^F \hookrightarrow \mathrm{Fun}([G],F). \tag{39}$$

However, as we will now see, this is not in general the natural inclusion obtained by taking characters, rather it is a twist of the natural inclusion.

Consider any object $M \in \mathscr{D}_G^F(G)$. Then, as defined in [Des14b, § 2.4.7] (see also Example 2.9), we have its character $\chi_M \in \mathrm{Fun}_G(R_{\mathrm{id},F})$. Let φ_M denote the F-twisted conjugation equivariance structure associated with M. We recall that for $(g,h) \in R_{\mathrm{id},F}$, $\chi_M(g,h) := \mathrm{tr}(M_h \xrightarrow{\varphi_M(g,h)} M_h)$. We further set

$$\psi_M(g) := \varphi_M(g,F(g)) : M_{F(g)} \xrightarrow{\cong} M_g \tag{40}$$

and obtain an object $(M,\psi_M) \in \mathscr{D}_G^F(G)^F$. Let us compute its associated trace function T_{M,ψ_M} in $\mathrm{Fun}_G(R_{F,F})$. Suppose that $(g,h) \in R_{F,F}$, i.e. $hF(g)F(h)^{-1} = g$. Then $T_{M,\psi_M}(g,h)$ is the trace of the automorphism

$$M_g \xrightarrow{\varphi_M(h^{-1},g)} M_{F(g)} \xrightarrow{\varphi_M(g,F(g))} M_g. \tag{41}$$

Hence $T_{M,\psi_M}(g,h) = \mathrm{tr}(\varphi_M(gh^{-1},g)) = \chi_M(gh^{-1},g)$. Now let $(g,h) \in R_{\mathrm{id},F}$. Then using Lemma 2.5, we see that

$$N_1^{-1*} T_{M,\psi_M}(g,h) = T_{M,\psi_M}(gh,h) = \chi_M(g,gh) = t_2^*\chi_M(g,h), \tag{42}$$

where t_2 is as in (15). Hence we see that if $W \in \mathrm{Irrep}(G,F)$, then we can canonically define $\mathrm{Sh}_1(W)$ and we have $\mathrm{Sh}_1(W) = t_2^*\chi_W$ where $\chi_W \in \mathrm{Fun}_G(R_{\mathrm{id},F})$ denotes the character of W.

3.3 Sheaf-function correspondence and twists

As we have seen before, the twist isomorphism $t_2 : R_{\mathrm{id},F} \longrightarrow R_{\mathrm{id},F}$ from (15) induces a unitary operator

$$t_2^* : \mathrm{Fun}([G],F) \longrightarrow \mathrm{Fun}([G],F). \tag{43}$$

In general, an irreducible character χ_W for $W \in \mathrm{Irrep}(G,F)$ may not be an eigenvector for the twisting operator defined above. However, as we will see below, if $(C,\psi) \in \mathscr{D}_G(G)^F$ is a conjugation equivariant Weil complex such that $\mathrm{End}(C) = \overline{\mathbb{Q}}_\ell$, then the trace of Frobenius function $T_{C,\psi} \in \mathrm{Fun}([G],F)$ is an eigenvector for the twisting operator.

First we recall from [BD14] that the category $\mathscr{D}_G(G)$ is equipped with a twist θ, i.e. an automorphism of the identity functor satisfying some properties. Hence for each $C \in \mathscr{D}_G(G)$ we have the twist $\theta_C : C \xrightarrow{\cong} C$. We now prove the following proposition.

PROPOSITION 3.4. *Let $(C,\psi) \in \mathscr{D}_G(G)^F$ be a Weil complex such that θ_C is a scalar. Then*

$$t_2^* T_{C,\psi} = \theta_C^{-1} \cdot T_{C,\psi}. \tag{44}$$

Proof. Let $(g,h) \in R_{\mathrm{id},F}$. Then we want to prove that

$$T_{C,\psi}(g,gh) = \theta_C^{-1} T_{C,\psi}(g,h). \tag{45}$$

We have a commutative diagram

$$\begin{tikzcd} M_g \arrow[r, "\varphi_C(h^{-1},g)"] \arrow[d, "\varphi_C(g,g)=\theta_C"'] & M_{F(g)} \arrow[r, "\psi(g)"] \arrow[d, equal] & M_g \arrow[d, equal] \\ M_g \arrow[r, "\varphi_C(h^{-1}g^{-1},g)"] & M_{F(g)} \arrow[r, "\psi(g)"] & M_g \end{tikzcd} \tag{46}$$

where φ_C denotes the G-equivariance structure associated with $C \in \mathscr{D}_G(G)$. The equality

$$\varphi_C(g,g) = \theta_C$$

follows from the definition of the twist θ_C (see [BD14]). The trace of the top row is $T_{C,\psi}(g,h)$ and the trace of the bottom row is $T_{C,\psi}(g,gh)$. Hence (45) holds. $\square$

4. Traces and twists in module categories

We have seen that the category $\mathscr{D}_G^F(G)$ is a $\mathscr{D}_G(G)$-module category. In this section, we will study some additional structures on $\mathscr{D}_G^F(G)$ known as twists and traces.

4.1 Twists in the category $\mathscr{D}_G^F(G)$

We know that the category $\mathscr{D}_G^F(G)$ is a $\mathscr{D}_G(G)$-module category and that the category $\mathscr{D}_G(G)$ is equipped with a twist $\theta : \mathrm{id}_{\mathscr{D}_G(G)} \to \mathrm{id}_{\mathscr{D}_G(G)}$. Explicitly, if $C \in \mathscr{D}_G(G)$ and φ_C is the conjugation equivariance structure associated with C, then

$$\theta_C(g) = \varphi_C(g,g) : C_g \xrightarrow{\cong} C_g.$$

We will now construct an analogous twist for the module category $\mathscr{D}_G^F(G)$, namely a natural transformation $\theta^F : \mathrm{id}_{\mathscr{D}_G^F(G)} \to F|_{\mathscr{D}_G^F(G)} = F^{-1*}|_{\mathscr{D}_G^F(G)}$.

Let $M \in \mathscr{D}_G^F(G)$ and let φ_M denote the G-equivariance structure of M. Then $\theta_M^F : M \to F(M)$ is defined by

$$\theta_M^F(g) = \varphi_M(F^{-1}(g),g) : M_g \xrightarrow{\cong} M_{F^{-1}(g)} = F(M)_g. \tag{47}$$

It is clear that the Frobenius preserves the twists, namely

$$F(\theta_C) = \theta_{F(C)} \quad \text{for } C \in \mathscr{D}_G(G) \text{ and} \tag{48}$$

$$F(\theta_M^F) = \theta_{F(M)}^F \quad \text{for } M \in \mathscr{D}_G^F(G). \tag{49}$$

Remark 4.1. For each integer m, we have a twist θ^{F^m} in the $\mathscr{D}_G(G)$-module category $\mathscr{D}_G^{F^m}(G)$. If we have $W \in \mathrm{Irrep}(G,F^m)^F$, with the choice of ψ_W as in § 2.6, then the composition

$$F^{m*}W_{\mathrm{loc}} \xrightarrow{F^{(m-1)^*}(\psi_W)} F^{(m-1)^*}W_{\mathrm{loc}} \to \cdots F^*W_{\mathrm{loc}} \xrightarrow{\psi_W} W_{\mathrm{loc}} \tag{50}$$

equals $F^{m*}\theta_{W_{\mathrm{loc}}}^{F^m}$.

It is also straightforward to check that the twist θ^F is compatible with the twist θ in the category $\mathscr{D}_G(G)$.

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LEMMA 4.2. *Let $C \in \mathscr{D}_G(G)$ and let $M \in \mathscr{D}_G^F(G)$. Then θ^F_{C*M} is equal to the composition*

$$C * M \xrightarrow{\beta_{C,M}} M * C \xrightarrow{\beta_{M,C}} F(C) * M \xrightarrow{\theta_{F(C)} * \theta^F_M} F(C) * F(M) \longrightarrow F(C * M).$$

Moreover, all the twists θ^{F^a} in $\mathscr{D}_G^{F^a}(G)$ for $a \in \mathbb{Z}$ are compatible with each other, i.e. if $L \in \mathscr{D}_G^{F^a}(G)$ and $M \in \mathscr{D}_G^{F^b}(G)$, then $\theta^{F^{a+b}}_{L*M}$ equals the composition

$$L{*}M \xrightarrow{\beta_{L,M}} F^a(M){*}L \xrightarrow{\beta_{F^a(M),L}} F^b(L){*}F^a(M) \xrightarrow{\theta^{F^a}_{F^b(L)} * \theta^{F^b}_{F^a(M)}} F^{a+b}(L){*}F^{a+b}(M) \to F^{a+b}(L{*}M).$$

4.2 Compatibility of twists and traces

In [Des14b, § 2.4.5], we have defined a normalized trace tr_F in the category $\mathscr{D}_G^F(G)$ which assigns a number in $\overline{\mathbb{Q}}_\ell$ to each endomorphism in $\mathscr{D}_G^F(G)$. We now prove the following theorem.

THEOREM 4.3. *For each object $M \in \mathscr{D}_G^F(G)$ we define the automorphism $\nu_{\theta^F_M}$ in $\mathscr{D}_G^{F^2}(G)$ to be the composition*

$$\nu_{\theta^F_M} : M * M \xrightarrow{\beta^{-1}_{M,F^*M}} M * F^*M \xrightarrow{\mathrm{id}_M * F^*(\theta^F_M)} M * M. \tag{51}$$

Then we have

$$\mathrm{tr}_F(\mathrm{id}_M) = \mathrm{tr}_{F^2}(\nu_{\theta^F_M}). \tag{52}$$

Proof. As usual, let φ_M denote the G-equivariance structure associated with $M \in \mathscr{D}_G^F(G)$. We will use the integral symbol ($\int$) to denote pushforward with compact supports; if $\mathcal{E} \in \mathscr{D}(X)$, we use $\int_X \mathcal{E}$ to denote $R\Gamma_c(X, \mathcal{E})$. Let $t \in G$. Using the definition of the convolution, the crossed braidings and the twists, we see that the stalk

$$\nu_{\theta^F_m}(t) : (M * M)_t \longrightarrow (M * M)_t$$

is given by the composition

$$\begin{aligned}
(M * M)_t &= \int_{h_1 F(h_2) = t} M_{h_1} \otimes M_{h_2} \\
&\xrightarrow{\int \varphi_M(h_2^{-1}, h_1) \otimes \mathrm{id}_M} \int_{h_1 F(h_2) = t} M_{h_2^{-1} h_1 F(h_2)} \otimes M_{h_2} \\
&= \int_{h_1 F(h_2) = t} F^* M_{F^{-1}(h_2^{-1} h_1) h_2} \otimes M_{h_2} = (M * F^*M)_t \\
&\xrightarrow{F^*(\theta^F_M) * \mathrm{id}_M = \int \varphi_M(F^{-1}(h_2^{-1} h_1) h_2, h_2^{-1} h_1 F(h_2)) \otimes \mathrm{id}_M} \int_{h_1 F(h_2) = t} M_{F^{-1}(h_2^{-1} h_1) h_2} \otimes M_{h_2} \\
&\times \xrightarrow{\tau = \mathrm{swap}} \int_{h_1 F(h_2) = t} M_{h_2} \otimes M_{F^{-1}(h_2^{-1} h_1) h_2} = (M * M)_t.
\end{aligned}$$

Consider the automorphism of the antidiagonal $\overline{\Delta}_t := \{(h_1, h_2) \in G \times G \mid h_1 F(h_2) = t\}$ that appears implicitly in the stalk of the automorphism $\nu_{\theta^F_m}$ at t as above, namely

$$(h_1, h_2) \mapsto (h_2, F^{-1}(h_2^{-1} h_1) h_2).$$

If we identify $\overline{\Delta}_t$ and G using the second projection, then the inverse of this automorphism corresponds to the Frobenius automorphism of G (as a scheme) defined by $h_2 \mapsto tF(h_2^{-1})$. The fixed point set of this Frobenius map is the set $\{(h,h) \mid hF(h) = t\} \subset \overline{\Delta}_t$. Hence by the Grothendieck–Lefschetz trace formula (the trace of the induced map on cohomology equals the sum of traces of the stalks of the map over all fixed points; see also [Boy13, Lemma 4.4(iii)]), we deduce that

$$\begin{aligned}\operatorname{tr}(\nu_{\theta_M^F}(t)) &= \sum_{hF(h)=t} \operatorname{tr}(\tau \circ (\varphi_M(1,h) \otimes \mathrm{id}) : M_h \otimes M_h \xrightarrow{\cong} M_h \otimes M_h) \\ &= \sum_{hF(h)=t} \operatorname{tr}(\tau : M_h \otimes M_h \xrightarrow{\cong} M_h \otimes M_h) \\ &= \sum_{hF(h)=t} \dim M_h.\end{aligned}$$

Now $M * M$ is an object of

$$\mathscr{D}_G^{F^2}(G) \cong \bigoplus_{\substack{\langle t\rangle \in \\ H^1(F^2,G)}} D^b\mathrm{Rep}(G^{tF^2}).$$

Hence by definition of the trace tr_{F^2} on $\mathscr{D}_G^{F^2}(G)$ (see [Des14b, § 2.4.5]) we have

$$\operatorname{tr}_{F^2}(\nu_{\theta_M^F}) = \sum_{\substack{\langle t\rangle \in \\ H^1(F^2,G)}} \frac{\operatorname{tr}(\nu_{\theta_M^F}(t))}{|G^{tF^2}|} \tag{53}$$

$$= \sum_{\substack{\langle t\rangle \in \\ H^1(F^2,G)}} \sum_{hF(h)=t} \frac{\dim M_h}{|G^{tF^2}|}. \tag{54}$$

Next note that $\Delta := \{(h,h) \mid h \in G\}$ is in fact a G-invariant subscheme of $R_{F,F}$ and (h,h) is mapped to $(h, hF(h))$ under the G-equivariant isomorphism $R_{F,F} \xrightarrow{t_2} R_{F,F^2}$. Hence we obtain

$$\operatorname{tr}_{F^2}(\nu_{\theta_M^F}(t)) = \sum_{\substack{\langle t\rangle \in \\ H^1(F^2,G)}} \sum_{(h,t)\in t_2(\Delta)} \frac{\dim M_h}{|G^{tF^2}|} \tag{55}$$

$$= \sum_{\substack{\langle t\rangle \in \\ H^1(F^2,G)}} \sum_{(h,t)\in t_2(\Delta)} \frac{\dim M_h}{|\mathrm{Stab}_G(h,t)| \cdot |G^{tF^2}\text{-orbit of } (h,t)|} \tag{56}$$

$$= \sum_{\langle (h,t)\rangle \in G\backslash t_2(\Delta)} \frac{\dim M_h}{|\mathrm{Stab}_G(h,t)|} \tag{57}$$

$$= \sum_{\langle (h,h)\rangle \in G\backslash \Delta} \frac{\dim M_h}{|\mathrm{Stab}_G(h,h)|} \tag{58}$$

$$= \sum_{\langle h\rangle \in H^1(G,F)} \frac{\operatorname{tr}(\mathrm{id}_{M_h})}{|G^{tF}|} \tag{59}$$

$$= \operatorname{tr}_F(\mathrm{id}_M) \tag{60}$$

as desired. □

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5. An inner product formula

For each integer $m > 0$, we have the mth Shintani basis $\{\mathrm{Sh}_m(W)\}_{W\in\mathrm{Irrep}(G,F^m)^F}$ of $\mathrm{Fun}([G],F)$ defined in § 3.1. On the other hand, we have the basis $\mathrm{Irrep}(G,F)$ consisting of the irreducible characters of all the pure inner forms. Our goal in this section is to describe the relationship between these two bases.

5.1 Twists and sheaf-function correspondence

We know that the category $\mathscr{D}_G^F(G)$ is equivalent to the direct sum of the bounded derived categories of the representation categories of all pure inner forms. Let us now observe that we have a natural identification of all the categories $\mathscr{D}_G^F(G)^{F^m}$ for all integers $m \geqslant 0$.

LEMMA 5.1. *For each $m \geqslant 0$, we have an equivalence of triangulated categories*

$$\mathscr{D}_G^F(G)^{\mathrm{id}_G} \xrightarrow{\cong} \mathscr{D}_G^F(G)^{F^m} \tag{61}$$

and hence the composition

$$\eta_m : \mathscr{D}_G^F(G) \longrightarrow \mathscr{D}_G^F(G)^{\mathrm{id}_G} \xrightarrow{\cong} \mathscr{D}_G^F(G)^{F^m} \tag{62}$$

where the first functor is defined by $M \mapsto (M, \mathrm{id}_M)$.

Proof. It is sufficient to construct an equivalence

$$\mathscr{D}_G^F(G)^{F^m} \xrightarrow{\cong} \mathscr{D}_G^F(G)^{F^{m+1}} \tag{63}$$

for each $m \geqslant 0$. Let $(M,\psi) \in \mathscr{D}_G^F(G)^{F^m}$, where $M \in \mathscr{D}_G^F(G)$ and $\psi : F^{m*}M \xrightarrow{\cong} M$. On the other hand, we have the twist $\theta^F_{F^{(m+1)^*}M} = F^{(m+1)^*}\theta^F_M : F^{(m+1)^*}M \xrightarrow{\cong} F^{m*}M$. Then

$$(M,\psi) \mapsto (M, \psi \circ F^{(m+1)^*}\theta^F_M) \tag{64}$$

defines the desired equivalence. □

Remark 5.2. From the proof, we see that if $M \in \mathscr{D}_G^F(G)$, then $\eta_m(M) = (M, \psi_{M,m}) \in \mathscr{D}_G^F(G)^{F^m}$ where

$$\psi_{M,m} : F^{m*}M \xrightarrow{F^{m*}\theta^F_M} F^{(m-1)^*}M \to \cdots F^*M \xrightarrow{F^*\theta^F_M} M. \tag{65}$$

LEMMA 5.3. *The diagram*

$$\begin{tikzcd}
\mathscr{D}_G^F(G)^{\mathrm{id}\,G} \arrow[r, "\cong"] \arrow[d] & \mathscr{D}_G^F(G)^F \arrow[r, "\cong"] \arrow[d] & \cdots \arrow[r, "\cong"] & \mathscr{D}_G^F(G)^{F^m} \arrow[r, "\cong"] \arrow[d] & {} \\
\mathrm{Fun}_G(R_{F,\mathrm{id}\,G}) \arrow[r, "\cong"] \arrow[d, "\tau^*"] & \mathrm{Fun}_G(R_{F,F}) \arrow[r, "\cong"] \arrow[d, "\tau^*"] & \cdots \arrow[r, "\cong"] & \mathrm{Fun}_G(R_{F,F^m}) \arrow[r, "\cong"] \arrow[d, "\tau^*"] & {} \\
\mathrm{Fun}([G],F) \arrow[r, "\cong"] & \mathrm{Fun}_G(R_{F,F}) \arrow[r, "\cong"] & \cdots \arrow[r, "\cong"] & \mathrm{Fun}_G(R_{F^m,F}) \arrow[r, "\cong"] & {}
\end{tikzcd} \tag{66}$$

is commutative, where the equivalences in the top row are as defined in the proof of the previous lemma, the isomorphisms in the bottom two rows are induced by the twists t_2, t_1 from (15), (14) respectively, and the top vertical arrows are defined by the sheaf-function correspondence from § 2.3.

Proof. This is straightforward to check from the definitions. □

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5.2 The inner product formula

Let $m \geqslant 0$ be an integer. Suppose that we have objects $L \in \mathscr{D}_G^{F^m}(G)$ and $M \in \mathscr{D}_G^F(G)$. Then we have the functorial crossed braiding isomorphisms (see § 2.8)

$$\beta_{M,F^*L}^{-1} : L * M \longrightarrow M * F^*L \tag{67}$$

and

$$\beta_{L,F^{m*}M}^{-1} : M * L \longrightarrow L * F^{m*}M \tag{68}$$

in the category $\mathscr{D}_G^{F^{m+1}}(G)$. If $L \in \mathscr{D}_G^{F^m}(G)^F$ and $M \in \mathscr{D}_G^F(G)^{F^m}$ then we obtain an automorphism

$$\zeta_{L,M} : L * M \xrightarrow{\beta_{M,F^*L}^{-1}} M * F^*L \xrightarrow{\beta_{F^*L,F^{m*}M}^{-1}} F^*L * F^{m*}M \xrightarrow{\psi_L * \psi_M} L * M \tag{69}$$

in $\mathscr{D}_G^{F^{m+1}}(G)$ where ψ_L and ψ_M are the equivariance isomorphisms associated with $L \in \mathscr{D}_G^{F^m}(G)^F$ and $M \in \mathscr{D}_G^F(G)^{F^m}$, respectively. Recall from [Des14b, § 2.4.5], that the category $\mathscr{D}_G^{F^{m+1}}(G)$ has a normalized trace denoted by $\mathrm{tr}_{F^{m+1}}$ in the notation from [Des14b, § 2.4.5]. The following theorem is a generalization of [Des14b, Theorem 2.14].

Theorem 5.4. *Let $M \in \mathscr{D}_G^F(G)$ and $L \in \mathscr{D}_G^{F^m}(G)^F$. Then we have the functions ${N_m^{-1}}^* T_{L,\psi_L}$ and $\chi_M \in \mathrm{Fun}([G], F)$. We also have the equality*

$$\langle {N_m^{-1}}^* T_{L,\psi_L}, \chi_M \rangle = \mathrm{tr}_{F^{m+1}}(\zeta_{L,\eta_m(M)}). \tag{70}$$

Corollary 5.5. *Let $V \in \mathrm{Irrep}(G, F)$ with the corresponding local system $V_{\mathrm{loc}} \in \mathscr{D}_G^F(G)$. Let $W \in \mathrm{Irrep}(G, F^m)^F$ and let $(W_{\mathrm{loc}}, \psi_W) \in D_G^{F^m}(G)^F$ where ψ_W is chosen as in § 2.6. Then we have the following relationship between the mth Shintani basis and $\mathrm{Irrep}(G, F)$:*

$$\langle \mathrm{Sh}_m(W), \chi_V \rangle = \mathrm{tr}_{F^{m+1}}(\zeta_{(W_{\mathrm{loc}},\psi_W),\eta_m(V_{\mathrm{loc}})}). \tag{71}$$

5.3 Proof of Theorem 5.4

The proof of Theorem 5.4 is similar to the proof of [Des14b, Theorem 2.14] as well as the proof of Theorem 4.3 above.

Let $\eta_m(M) = (M, \psi_M)$. We will often abuse notation and denote this object simply as $M \in \mathscr{D}_G^F(G)^{F^m}$. As before, let φ_L, φ_M denote the equivariance structures on L, M corresponding to the twisted conjugations actions. Let us compute the stalk of the automorphism $\zeta_{L,M}$ at a point $t \in G$. By the definition of the crossed braidings, the stalk

$$\zeta_{L,M}(t) : (L * M)_t \longrightarrow (L * M)_t$$

is given by the composition

$$\begin{aligned}
(L * M)_t &= \int_{h_1 F^m(h_2) = t} L_{h_1} \otimes M_{h_2} \\
&\xrightarrow{\int \varphi_L(h_2^{-1}, h_1) \otimes \mathrm{id}_M} \int_{h_1 F^m(h_2) = t} L_{h_2^{-1} h_1 F^m(h_2)} \otimes M_{h_2} = (M * F^*L)_t \\
&\xrightarrow{\int \mathrm{id}_L \otimes \varphi_M(F^{-1}(F^m(h_2)^{-1} h_1^{-1} h_2), h_2)} \int_{h_1 F^m(h_2) = t} F^*L_{F^{-1}(h_2^{-1} h_1 F^m(h_2))} \otimes M_{F^{-1}(F^m(h_2)^{-1} h_1^{-1} h_2) h_1 F^m(h_2)} \\
&= (F^*L * F^{m*}M)_t
\end{aligned}$$

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$$\int_{h_1F^m(h_2)=t} L_{F^{-1}(h_2^{-1}h_1F^m(h_2))}\otimes M_{F^{-1}(h_2^{-1}F^{-m}(h_1^{-1}h_2))F^{-m}(h_1)h_2} \xrightarrow{\int \psi_L(F^{-1}(h_2^{-1}h_1F^m(h_2)))\otimes\psi_M(F^{-1}(h_2^{-1}F^{-m}(h_1^{-1}h_2))F^{-m}(h_1)h_2)} (L*M)_t.$$

Consider the automorphism of the antidiagonal $\overline{\Delta}_t := \{(h_1,h_2)\in G\times G \mid h_1F^m(h_2)=t\}$ that appears implicitly in the stalk of the automorphism $\zeta_{L,M}$ as above, namely

$$(h_1,h_2)\mapsto (F^{-1}(h_2^{-1}h_1F^m(h_2)), F^{-1}(h_2^{-1}F^{-m}(h_1^{-1}h_2))F^{-m}(h_1)h_2).$$

The fixed point set of this automorphism is the finite set $\{(h_1,h_2)\in\overline{\Delta}_t \mid (h_1,h_2)\in R_{F^m,F}\}$. If we identify $\overline{\Delta}_t$ and G using the first projection, then the inverse of this automorphism corresponds to the Frobenius automorphism of G (as a scheme) defined by $h_1\mapsto tF^{m+1}(h_1)F^m(t)^{-1}$. Hence by the Grothendieck–Lefschetz trace formula and using the definition of the trace functions $T_{L,\psi_L}, T_{M,\psi_M}$, we deduce that

$$\mathrm{tr}(\zeta_{L,M}(t)) = \sum_{\substack{(h_1,h_2)\in R_{F^m,F}\\ h_1F^m(h_2)=t}} T_{L,\psi_L}(h_1,h_2)\cdot T_{M,\psi_M}(h_2,h_1). \tag{72}$$

Now $L*M$ is an object of

$$\mathscr{D}_G^{F^{m+1}}(G)\cong \bigoplus_{\substack{\langle t\rangle\in\\ H^1(F^{m+1},G)}} D^b\mathrm{Rep}(G^{tF^{m+1}}).$$

Hence by definition of the trace $\mathrm{tr}_{F^{m+1}}$ on $\mathscr{D}_G^{F^{m+1}}(G)$ (see [Des14b, § 2.4.5]) we have

$$\mathrm{tr}_{F^{m+1}}(\zeta_{L,M}) = \sum_{\substack{\langle t\rangle\in\\ H^1(F^{m+1},G)}} \frac{\mathrm{tr}(\zeta_{L,M}(t))}{|G^{tF^{m+1}}|} \tag{73}$$

$$= \sum_{\substack{\langle t\rangle\in\\ H^1(F^{m+1},G)}} \sum_{\substack{(h_1,h_2)\in R_{F^m,F}\\ h_1F^m(h_2)=t}} \frac{T_{L,\psi_L}(h_1,h_2)\cdot T_{M,\psi_M}(h_2,h_1)}{|G^{tF^{m+1}}|} \tag{74}$$

$$= \sum_{\substack{\langle t\rangle\in\\ H^1(F^{m+1},G)}} \sum_{\substack{(h,t)=(h_1,h_1F^m(h_2))\in\\ R_{F^m,F^{m+1}}}} \frac{T_{L,\psi_L}(h,F^{-m}(h^{-1}t))\cdot T_{M,\psi_M}(F^{-m}(h^{-1}t),h)}{|G^{tF^{m+1}}|}. \tag{75}$$

Note that in the last equality, we used the G-equivariant isomorphism $t_2 : R_{F^m,F}\xrightarrow{\cong} R_{F^m,F^{m+1}}$ defined by (15). Hence continuing further, we get

$$\mathrm{tr}_{F^{m+1}}(\zeta_{L,M}) = \sum_{\substack{\langle t\rangle\in\\ H^1(F^{m+1},G)}} \sum_{\substack{(h,t)\in\\ R_{F^m,F^{m+1}}}} \frac{T_{L,\psi_L}(h,F^{-m}(h^{-1}t))\cdot T_{M,\psi_M}(F^{-m}(h^{-1}t),h)}{|\mathrm{Stab}_G(h,t)|\cdot|G^{tF^{m+1}}\text{-orbit of }(h,t)|} \tag{76}$$

$$= \sum_{\langle(h,t)\rangle\in G\backslash R_{F^m,F^{m+1}}} \frac{T_{L,\psi_L}(h,F^{-m}(h^{-1}t))\cdot T_{M,\psi_M}(F^{-m}(h^{-1}t),h)}{|\mathrm{Stab}_G(h,t)|} \tag{77}$$

$$= \sum_{\langle(h_1,h_2)\rangle\in G\backslash R_{F^m,F}} \frac{T_{L,\psi_L}(h_1,h_2)\cdot T_{M,\psi_M}(h_2,h_1)}{|\mathrm{Stab}_G(h_1,h_2)|} \tag{78}$$

$$= \sum_{\langle (g,h)\rangle \in G\backslash R_{\mathrm{id}_G,F}} \frac{N_m^{-1*}T_{L,\psi_L}(g,h)\cdot N_m^{-1*}\tau^* T_{M,\psi_M}(g,h)}{|\mathrm{Stab}_G(g,h)|} \tag{79}$$

$$= \sum_{\langle (g,h)\rangle \in G\backslash R_{\mathrm{id}_G,F}} \frac{N_m^{-1*}T_{L,\psi_L}(g,h)\cdot \chi_M(g^{-1},h)}{|\mathrm{Stab}_G(g,h)|} \tag{80}$$

$$= \langle N_m^{-1*}T_{L,\psi_L}, \chi_M\rangle \tag{81}$$

as desired. Here in (80), we have used Lemma 5.3 and Example 2.9.

6. Shintani descent for neutrally unipotent groups

In the remainder of this paper, we restrict our attention to neutrally unipotent groups G over $\mathtt{k}$ equipped with an $\mathbb{F}_q$-Frobenius F. In this section, we will describe a refined version (see Theorem 6.6) of Theorem 1.7 and deduce that the theorem in fact follows from this refinement. We will prove Theorem 6.6 in § 8. We will use the theory of character sheaves on neutrally unipotent groups. Let us begin by recalling some aspects of this theory.

6.1 $\mathbb{L}$-packets of characters of neutrally unipotent groups

In [Boy13] a partitioning of the set $\mathrm{Irrep}(G,F)$ of irreducible characters into $\mathbb{L}$-packets has been constructed. Let us recall this notion. Let $\widehat{G}$ denote the set of isomorphism classes of minimal idempotents in the braided monoidal category $\mathscr{D}_G(G)$. The $\mathbb{L}$-packets of irreducible characters are parametrized by the set $\widehat{G}^F$ of F-stable minimal idempotents in $\mathscr{D}_G(G)$.

Definition 6.1. Let $e \in \widehat{G}^F$ and let $\psi_e : F^*e \xrightarrow{\cong} e$ be the unique isomorphism such that

$$T_e := T_{e,\psi_e} \in \mathrm{Fun}([G],F)$$

is an idempotent. Let $W \in \mathrm{Irrep}(G,F)$. We say that W lies in the $\mathbb{L}$-packet associated with e, or equivalently that $W \in \mathrm{Irrep}_e(G,F)$, if and only if the following equivalent (see [Des14b, Theorem 2.27]) conditions hold.

(i) The character χ_W lies in $T_e\mathrm{Fun}([G],F) \subset \mathrm{Fun}([G],F)$.

(ii) The dual (i.e. 'complex conjugate') idempotent $\overline{T_e} \in \mathrm{Fun}([G],F)$ acts as the identity on W. To be more precise, suppose that W is an irreducible representation of a pure inner form G^{gF}. Then we want the idempotent $\overline{T_e(\cdot,g)} \in \mathrm{Fun}(G^{gF})^{G^{gF}}$ to act trivially on W.

(iii) The local system $W_{\mathrm{loc}} \in e\mathscr{D}_G^F(G) \subset \mathscr{D}_G^F(G)$.

In [BD14], character sheaves and their $\mathbb{L}$-packets are also defined. The $\mathbb{L}$-packets of character sheaves are parametrized by $\widehat{G}$. Let us recall this.

Definition 6.2. Let $e \in \mathscr{D}_G(G)$ be a minimal idempotent. Let $\mathcal{M}_{G,e} \subset e\mathscr{D}_G(G)$ be the modular category associated with e as defined in [BD14]. Then the set $CS_e(G)$ of character sheaves in the $\mathbb{L}$-packet associated with e is defined to be the set of simple objects (up to isomorphism) of the modular category $\mathcal{M}_{G,e} \subset \mathscr{D}_G(G)$. The set $CS(G)$ of all character sheaves on G is defined to be the union (which turns out to be disjoint) of all the sets $CS_e(G)$ as e ranges over $\widehat{G}$.

We recall the following results from [Boy13].

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Theorem 6.3. (i) *We have a partition*

$$\mathrm{Irrep}(G,F)=\coprod_{e\in\widehat{G}^F}\mathrm{Irrep}_e(G,F) \tag{82}$$

and an isomorphism of algebras

$$\mathrm{Fun}([G],F)=\bigoplus_{e\in\widehat{G}^F}T_e\mathrm{Fun}([G],F). \tag{83}$$

(ii) *The set* $\mathrm{Irrep}_e(G,F)$ *is (after taking characters of the irreducible representations) an orthonormal basis of* $T_e\mathrm{Fun}([G],F)$.

(iii) *For each* $C\in CS_e(G)^F$, *we choose* $\psi_C:F^*C\xrightarrow{\cong}C$ *so that*[3] $\|T_{C,\psi_C}\|=1$. *Then the set* $\{T_{C,\psi_C}\}_{C\in CS_e(G)^F}$ *is also an orthonormal basis of* $T_e\mathrm{Fun}([G],F)$.

6.2 Shintani descent and $\mathbb{L}$-packets

In this section we show that Shintani descent respects the $\mathbb{L}$-packet decomposition. In particular, we will show that to prove Theorem 1.7, it is enough to restrict our attention to each $\mathbb{L}$-packet.

For any positive integer m we have

$$\mathrm{Irrep}(G,F^m)=\coprod_{e\in\widehat{G}^{F^m}}\mathrm{Irrep}_e(G,F^m). \tag{84}$$

Moreover, by Definition 6.1(iii) and the fact that F^* preserves convolution, the permutation of the set $\mathrm{Irrep}(G,F^m)$ induced by F (see §§ 2.4, 2.6) preserves this $\mathbb{L}$-packet decomposition:

$$F^*:\mathrm{Irrep}_e(G,F^m)\xrightarrow{\cong}\mathrm{Irrep}_{F^*e}(G,F^m). \tag{85}$$

Hence

$$\mathrm{Irrep}(G,F^m)^F=\coprod_{e\in\widehat{G}^F}\mathrm{Irrep}_e(G,F^m)^F. \tag{86}$$

Recall that Shintani descent is a map $\mathrm{Sh}_m:\mathrm{Irrep}(G,F^m)^F\hookrightarrow\mathrm{Fun}([G],F)$.

Proposition 6.4. *Let* $e\in\widehat{G}^F$. *Then we have* $\mathrm{Sh}_m(\mathrm{Irrep}_e(G,F^m)^F)\subset T_e\mathrm{Fun}([G],F)$ *and the image is an orthonormal basis of* $T_e\mathrm{Fun}([G],F)$. *In other words, Shintani descent preserves* $\mathbb{L}$-*packets.*

Proof. Let $W\in\mathrm{Irrep}_e(G,F^m)^F$ and let ψ_W be an intertwiner as in § 2.6. Since W lies in the $\mathbb{L}$-packet associated with e, $W_{\mathrm{loc}}\in e\mathscr{D}_G^{F^m}(G)$, i.e. $e*W_{\mathrm{loc}}\cong W_{\mathrm{loc}}$. Hence by Proposition 2.16, $T_{W_{\mathrm{loc}},\psi_W}\in T_e\mathrm{Fun}_G(R_{F^m,F})\subset\mathrm{Fun}_G(R_{F^m,F})$. Now by Lemma 3.1,

$$N_m^{-1*}:\mathrm{Fun}_G(R_{F^m,F})\to\mathrm{Fun}([G],F)$$

is a $\mathrm{Fun}([G],F)$-module isomorphism. Hence $\mathrm{Sh}_m(W)=N_m^{-1*}T_{W_{\mathrm{loc}},\psi_W}\in T_e\mathrm{Fun}([G],F)$ as desired. Moreover, since $\mathrm{Sh}_m(\mathrm{Irrep}(G,F^m)^F)$ is an orthonormal basis of $\mathrm{Fun}([G],F)$, by Theorem 6.3(i) and (86) it follows that $\mathrm{Sh}_m(\mathrm{Irrep}_e(G,F^m)^F)\subset T_e\mathrm{Fun}([G],F)$ must in fact be an orthonormal basis. □

Remark 6.5. We say that the image $\mathrm{Sh}_m(\mathrm{Irrep}_e(G,F^m)^F)$ is the mth Shintani basis of the subspace $T_e\mathrm{Fun}([G],F)$ associated with the minimal idempotent e.

[3] It is possible to impose some additional conditions so that ψ_C will be uniquely determined up to scaling by roots of unity.

6.3 A refined version of Theorem 1.7

We can now state the refined version of our main result, Theorem 1.7, and deduce the main theorem from this refinement.

Theorem 6.6. *Let $e \in \widehat{G}^F$ be an F-stable minimal idempotent. Then we have the following results.*

(i) *There exists a positive integer m_0 such that if m is divisible by m_0, then the mth Shintani basis of $T_e\mathrm{Fun}([G], F)$ (see Remark 6.5) is (up to scaling by roots of unity) independent of m. We define almost characters associated with e to be the elements of this common Shintani basis.*

(i′) *The mth Shintani basis of $T_e\mathrm{Fun}([G], F)$ only depends (up to scaling by roots of unity) on the residue of m modulo m_0.*

(ii) *The almost characters associated with e as defined above are eigenvectors for the twisting operator Θ (see (10)).*

(iii) *The mapping which takes an F-stable character sheaf $C \in CS_e(G)^F$ to its associated trace of Frobenius function T_{C,ψ_C} in $T_e\mathrm{Fun}([G], F)$ defines a bijection from the set $CS_e(G)^F$ to the set of almost characters associated with e.*

We will prove the theorem in § 8 below.

To prove Theorem 1.7(i) from this, it suffices to take a common multiple of all the positive integers m_0 corresponding to the minimal idempotents e in the finite set $\widehat{G}^F$. Parts (ii) and (iii) of Theorem 1.7 also follow readily from the above refinement.

7. Categorical preliminaries

We fix an F-stable minimal idempotent e in $\mathscr{D}_G(G)$. Note that we have the integer n_e from [BD14, Theorem 1.15] such that the category $\mathcal{M}_{G,e}[-n_e] \subset \mathscr{D}_G(G)$ consists of perverse sheaves.

Let $\mathcal{M}_{GF,e} \subset e\mathscr{D}_G(GF) \cong e\mathscr{D}_G^F(G)$ be the full subcategory formed by those objects whose underlying $\overline{\mathbb{Q}}_\ell$-complex is a perverse sheaf shifted by n_e. By [Des14b, Theorem 2.27], $\mathcal{M}_{GF,e}$ is an invertible $\mathcal{M}_{G,e}$-module category. By the results of [BD14, Des14b] we also have $e\mathscr{D}_G(G) \cong D^b\mathcal{M}_{G,e}$ and $e\mathscr{D}_G^{F^m}(G) \cong D^b\mathcal{M}_{GF^m,e}$.

To complete the proof of Theorem 6.6, we will use some categorical notions and results from [ENO10]. In this section we recall some of these facts and apply them in our setting of neutrally unipotent groups.

7.1 Twists in module categories

By [ENO10, Theorem 5.2], we have an equivalence of groupoids

$$\underline{\mathrm{Pic}}(\mathcal{M}_{G,e}) \cong \underline{\mathrm{EqBr}}(\mathcal{M}_{G,e}). \tag{87}$$

Under this equivalence the invertible $\mathcal{M}_{G,e}$-module category $\mathcal{M}_{GF,e}$ corresponds to the modular autoequivalence $F = F^{-1*} : \mathcal{M}_{G,e} \to \mathcal{M}_{G,e}$ since for $C \in \mathcal{M}_{G,e} \subset e\mathscr{D}_G(G)$ and $M \in \mathcal{M}_{GF,e} \subset e\mathscr{D}_G^F(G)$ we have the functorial crossed braiding isomorphisms $\beta_{M,C} : M * C \to F(C) * M$.

Let us consider the semidirect product $G\langle F\rangle = G \rtimes \mathbb{Z}$. The categories

$$e\mathscr{D}_G(G\langle F\rangle) := \bigoplus_{m\in\mathbb{Z}} e\mathscr{D}_G(GF^m) \text{ and} \tag{88}$$

$$\mathcal{M}_{G\langle F\rangle,e} := \bigoplus_{m\in\mathbb{Z}} \mathcal{M}_{GF^m,e} \tag{89}$$

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are (infinite) spherical braided $\mathbb{Z}$-crossed categories with trivial components $e\mathscr{D}_G(G)$ and $\mathcal{M}_{G,e}$ respectively with a (rigid) duality functor which we denote by $(\cdot)^\vee$ (cf. [Des10, § 2.3]) and a natural isomorphism $(\cdot)^{\vee\vee} \cong \mathrm{id}$ of monoidal functors. In particular, for each $M \in e\mathscr{D}_G^F(G)$ we have a twist $\theta_M^{F,e}$ defined as the composition

$$\theta_M^{F,e} : M \xrightarrow{\mathrm{id}_M * \mathrm{coev}_M} M * M * M^\vee \xrightarrow{\beta_{M,M}} F(M) * M * M^\vee \xrightarrow{\mathrm{id}_{F(M)} * \mathrm{ev}_{M^\vee}} F(M). \tag{90}$$

From [BD14, Des10], it is known that the twist $\theta : \mathrm{id}_{\mathscr{D}_G(G)} \to \mathrm{id}_{\mathscr{D}_G(G)}$ in $\mathscr{D}_G(G)$ defines the spherical structure of the modular category $\mathcal{M}_{G,e} \subset e\mathscr{D}_G(G)$. Similarly, we can define twists $\theta^{F^m,e}$ in each $e\mathscr{D}_G(G)$-module category $e\mathscr{D}_G^{F^m}(G)$. Then using the properties of spherical braided crossed categories (e.g. using string diagrams to aid visualization), we can prove analogues of Lemma 4.2 and Theorem 4.3.

LEMMA 7.1. (i) *Let $C \in e\mathscr{D}_G(G)$ and let $M \in e\mathscr{D}_G^F(G)$. Then $\theta_{C*M}^{F,e}$ is equal to the composition*

$$C * M \xrightarrow{\beta_{C,M}} M * C \xrightarrow{\beta_{M,C}} F(C) * M \xrightarrow{\theta_{F(C)} * \theta_M^{F,e}} F(C) * F(M) \longrightarrow F(C * M). \tag{91}$$

(ii) *In fact, all the twists $\theta^{F^a,e}$ in $e\mathscr{D}_G^{F^a}(G)$ for $a \in \mathbb{Z}$ are compatible with each other, namely if $L \in e\mathscr{D}_G^{F^a}(G)$ and $M \in e\mathscr{D}_G^{F^b}(G)$ then $\theta_{L*M}^{F^{a+b},e}$ equals the composition*

$$\begin{aligned} L * M \xrightarrow{\beta_{L,M}} F^a(M) * L \xrightarrow{\beta_{F^a(M),L}} F^b(L) * F^a(M) \xrightarrow{\theta_{F^b(L)}^{F^a,e} * \theta_{F^a(M)}^{F^b,e}} F^{a+b}(L) \\ * F^{a+b}(M) \longrightarrow F^{a+b}(L * M). \end{aligned} \tag{92}$$

(iii) *For $M \in e\mathscr{D}_G^F(G)$ we define the automorphism $\nu_{\theta_M^{F,e}}$ in $e\mathscr{D}_G^{F^2}(G)$ to be the composition*

$$\nu_{\theta_M^{F,e}} : M * M \xrightarrow{\beta_{M,F^*M}^{-1}} M * F^*M \xrightarrow{\mathrm{id}_M * F^*(\theta_M^{F,e})} M * M. \tag{93}$$

Then we have

$$\mathrm{tr}_{F,e}(\mathrm{id}_M) = \mathrm{tr}_{F^2,e}(\nu_{\theta_M^{F,e}}), \tag{94}$$

where $\mathrm{tr}_{F,e}$ denotes the $e\mathscr{D}_G(G)$-module trace (cf. [Sch13]) in $e\mathscr{D}_G^F(G)$ coming from the natural spherical structure.

Combining this with Lemma 4.2, we conclude that both the twists θ^F and $\theta^{F,e}$ in the invertible $\mathcal{M}_{G,e}$-module category $\mathcal{M}_{GF,e}$ are compatible with the twist θ in $\mathcal{M}_{G,e}$. Hence we must have the following proposition.

PROPOSITION 7.2. *For each $m \in \mathbb{Z}$, there is a constant $c_{F^m,e} \in \overline{\mathbb{Q}}_\ell^\times$ such that for each $M \in e\mathscr{D}_G^{F^m}(G)$, we have $\theta_M^{F^m} = c_{F^m,e} \cdot \theta_M^{F^m,e}$ and that $c_{F^m,e} = (c_{F,e})^m$ for each $m \in \mathbb{Z}$. In fact we must have*

$$c_{F,e} = \sigma_{F,e} \cdot \frac{q^{\dim G}}{q^{d_e}}, \tag{95}$$

where $d_e := (\dim G - n_e)/2$ is called the functional dimension of e and where $\sigma_{F,e} = \pm 1$.

Proof. Equality (95) is proved by (99) below. The remaining part of the proposition is clear from the previous results and the uniqueness (cf. [Sch13]) of module traces up to scaling. □

7.2 Positivity of spherical structure

By [Des14a, Theorem 2.17], the Frobenius–Perron dimension of $\mathcal{M}_{G,e}$ is an integer. In this section, let us choose an identification $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Then by [DGNO10, Corollary 2.24] the braided monoidal category $\mathcal{M}_{G,e}$ has a unique positive (with respect to our chosen identification) spherical structure. Let us denote the corresponding twist by $\theta^+ : \mathrm{id}_{\mathcal{M}_{G,e}} \to \mathrm{id}_{\mathcal{M}_{G,e}}$. Similarly, the crossed braided category $\mathcal{M}_{G\langle F\rangle,e}$ also has a unique positive spherical structure. Using this, we obtain new twists $\theta^{F^m,e,+}$ in the module categories $\mathcal{M}_{GF^m,e}$ for each $m \in \mathbb{Z}$. Let $\mathcal{M}^+_{G\langle F\rangle,e}$ denote the braided $\mathbb{Z}$-crossed category equipped with this positive spherical structure. For each $m \in \mathbb{Z}$, let $\mathrm{tr}^+_{F^m,e}$ denote the $\mathcal{M}^+_{G,e}$-module trace in $\mathcal{M}^+_{GF^m,e}$ corresponding to the positive spherical structure. Using [DGNO10, § 2.4.3] we deduce the following lemma.

Lemma 7.3. *For $m \in \mathbb{Z}$ and a simple object $M \in \mathcal{M}_{GF^m,e}$, we must have*

$$\theta^{F^m,e,+}_M = \sigma_M \theta^{F^m,e}_M : M \xrightarrow{\cong} F^m(M)$$

where $\sigma_M = \pm 1$.

Remark 7.4. Since $\theta^+, \theta^{F^m,e,+}, \mathrm{tr}^+_{F^m,e}$ are all defined using a spherical structure on a braided $\mathbb{Z}$-crossed category, they must satisfy analogues of Lemma 7.1. In particular, for each $M \in \mathcal{M}^+_{GF,e}$ we must have

$$\mathrm{tr}^+_{F,e}(\mathrm{id}_M) = \mathrm{tr}^+_{F^2,e}(\nu_{\theta^{F,e,+}_M}). \tag{96}$$

Let $M \in \mathcal{M}_{GF,e}$ be a simple object. By Theorem 4.3, we have

$$\mathrm{tr}_F(\mathrm{id}_M) = \mathrm{tr}_{F^2}(\nu_{\theta^F_M}) = \mathrm{tr}_{F^2}(\nu_{\sigma_M\cdot c_{F,e}\cdot\theta^{F,e,+}_M}) = \sigma_M \cdot c_{F,e} \cdot \mathrm{tr}_{F^2}(\nu_{\theta^{F,e,+}_M}). \tag{97}$$

In [Des14b, Theorem 2.30], we have described the explicit relation between the trace tr_F and $\mathrm{tr}^+_{F,e}$. Using this, we obtain

$$\frac{(-1)^{2d_e}}{\sqrt{\dim M_{G,e}}} \cdot \frac{q^{d_e}}{q^{\dim G}} \cdot \mathrm{tr}^+_{F,e}(\mathrm{id}_M) = \frac{(-1)^{2d_e}}{\sqrt{\dim M_{G,e}}} \cdot \frac{q^{2d_e}}{q^{2\dim G}} \cdot \sigma_M \cdot c_{F,e} \cdot \mathrm{tr}^+_{F^2,e}(\nu_{\theta^{F,e,+}_M}), \tag{98}$$

where d_e is the functional dimension of e (see Proposition 7.2). Comparing with (96), we obtain

$$c_{F,e} = \sigma_M \cdot \frac{q^{\dim G}}{q^{d_e}}. \tag{99}$$

In particular, this means that σ_M does not depend on M and we set $\sigma_{F,e} = \sigma_M$. This implies that the natural $\mathcal{M}_{G,e}$-module trace $\mathrm{tr}_{F,e}$ in $\mathcal{M}_{GF,e}$ is either purely positive or purely negative. Hence the natural spherical structure on $\mathcal{M}_{G,e}$ must in fact be positive. Moreover, this is true with respect to any identification $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. This means that the weakly integral modular category $\mathcal{M}_{G,e}$ must actually be integral. Hence we obtain the following result which confirms a conjecture from [Des14b].

Theorem 7.5. *Let G be any neutrally unipotent group defined over $\mathtt{k}$ and let $e \in \mathscr{D}_G(G)$ be any minimal idempotent. Then the modular category $\mathcal{M}_{G,e}$ is positive integral.*

Proof. By the results of [BD14], the minimal idempotent e can be obtained from some admissible pair[4]$(H,\mathcal{L})$. Now the subgroup H must be stable under some Frobenius map $F: G\to G$. In fact we may choose a Frobenius F so that the multiplicative local system $\mathcal{L}\in H^*$ (the Serre dual of H; cf. [BD14]) is also F-stable. Then the minimal idempotent e is also F-stable for such an F. Now we are in the situation where we can apply our previous results to deduce the theorem. $\square$

Using the observations of this section and [Des14b, Theorem 2.30], we see that the following corollary holds.

COROLLARY 7.6. *There is the following relationship between the $e\mathscr{D}_G(G)$-module traces tr_F and $\mathrm{tr}_{F,e}$ in the category $e\mathscr{D}_G^F(G)$:*

$$\mathrm{tr}_{F,e}=\sigma_{F,e}\cdot(-1)^{2d_e}\sqrt{\dim\mathcal{M}_{G,e}}\cdot\frac{q^{\dim G}}{q^{d_e}}\cdot\mathrm{tr}_F=(-1)^{2d_e}\sqrt{\dim\mathcal{M}_{G,e}}\cdot c_{F,e}\cdot\mathrm{tr}_F. \tag{100}$$

7.3 Braided monoidal actions and braided crossed categories

In this section we will use the notation and results from [ENO10]. For example, $\underline{\underline{\mathrm{Pic}}}(\mathcal{M}_{G,e})$ denotes the categorical 2-group of invertible $\mathcal{M}_{G,e}$-module categories, and by truncating it we obtain the categorical 1-group $\underline{\mathrm{Pic}}(\mathcal{M}_{G,e})$ and the ordinary group $\mathrm{Pic}(\mathcal{M}_{G,e})$. We refer to [ENO10] for details. Now by [ENO10, Theorem 4.5], we must have $F^{n'}\cong\mathrm{id}_{\mathcal{M}_{G,e}}$ as braided monoidal functors for some positive integer n'. This gives us a group homomorphism $\mathbb{Z}/n'\mathbb{Z}\to\mathrm{Pic}(\mathcal{M}_{G,e})\cong\mathrm{EqBr}(\mathcal{M}_{G,e})$. However, this may not necessarily give rise to a braided monoidal action of $\mathbb{Z}/n'\mathbb{Z}$ on $\mathcal{M}_{G,e}$. By [ENO10, § 8], the obstruction is described by a 3-cocycle in $H^3(\mathbb{Z}/n'\mathbb{Z},Z)$ where Z is the group of automorphisms of $\mathrm{id}_{\mathcal{M}_{G,e}}$ considered as a braided tensor functor or equivalently $Z=\mathrm{Aut}_{\mathcal{M}_{G,e}\text{-Mod}}(\mathcal{M}_{G,e})$. We can trivialize this 3-cocycle by choosing a suitable multiple n of n'. This means that we have a natural equivalence $\mathrm{id}_{\mathcal{M}_{G,e}}\xrightarrow{\xi}F^n$ (of braided monoidal functors) such that this defines a braided monoidal action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathcal{M}_{G,e}$ or equivalently a map

$$\mathbb{Z}/n\mathbb{Z}\longrightarrow\underline{\mathrm{EqBr}}(\mathcal{M}_{G,e})\cong\underline{\mathrm{Pic}}(\mathcal{M}_{G,e}). \tag{101}$$

Now note that in fact the cohomology group $H^4(\mathbb{Z}/n\mathbb{Z},\overline{\mathbb{Q}}_\ell^\times)$ is trivial. Hence by [ENO10, § 8] we can in fact lift the map (101) to obtain a map

$$\mathbb{Z}/n\mathbb{Z}\longrightarrow\underline{\underline{\mathrm{Pic}}}(\mathcal{M}_{G,e}). \tag{102}$$

In other words, we can put a structure of a braided $\mathbb{Z}/n\mathbb{Z}$-crossed category on the category

$$\mathscr{D}:=\bigoplus_{i=0}^{n-1}\mathcal{M}_{GF^i,e}. \tag{103}$$

This means that we have an induced action of $\mathbb{Z}/n\mathbb{Z}$ on each of the categories $\mathcal{M}_{GF^a,e}$ compatible with the braided monoidal action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathcal{M}_{G,e}$. We have well-defined natural isomorphisms $F^a\xrightarrow{\xi_a^m}F^{a+m}$ for each $a\in\mathbb{Z}$ and $m\in n\mathbb{Z}$. Also we have the corresponding equivalences of $\mathcal{M}_{G,e}$-module categories

$$\widetilde{\xi}_a^m:\mathcal{M}_{GF^a,e}\xrightarrow{\cong}\mathcal{M}_{GF^{a+m},e} \tag{104}$$

for $a\in\mathbb{Z},m\in n\mathbb{Z}$, and these satisfy certain compatibility conditions. The functor $\widetilde{\xi}_a^m$ commutes with the actions of $\mathbb{Z}/n\mathbb{Z}$.

[4] We refer to [BD14] for a precise definition.

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Note that we have the equivalence of $\mathcal{M}_{G,e}$-module categories

$$\widetilde{\xi} = \widetilde{\xi}_0^n : \mathcal{M}_{G,e} \xrightarrow{\cong} \mathcal{M}_{GF^n,e}. \tag{105}$$

In particular, we have the object $\widetilde{\xi}e \in \mathcal{M}_{GF^n,e}$, and the functor $\widetilde{\xi}$ above is canonically equivalent to the functors $C \mapsto \widetilde{\xi}e * C$ and $C \mapsto C * \widetilde{\xi}e$. And similarly, for each $a \in \mathbb{Z}, m \in n\mathbb{Z}$ and $M \in \mathcal{M}_{GF^a,e}$, we have natural isomorphisms $(\widetilde{\xi}e)^{*m/n} * M \cong \widetilde{\xi}_a^m(M) \cong M * (\widetilde{\xi}e)^{*m/n} \xrightarrow{\cong} F^a(\widetilde{\xi}e)^{*m/n} * M$ giving us a canonical isomorphism $\widetilde{\xi}e \cong F^a(\widetilde{\xi}e)$ for each $a \in \mathbb{Z}$.

Remark 7.7. The fact that $\bigoplus_{i\in\mathbb{Z}} \mathcal{M}_{GF^i,e}$ is a spherical braided $\mathbb{Z}$-crossed category implies that each $\mathcal{M}_{GF^i,e}$ is equipped with a normalized $\mathcal{M}_{G,e}$-module trace (in the sense of [Des14b, § 2.3.1 (2)]). Hence under the $\mathcal{M}_{G,e}$-module equivalence $\widetilde{\xi}_0^n$, the $\mathcal{M}_{G,e}$-module traces on $\mathcal{M}_{G,e}$ and $\mathcal{M}_{GF^n,e}$ must either agree or be negatives of each other. In any case the traces on $\mathcal{M}_{G,e}$ and $\mathcal{M}_{GF^{2n},e}$ must agree. Hence, replacing n by $2n$ if required, we may assume that the traces agree and hence we obtain a spherical structure on the category $\mathscr{D}$.

Note that all the structures above can be extended to the corresponding bounded derived categories. Recall from (69) that for $C \in e\mathscr{D}_G(G)^F$ and $M \in e\mathscr{D}_G^F(G)$ we have the composition

$$\zeta_{C,M} : C * M \xrightarrow{\beta^{-1}_{M,F^*C}} M * F^*C \xrightarrow{\beta^{-1}_{F^*C,M}} F^*C * M \xrightarrow{\psi_C * \mathrm{id}_M} C * M \tag{106}$$

in the category $e\mathscr{D}_G^F(G)$. Then we have the following lemma.

Lemma 7.8. *Let $m \in n\mathbb{Z}$. Then the automorphism $\widetilde{\xi}_1^m(\zeta_{C,M})$ in $e\mathscr{D}_G^{F^{m+1}}(G)$ is equal to the composition*[5]

$$\widetilde{\xi}_1^m(\zeta_{C,M}) : \widetilde{\xi}_0^m C * M \xrightarrow{\beta^{-1}} M * F^*\widetilde{\xi}_0^m C \xrightarrow{\beta^{-1}} F^*\widetilde{\xi}_0^m C * F^{m*}M \xrightarrow{\widetilde{\xi}_0^m(\psi_C)*(\xi_0^m)_{F^{m*}M}} \widetilde{\xi}_0^m C * M. \tag{107}$$

7.4 Certain twists are roots of unity

Recall from (90) that we have the twist $\theta^{F,e}$ in the $\mathcal{M}_{G,e}$-module category $\mathcal{M}_{GF,e}$. For each $M \in \mathcal{M}_{GF,e}$ we have the following automorphism of M:

$$\omega_M : M \xrightarrow{\theta_M^{F,e}} F(M) \xrightarrow{F(\theta_M^{F,e})} \cdots \xrightarrow{F^{n-1}(\theta_M^{F,e})} F^n(M) \xrightarrow{(\xi_0^n)^{-1}_M} M. \tag{108}$$

Lemma 7.9. *If $M \in \mathcal{M}_{GF,e}$ is a simple object then ω_M is a root of unity.*

Proof. We have the spherical braided $\mathbb{Z}/n\mathbb{Z}$-crossed category $\mathscr{D}$. Taking the equivariantization we obtain the modular category $\mathscr{D}^{\mathbb{Z}/n\mathbb{Z}}$. We choose an isomorphism $M \xrightarrow{\psi_M} F(M)$ such that (M, ψ_M) defines a $\mathbb{Z}/n\mathbb{Z}$-equivariant object in the category $\mathcal{M}_{GF,e}^{\mathbb{Z}/n\mathbb{Z}} \subset \mathscr{D}^{\mathbb{Z}/n\mathbb{Z}}$. Or in other words, we must have

$$\mathrm{id}_M : M \xrightarrow{\psi_M} F(M) \xrightarrow{F(\psi_M)} \cdots \xrightarrow{F^{n-1}(\psi_M)} F^n(M) \xrightarrow{(\xi_0^n)^{-1}_M} M. \tag{109}$$

Since $\mathscr{D}^{\mathbb{Z}/n\mathbb{Z}}$ is a modular category, it is equipped with a twist θ'. The twist θ'_{M,ψ_M} is equal to the composition $M \xrightarrow{\theta_M^{F,e}} F(M) \xrightarrow{\psi_M^{-1}} M$. We know that twists of simple objects in a modular category are roots of unity. Hence θ'_{M,ψ_M} is a root of unity. On the other hand, by comparing (108) and (109) we see that $\omega_M = (\theta'_{M,\psi_M})^n$. Hence ω_M is also a root of unity. □

[5] For brevity, we may occasionally omit the lower indices of the crossed braiding isomorphisms.

Corollary 7.10. *There exists a positive integer m_0 divisible by n such that, for each $m \in m_0\mathbb{Z} \subset n\mathbb{Z}$ and each simple object $M \in \mathcal{M}_{GF,e}$, the composition*

$$M \xrightarrow{\theta_M^{F,e}} F(M) \xrightarrow{F(\theta_M^{F,e})} \cdots \xrightarrow{F^{m-1}(\theta_M^{F,e})} F^m(M) \xrightarrow{(\xi_0^m)_M^{-1}} M \tag{110}$$

is equal to the identity, or equivalently (see Lemma 7.2), such that the composition

$$M \xrightarrow{\theta_M^{F}} F(M) \xrightarrow{F(\theta_M^{F})} \cdots \xrightarrow{F^{m-1}(\theta_M^{F})} F^m(M) \xrightarrow{(\xi_0^m)_M^{-1}} M \tag{111}$$

equals scaling by $(c_{F,e})^m$.

Proof. Since $\mathcal{M}_{GF,e}$ has finitely many simple objects, there exists a positive integer b such that $\omega_M^b = 1$ for all simples M. Then it is easy to check that $m_0 = nb$ satisfies the desired property. □

8. Completion of the proof

We can now complete the proof of Theorem 6.6. Let $M \in \mathscr{D}_G^F(G)$ and let m be any integer. Then we have the object $\eta_m(M) = (M, \psi_{M,m}) \in \mathscr{D}_G^F(G)^{F^m}$ defined by Lemma 5.1. Suppose that $m \in m_0\mathbb{Z}$, where m_0 is as in Corollary 7.10. Applying F^{m*} (which is inverse to the functor F^m; cf. § 7.1) to (111) and comparing with (65), we see that

$$\psi_{M,m} = (c_{F,e})^m \cdot F^{m*}(\xi_0^m)_M = (c_{F,e})^m \cdot (\xi_0^m)_{F^{m*}M} : F^{m*}M \to M \tag{112}$$

for each $m \in m_0\mathbb{Z}$.

We will now prove that the integer m_0 above satisfies the desired properties of Theorem 6.6. Let m be any positive multiple of m_0. Let $C \in \mathcal{M}_{G,e}$ be an F-stable simple object. We choose an F-equivariance isomorphism $\psi_{C,m} : F^*C \xrightarrow{\cong} C$ such that the composition

$$C \xrightarrow{F(\psi_{C,m})} F(C) \xrightarrow{F^2(\psi_{C,m})} \cdots \xrightarrow{F^m(\psi_{C,m})} F^m(C) \xrightarrow{(\xi_0^m)_C^{-1}} C \tag{113}$$

equals the twist $\theta_C : C \to C$. Consider the F-stable object $L_m := \widetilde{\xi}_0^m(C)$ in $\mathcal{M}_{GF^m,e} \subset e\mathscr{D}_G^{F^m}(G)$. It is equipped with the associated F-equivariance isomorphism

$$\psi_{L_m} := c_{F,e} \cdot \widetilde{\xi}_0^m(\psi_{C,m}) : F^*L_m = F^*\widetilde{\xi}_0^m(C) = \widetilde{\xi}_0^m(F^*C) \xrightarrow{\cong} \widetilde{\xi}_0^m(C) = L_m. \tag{114}$$

The reason for introducing the scaling factor $c_{F,e}$ in the definition of ψ_{L_m} is that now the composition

$$L_m \xrightarrow{F(\psi_{L_m})} F(L_m) \xrightarrow{F^2(\psi_{L_m})} \cdots \xrightarrow{F^m(\psi_{L_m})} F^m(L_m) \tag{115}$$

equals $(c_{F,e})^m \cdot \theta_{L_m}^{F^m,e} = c_{F^m,e} \cdot \theta_{L_m}^{F^m,e} = \theta_{L_m}^{F^m}$. Now $L_m[-\dim G - n_e]$ is an F-stable local system, say equal to $W_{m\mathrm{loc}}$ where $W_m \in \mathrm{Irrep}(G, F^m)^F$. Then by (115) the F-equivariance structure

$$\psi_{W_m} := \psi_{L_m}[-\dim G - n_e] : F^*W_{m\mathrm{loc}} \xrightarrow{\cong} W_{m\mathrm{loc}} \tag{116}$$

satisfies the desired property as described in § 2.6 and Remark 4.1. In particular, we must have

$$\|T_{W_m,\psi_{W_m}}\| = \|T_{L_m,\psi_{L_m}}\| = \|c_{F,e} \cdot T_{C,\psi_{C,m}}\| = \|T_{C,c_{F,e}\cdot\psi_{C,m}}\| = 1, \tag{117}$$

where $\|\cdot\|$ denotes the norm with respect to the Hermitian inner products on the corresponding function spaces.

Combining Lemma 7.8 and (112), we see that the composition (see (69))

$$\zeta_{(\widetilde{\xi}_0^m C, \widetilde{\xi}_0^m(\psi_{C,m})),\eta_m M} : \widetilde{\xi}_0^m C * M \xrightarrow{\beta^{-1}} M * F^*\widetilde{\xi}_0^m C \xrightarrow{\beta^{-1}} F^*\widetilde{\xi}_0^m C * F^{m*}M \xrightarrow{\widetilde{\xi}_0^m(\psi_C)*\psi_{M,m}} \widetilde{\xi}_0^m C * M \tag{118}$$

is equal to $(c_{F,e})^m \cdot \widetilde{\xi}_1^m(\zeta_{(C,\psi_{C,m}),M})$. Hence we see that

$$\zeta_{(L_m,\psi_{L_m}),\eta_m M} = (c_{F,e})^{m+1} \cdot \widetilde{\xi}_1^m(\zeta_{(C,\psi_{C,m}),M}) \tag{119}$$

and that

$$\zeta_{(W_{m\,\mathrm{loc}},\psi_{W_m}),\eta_m M} = (-1)^{2d_e} \cdot (c_{F,e})^{m+1} \cdot \widetilde{\xi}_1^m(\zeta_{(C,\psi_{C,m}),M}). \tag{120}$$

Now under the identification

$$\widetilde{\xi}_1^m : e\mathscr{D}_G^F(G) \xrightarrow{\cong} e\mathscr{D}_G^{F^{m+1}}(G), \tag{121}$$

the traces $\mathrm{tr}_{F,e}$ and $\mathrm{tr}_{F^{m+1},e}$ defined using the spherical structure on $e\mathscr{D}_G(G\langle F\rangle)$ agree. Hence

$$\mathrm{tr}_{F^{m+1},e}(\zeta_{(W_{m\,\mathrm{loc}},\psi_{W_m}),\eta_m M}) = (-1)^{2d_e} \cdot (c_{F,e})^{m+1} \cdot \mathrm{tr}_{F,e}(\zeta_{(C,\psi_{C,m}),M}). \tag{122}$$

Hence, using Corollary 7.6, we obtain

$$\mathrm{tr}_{F^{m+1}}(\zeta_{(W_{m\,\mathrm{loc}},\psi_{W_m}),\eta_m M}) = \frac{1}{\sqrt{\dim \mathcal{M}_{G,e}}} \cdot \mathrm{tr}_{F,e}(\zeta_{(C,\psi_{C,m}),M}) \tag{123}$$

$$= (-1)^{2d_e} c_{F,e} \cdot \mathrm{tr}_F(\zeta_{(C,\psi_{C,m}),M}) = (-1)^{2d_e}\, \mathrm{tr}_F(\zeta_{(C,c_{F,e}\cdot\psi_{C,m}),M}). \tag{124}$$

Now let $V \in \mathrm{Irrep}_e(G,F)$ and set $M = V_{\mathrm{loc}} \in e\mathscr{D}_G^F(G)$. Then from Theorem 5.4 and Corollary 5.5 we deduce that

$$\langle \mathrm{Sh}_m(W_m), \chi_V\rangle = (-1)^{2d_e}\langle T_{C,c_{F,e}\cdot\psi_{C,m}}, \chi_V\rangle. \tag{125}$$

Now the twist θ_C is a root of unity. Hence the F-equivariance isomorphisms $\psi_{C,m} : F^*C \xrightarrow{\cong} C$ (chosen according to (113)) for different $m \in m_0\mathbb{Z}$ differ from each other only up to scaling by roots of unity. Hence $\mathrm{Sh}_m(W_m)$ for different values of m (in $m_0\mathbb{Z}_{>0}$) as well as the trace of Frobenius functions $T_{C,c_{F,e}\cdot\psi_{C,m}} = c_{F,e}\cdot T_{C,\psi_{C,m}}$ all differ from each other only up to scalings by roots of unity. This completes the proof of Theorem 6.6(i) and (iii). The proof of (i′) is similar. Theorem 6.6(ii) also follows from Proposition 3.4.

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