BULL. AUSTRAL. MATH. SOC. 46E35, 35B60
VOL. 64 (2001) [149–156]

UNIQUE CONTINUATION FOR NON-NEGATIVE SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

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Dedicated to Filippo Chiarenza

The aim of this note is to prove the unique continuation property for non-negative solutions of the quasilinear elliptic equation

$$(*)\qquad \operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u).$$

We allow the coefficients to belong to a generalised Kato class.

1. INTRODUCTION

In his paper on Schrödinger semigroups [**12**] Simon formulated the following conjecture

> Let Ω be a bounded subset of $\mathbb{R}^n$ and V a function defined in Ω whose extension with zero values outside Ω belongs to the Stummel-Kato class $S(\mathbb{R}^n)$ (see Definition 2.2). Then the Schrödinger operator $H = -\Delta + V$ has the unique continuation property,

that is, if $u \in H^1(\Omega)$ is a solution of equation $Hu = 0$ which vanishes of infinite order at one point $x_0 \in \Omega$ (see Definition 4.2), then u must be identically zero in Ω.

A positive answer to Simon's conjecture was given by Fabes, Garofalo and Lin in [**5**] for radial potentials V.

At the same time Chanillo and Sawyer in [**1**] proved the unique continuation property for solutions of the inequality $|\Delta u| \leqslant |V|\,|u|$, assuming V in the *Morrey space* $L^{r,n-2r}(\mathbb{R}^n)$ with $r > (n-1)/2$ (see Definition 2.1).

In this note, following an idea of Chiarenza and Garofalo (see [**3**]), we extend both the above results to the non-negative solutions of a quasilinear elliptic equation of the form

$$(1.1)\qquad \operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u).$$

Received 16th November, 2000
The author wish to express his gratitude to Professor Richard Wheeden and Professor Nicola Garofalo for some useful talks and suggestions.

https://doi.org/10.1017/S0004972700019766 Published online by Cambridge University Press

Precisely we show that a non-negative solution u, $u \not\equiv 0$, of (1.1) cannot have a zero of infinite order, assuming that suitable powers of the coefficients of (1.1) belong to the Morrey space $L^{r,n-pr}(\mathbb{R}^n)$, with $r \in (1, n/p)$, or to the function space $\widetilde{M}_p(\mathbb{R}^n)$ (see Theorem 5.1). We denote by $\widetilde{M}_p(\mathbb{R}^n)$ a generalisation of the Stummel-Kato class (see and Remark 2.5).

We point out that a crucial role in the proof of the Theorem 5.1 is played by Fefferman's inequality

$$\int_{\mathbb{R}^n} |u(x)|^p |V(x)|\, dx \leqslant c \int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{1.2}$$

where c is a positive constant depending on some norm of V. In Section 3 we give a new proof of (1.2) assuming $V \in \widetilde{M}_p$.

2 Some function spaces

We begin this section giving some definitions.

Definition 2.1: (*Morrey spaces*) Let $q \geqslant 1$, $\lambda \in (0, n)$. We say that $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $L^{q,\lambda}(\mathbb{R}^n)$ if

$$\sup_{\substack{x \in \mathbb{R}^n \\ \rho > 0}} \frac{1}{\rho^\lambda} \int_{B(x,\rho)} |f(y)|^q\, dy \equiv \|f\|^q_{q,\lambda} < +\infty.$$

Here and in the following, we denote by $B(x,\rho)$ the ball centred at x with radius ρ. Whenever x is not relevant we shall write B_ρ.

Definition 2.2: (*Stummel-Kato class*) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For any $r > 0$ we set

$$\eta(r) \equiv \sup_{x \in \mathbb{R}^n} \int_{B(x,r)} \frac{|f(y)|}{|x - y|^{n-2}}\, dy.$$

We say that f belongs to $S(\mathbb{R}^n)$ if

$$\lim_{r \to 0} \eta(r) = 0.$$

Definition 2.3: Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For $p \in (1, n)$ and $r > 0$ we set

$$\phi(r) \equiv \sup_{x \in \mathbb{R}^n} \left(\int_{|x-y|<r} \frac{1}{|x - y|^{n-1}} \left(\int_{|z-x|<r} \frac{|f(z)|}{|z - y|^{n-1}}\, dz \right)^{1/(p-1)} dy \right)^{(p-1)}.$$

We say that f belongs to the function space $\widetilde{M}_p(\mathbb{R}^n)$ if

$$\phi(r) < +\infty, \qquad \forall r > 0.$$

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DEFINITION 2.4: We say that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ belongs to the function space $M_p(\mathbb{R}^n)$ if

$$\lim_{r\to 0} \phi(r) = 0,$$

where $\phi(r)$ is defined as in Definition 2.3.

Some comments are now in order.

REMARK 2.5. We have

(i) $M_p(\mathbb{R}^n) \subset \widetilde{M}_p(\mathbb{R}^n)$;

(ii) $M_2(\mathbb{R}^n) \equiv S(\mathbb{R}^n)$.

(i) is trivial. Concerning (ii), Fubini's theorem implies

$$\int_{|x-y|<r} \frac{1}{|x-y|^{n-1}} \left(\int_{|z-x|<r} \frac{|f(z)|}{|z-y|^{n-1}}\, dz \right) dy = \int_{|z-x|<r} |f(z)| \int_{|x-y|<r} \frac{1}{|x-y|^{n-1}} \frac{1}{|z-y|^{n-1}}\, dy\, dz.$$

Since

$$\int_{|x-y|<r} \frac{1}{|x-y|^{n-1}} \frac{1}{|z-y|^{n-1}}\, dy \sim \frac{1}{|z-x|^{n-2}},$$

we get the conclusion.

Therefore both the function spaces $M_p(\mathbb{R}^n)$ and $\widetilde{M}_p(\mathbb{R}^n)$ are generalisations of $S(\mathbb{R}^n)$.

3. ON FEFFERMAN'S INEQUALITY

In this section we recall some known results concerning Fefferman's inequality

$$\int_{\mathbb{R}^n} |u(x)|^p |f(x)|\, dx \leqslant c \int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{3.1}$$

and give a new proof assuming $f \in \widetilde{M}_p(\mathbb{R}^n)$.

In [7] Fefferman proved (3.1), in the case $p = 2$, assuming $f \in L^{r,n-2r}(\mathbb{R}^n)$, with $1 < r \leqslant n/2$.

Later in [10] Schechter showed the same result taking f in the Stummel-Kato class $S(\mathbb{R}^n)$.

We stress that it is not possible to compare the assumptions $f \in L^{r,n-2r}(\mathbb{R}^n)$ and $f \in S(\mathbb{R}^n)$.

Chiarenza and Frasca [2] generalised Fefferman's result proving (3.1) under the assumption $V \in L^{r,n-pr}(\mathbb{R}^n)$ with $r \in (1, n/p)$ and $p \in (1, n)$. Namely they proved the following

Theorem 3.1. (See [2, p.407].) *Assume $1 < p < n$, $1 < r \leqslant n/p$, $f \in L^{r,n-pr}(\mathbb{R}^n)$. Then there exists a constant c depending on n and p such that*

$$\int_{\mathbb{R}^n} |u^p(x)|\,|f(x)|\,dx \leqslant c\|f\|_{r,n-pr} \int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx, \quad \forall u \in C_0^\infty(\mathbb{R}^n).$$

In the following theorem we provide a generalisation of Schecter's result, proving (3.1) under the assumption $f \in \widetilde{M}_p(\mathbb{R}^n)$, $p \in (1,n)$.

Theorem 3.2. *Assume $f \in \widetilde{M}_p(\mathbb{R}^n)$. Then for any $r > 0$ there exists a positive constant $c(n,p)$ such that*

$$\int_{\mathbb{R}^n} |f(x)|\,|u(x)|^p\,dx \leqslant c(n,p)\phi(2r) \int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx$$

for any $u \in C_0^\infty(\mathbb{R}^n)$ supported in $B(x_0,r)$.

Proof: For any $u \in C_0^\infty(\mathbb{R}^n)$ supported in $B(x_0,r)$, using the well known inequality

$$|u(x)| \leqslant c(n,p) \int_{B(x_0,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy \tag{3.2}$$

and Fubini's theorem, we have

(3.3)

$$\begin{aligned}
&\int_{\mathbb{R}^n} |f(x)|\,|u(x)|^p\,dx \\
&\quad = \int_{B(x_0,r)} |f(x)|\,|u(x)|^p\,dx \\
&\quad \leqslant c(n,p) \int_{B(x_0,r)} |f(x)|\,|u(x)|^{p-1} \left(\int_{B(x_0,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy \right) dx \\
&\quad \leqslant c(n,p) \int_{B(x_0,r)} |\nabla u(y)| \left(\int_{B(x_0,r)} |f(x)|\,|u(x)|^{p-1} \frac{1}{|x-y|^{n-1}}\,dx \right) dy \\
&\quad \leqslant c(n,p) \left(\int_{B(x_0,r)} |\nabla u(y)|^p\,dy \right)^{1/p} \cdot \\
&\qquad\quad \cdot \left[\int_{B(x_0,r)} \left(\int_{B(x_0,r)} |f(x)|\,|u(x)|^{p-1} \frac{1}{|x-y|^{n-1}}\,dx \right)^{p/(p-1)} dy \right]^{(p-1)/p}.
\end{aligned}$$

We also have

(3.4)

$$\begin{aligned}
&\int_{B(x_0,r)}\left(\int_{B(x_0,r)}|f(x)|\,|u(x)|^{p-1}\frac{1}{|x-y|^{n-1}}\,dx\right)^{p/(p-1)}dy\\
&\quad\leqslant\int_{B(x_0,r)}\left(\int_{B(x_0,r)}\frac{|f(z)|}{|z-y|^{n-1}}\,dx\right)^{1/(p-1)}\int_{B(x_0,r)}\frac{|f(x)|\,|u(x)|^p}{|x-y|^{n-1}}\,dx\,dy\\
&\quad=\int_{B(x_0,r)}|f(x)|\,|u(x)|^p\int_{B(x_0,r)}\frac{1}{|x-y|^{n-1}}\left(\int_{B(x_0,r)}\frac{|f(z)|}{|z-y|^{n-1}}\,dz\right)^{1/(p-1)}dy\,dx\\
&\quad\leqslant\phi^{1/(p-1)}(2r)\int_{B(x_0,r)}|f(x)|\,|u(x)|^p\,dx.
\end{aligned}$$

By (3.3) and (3.4) we obtain the desired conclusion. □

REMARK 3.3. We note that proceeding as in Theorem 3.2 using the representation formula (see, for example [**6**])

$$|u(x)-u_{B_R}|\leqslant c\int_{B_R}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy,$$

instead of (3.2), it is possible to obtain a Poincaré type inequality. Namely

THEOREM 3.4. *Suppose u is a Lipschitz continuous function on $\overline{B}_R$, the closure of B_R, and f is a function defined on B_R whose extension with zero values outside B_R belongs to $\widetilde{M}_p(\mathbb{R}^n)$. Then there exists a positive constant c such that*

$$\int_{B_R}|f(x)|\,|u(x)-u_{B_R}|^p\,dx\leqslant c\phi(2R)\int_{B_R}|\nabla u(x)|^p\,dx$$

where u_{B_R} is the average $(1/|B_R|)\int_{B_R}u(x)\,dx$ where $|B_R|$ is the Lebesgue measure of B_R.

4. ASSUMPTIONS AND PRELIMINARY RESULTS

Let Ω be a bounded open set in $\mathbb{R}^n$. The equation we consider is of the form

$$\operatorname{div}A(x,u,\nabla u)=B(x,u,\nabla u), \tag{4.1}$$

where

$$A(x,u,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$$

and

$$B(x,u,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$$

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are two continuous functions satisfying the following conditions

$$\text{(4.2)}\qquad \begin{cases} |A(x,u,\xi)| \leqslant a|\xi|^{p-1} + b(x)|u|^{p-1} \\ |B(x,u,\xi)| \leqslant c(x)|\xi|^{p-1} + d(x)|u|^{p-1} \\ \xi A(x,u,\xi) \geqslant |\xi|^p - d(x)|u|^p \end{cases}$$

for almost all, $x \in \Omega$, $\forall u \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^n$. We assume that p is a fixed number in $(1,n)$, a is a positive constant and b, c and d are measurable functions in Ω whose extension with zero values outside Ω are such that

$$\text{(4.3)}\qquad b^{p/(p-1)}, c^p, d \in M_p(\mathbb{R}^n),$$

or

$$\text{(4.3)}'\qquad b^{p/(p-1)}, c^p, d \in L^{r,n-pr}(\mathbb{R}^n) \quad r \in (1, n/p).$$

DEFINITION 4.1: We say that a function $u \in H^{1,p}_{\mathrm{loc}}(\Omega)$ is a local weak solution of (4.1) in Ω if

$$\text{(4.4)}\qquad \int_\Omega \Big\{ A\big(x,u(x),\nabla u(x)\big)\nabla\phi(x) + B(x,u(x),\nabla u(x))\phi(x) \Big\}\,dx = 0$$

for every $\phi \in C_0^\infty(\Omega)$.

We remark that Definition 4.1 is meaningful by Theorem 3.1 or Theorem 3.2.

To state our result we need one more definition.

DEFINITION 4.2. *Assume $w \in L^1_{\mathrm{loc}}(\Omega)$, $w \geqslant 0$ almost everywhere in Ω. We say that w has a zero of infinite order at $x_0 \in \Omega$ if*

$$\lim_{\sigma\to 0} \frac{\int_{B(x_0,\sigma)} w(x)\,dx}{|B(x_0,\sigma)|^k} = 0 \quad \forall k > 0.$$

The following two lemmas are known.

LEMMA 4.3. (See [9].) *Assume $w \in L^1_{\mathrm{loc}}(\Omega)$, $w \geqslant 0$ almost everywhere in Ω, $w \not\equiv 0$. If*

$$\exists C > 0 : \int_{B(x_0,2\sigma)} w(x)\,dx \leqslant C \int_{B(x_0,\sigma)} w(x)\,dx \quad \forall \sigma > 0,$$

then $w(x)$ has no zero of infinite order in Ω.

LEMMA 4.4. (See [4] and [8].) *Let $B_{\tilde r} \subset \mathbb{R}^n$, $u \in H^{1,p}(B_{\tilde r})$ be and assume that for all $B_r \subset B_{\tilde r}$ there exists a constant K such that*

$$\left(\int_{B_r} |\nabla u(x)|^p\,dx\right)^{1/p} \leqslant K r^{(n-p)/p}.$$

Then there exist two positive constants δ and C, depending on K, p, n, such that

$$\left(\int_{B_{\tilde r}} e^{\delta u(x)}\,dx\right)\left(\int_{B_{\tilde r}} e^{-\delta u(x)}\,dx\right) \leqslant C|B_{\tilde r}|^2.$$

5. Unique continuation

In this section we state and prove our result, namely

Theorem 5.1. *Let $u \in H^1(\Omega)$, $u \geqslant 0$, $u \not\equiv 0$, be a solution of* (4.1) *satisfying* (4.2) *and* (4.3) *or* (4.2) *and* (4.3)$'$.

Then u has no zero of infinite order in Ω.

Proof: Let $x_0 \in \Omega$, let $B(x_0, R)$ be a ball such that $B(x_0, 2R)$ is contained in Ω. Consider any B_h contained in $B(x_0, R)$. Let η be a non negative smooth function with support in B_{2h}. Using $\phi = \eta^p u^{1-p}$ as test function in (4.4) we get (see [**11**])

$$\int_\Omega \left|\nabla \log u(x)\right|^p \eta^p(x)\,dx \leqslant C_1(p,a)\left\{\int_\Omega \left|\nabla\eta(x)\right|^p dx + \int_\Omega V(x)\eta^p(x)\,dx\right\}, \tag{5.1}$$

where V is defined by

$$V = b^{p/(p-1)} + c^p + d.$$

By Theorem 3.1 or Theorem 3.2, we have

$$\int_\Omega V(x)\eta^p(x)\,dx \leqslant C_2(\operatorname{spt}\eta)\int_\Omega \left|\nabla\eta(x)\right|^p dx.$$

Inserting this inequality in (5.1), we obtain

$$\int_\Omega \eta^p(x)\left|\nabla \log u(x)\right|^p dx \leqslant C_3(p, a, \operatorname{diam}\Omega)\int_\Omega \left|\nabla\eta(x)\right|^p dx. \tag{5.2}$$

Choosing η so that $\eta = 1$ in B_h and $|\nabla\eta| \leqslant 3/h$, by (5.2) we have

$$\int_{B_h} \left|\nabla \log u(x)\right|^p dx \leqslant C_4(p, a, \operatorname{diam}\Omega) h^{n-p}. \tag{5.3}$$

Therefore, by Lemma 4.4, we have

$$\int_{B_h} u^\delta(x)\,dx \int_{B_h} u^{-\delta}(x)\,dx \leqslant C|B_h|^2,$$

that is, u^δ belongs to the Muckenhoupt class A_2 for some $\delta > 0$ (see [**3**] and [**6**]). Now it is well known that A_2 implies the *doubling property* for u_δ, that is, the assumption of Lemma 4.3. So the conclusion follows for u^δ and hence also for u. ∎

https://doi.org/10.1017/S0004972700019766 Published online by Cambridge University Press

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