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ON THE ARITHMETIC PROPERTIES OF THE VALUES OF G-FUNCTIONS

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Abstract

In a recent paper Chudnovsky considered the arithmetic properties of certain values of classical Siegel G-function solutions of a system of linear homogeneous differential equations without any restrictive conditions. The present paper generalizes some results of Chudnovsky in both the archimedian and the p-adic case.

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In 1929 Siegel [9] developed a method for studying the arithmetic properties of the values of certain classes of analytic functions, called E- and G-functions. Later this method has been applied to G-functions by Nurmagomedov [8], Galochkin [5], Flicker [4], Väänänen ([10], [11], [12], [13]), Matveev [6] and Xu ([15], [16]), for example, but their results use the additional Galochkin's [5] condition. This is replaced by another condition in an important work of Bombieri [2]. Then, in a recent paper, Chudnovsky [3], using ingenious new ideas, succeeded in considering the arithmetic properties of the values of classical G-function solutions of a system of linear homogeneous differential equations without any restrictive conditions.

In Väänänen and Xu [14] some generalizations of certain results of Chudnovsky [3] are obtained. The purpose of this paper is to generalize Theorem II of Chudnovsky [3], in both the archimedian and the p-adic case. Our proof

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follows closely the main lines of Väänänen ([12], [13]), but here we essentially use the ideas of Chudnovsky [3] in the construction of approximation forms and apply the local to global technique, as in Bombieri [2].

2. Notations and results

Let K be an algebraic number field of degree d over $\mathbf{Q}$, and let O_K denote the domain of integers in K. For every place v of K we write $d_v = [K_v : \mathbf{Q}_v]$. If the finite place v of K lies over the prime number p, we write $v|p$, for infinite place v of K we write $v|\infty$. We normalize the absolute value $|\ |_v$ so that

(i) if $v|p$, then $|p|_v = p^{-d_v/d}$,

(ii) if $v|\infty$, then $|x|_v = |x|^{d_v/d}$,

here $|\ |$ denotes the ordinary absolute value in $\mathbf{R}$ or $\mathbf{C}$.

The absolute height $h(x)$ of $x \in K$ is defined by the formula

$$h(x) = \prod_v \max(1, |x|_v).$$

For any polynomial $P(z) = \sum_{i=0}^n p_i z^i \in K[z]$ we denote

$$|P|_v = \max\left(1, \max_i |p_i|_v\right),$$

and define the absolute height of P by $h(P) = \prod_v |p|_v$.

We write $\log a = \log\max(1, a)$ for all $a \geq 0$, and denote

$$\alpha_v = \begin{cases} 1, & \text{if } v|p, \\ 0, & \text{if } v|\infty, \end{cases} \qquad \beta_v = \begin{cases} 0, & \text{if } v|p, \\ d_v/d, & \text{if } v|\infty. \end{cases}$$

The power series

$$y_i(z) = \sum_{m=0}^{\infty} a_{m,i} z^m, \qquad i = 1, \dots, n, \tag{1}$$

are said to belong to the class $KG(\gamma, C, C_0)$, $\gamma, C, C_0 \geq 1$, if the following conditions are satisfied:

(i) $a_{m,i} \in K$, $i = 1, \dots, n$, $m = 0, 1, \dots$;

(ii) $\max_i |a_{m,i}|_v \leq \gamma^{\beta_v} C^{\beta_v m}$, $m = 0, 1, \dots$, for every $v|\infty$;

(iii) there exists a sequence of natural numbers (r_l) such that

$$r_l a_{m,i} \in O_K, \quad i = 1, \dots, n, \quad m = 0, 1, \dots, l, \quad l = 0, 1, \dots,$$

and

$$r_l \leq \gamma C_0^l, \qquad l = 0, 1, \dots.$$

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By (iii),

$$\max_{\substack{1\le i\le n\\ 0\le m\le l}} |a_{m,i}|_v \le 1/|r_l|_v \le r_l \le \gamma C_0^l$$

for every finite place v of K and $l = 0, 1, \ldots$. Thus the functions (1) are v-adically convergent in

$$|z|_v < C^{-\beta_v} C_0^{-\alpha_v}.$$

In the following we suppose that the functions (1) satisfy a system of linear differential equations

$$\frac{d}{dz}Y = AY, \tag{2}$$

where $Y = (y_1(z), \ldots, y_n(z))^t$, $A = (A_{ij}(z))_{n\times n}$, $A_{ij} \in K(z)$. Let $T(z) \in K[z]$ denote the common denominator of A_{ij}, and put

$$s = \max(\deg T, \deg TA_{ij}, i, j = 1, \ldots, n).$$

In the following theorem we shall estimate the v-value of

$$P(y_1(\theta), \ldots, y_n(\theta)),$$

$\theta \in K$, $P \in K[x_1, \ldots, x_n]$. In writing $|P(y_1(\theta), \ldots, y_n(\theta)|_v$ we consider all the coefficients of P and the power series y_i as elements of the corresponding completion K_v, and thus this v-value is defined for all $|\theta|_v < C^{-\beta_v} C_0^{-\alpha_v}$.

THEOREM. *Assume that the functions* (1) *satisfying* (2) *are algebraically independent over* $K(z)$ *and belong to the class* $KG(\gamma, C, C_0)$. *Let*

$$P \in K[x_1, \ldots, x_n], \qquad P \not\equiv 0,$$

be a polynomial of degree at most λ *and height* $h(P)$. *There then exist positive constants* c, Λ, *depending only on the functions* (1) *and* n, *such that, for any* $\theta \in K$ *of height* $h(\theta) \le h \ge e^e$ *satisfying*

$$\begin{gathered}\theta T(\theta) \ne 0, \quad \log h \ge (1 + \max(3, \lambda))^{4n} \log\log h, \\ |\theta|_v < e^{-c\lambda(\log h)^{(4n-1)/4n}(\log\log h)^{1/4n}}\end{gathered} \tag{3}$$

we have

$$|P(y_1(\theta), \ldots, y_n(\theta))|_v > h(P)^{-\Lambda(\log h)^{1/4}(\log\log h)^{-1/4}}$$

for all $h(P) \ge H$, *where*

$$\log H = \max\left\{\bar{c}\lambda(\log h)^{2-1/4n}(\log\log h)^{(1-4n)/4n}, \log\max_i(1, |y_i(\theta)|_v)\right\}$$

with a constant $\bar{c} > 0$ *depending only on* (2).

We note that our condition (3) slightly sharpens the corresponding condition of Chudnovsky [3]. In fact, we shall prove the above estimate for $|P(\theta)|_v$

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under a condition (3)′ (see Section 4), which is better that (3) in some cases. Further, it should be noted that $4n$ in (3) can be replaced by $2n$ under the restrictive conditions used in Bombieri [2] or Väänänen [12], and then also $1/4$ in the conclusion can be replaced by $1/2$.

The authors thank the referee for his useful suggestions and advise. In particular, a bound for H was not given in the original manuscript.

3. Lemmas

Let

$$(4) \qquad g_0(z) \equiv 1, \qquad g_1(z), \dots, g_m(z)$$

denote the power-products

$$y_1^{k_1}(z) \cdots y_n^{k_n}(z), \qquad 0 \le k_1 + \cdots + k_n \le S, \qquad m = \binom{n+S}{S} - 1,$$

where S is a natural number ≥ 3. As in [12], we see that the functions (4) belong to the class $KG((2\gamma)^S, 2C, C_0^{1+(\log S)^u})$ for some u satisfying $0 \le u \le 1$. Thus these functions are v-adically defined in $|z|_v < (2C)^{-\beta_v} C_0^{-\alpha_v(1+(\log S)^u)}$. Further, the functions (4) satisfy a system of linear differential equations of type (2), where the rational function coefficients again have a common denominator $T(z)$.

First we give a lemma on Padé approximations of the second kind (for the definition, see Chudnovsky [3]).

LEMMA 1. *For any δ, $0 < \delta < 1/m$, and an arbitrary positive integer D and $M = [(m^{-1} - \delta)D]$, there exists a system $(Q(x); P_1(z), \dots, P_m(z))$ of Padé approximations of the second kind with parameters (D, D, M) for the functions* (4) *such that $Q(z)$, $P_i(z) \in K[z]$ and*

$$\log h(Q) \le ((\delta m)^{-1} - 1)(1 + m^{-1} - \delta)D(\log 2C + (1 + (\log S)^u)\log C_0) + (\delta m)^{-1}(\log 2(D+1) + 2S\log 2\gamma + \log\Gamma),$$

where Γ is a positive constant depending only on K.

PROOF. The proof is completely analogous to that of Lemma 4 in Väänänen and Xu [14], using Siegel's lemma in the form given by Bombieri [2].

The following result is Theorem 1.1 of Chudnovsky [3].

Lemma 2. *Let $(Q(z); P_1(z), \dots, P_n(z))$ be a system constructed in Lemma 1. Let $k \in N$ and suppose that $M \geq k(s+1)$. We define*

$$Q^{\langle k \rangle}(z) = T^k(z)\left(\frac{d}{dz}\right)^k Q(z)/k!,$$
$$P_i^{\langle k \rangle}(z) = Q^{\langle k \rangle}(z) g_i(z)]_{D+ks}, \qquad i = 1, \dots, m$$

(this means that $P_i^{\langle k \rangle}(z)$ is a polynomial of degree $\leq D+ks$ such that the order of zero of $Q^{\langle k \rangle}(z)g_i(z) - P_i^{\langle k \rangle}(z)$ at $z = 0$ is at least $D+ks+1$). Then $(Q^{\langle k \rangle}(z); P_1^{\langle k \rangle}(z), \dots, P_m^{\langle k \rangle}(z))$ is a system of Padé approximations of the second kind with parameters $(D+ks, D+ks, M-k(s+1))$ for the functions (4).

Analogously to Lemma 5 of Väänänen and Xu [14], we now have the following

Lemma 3. *Let $(Q(z); P_1(z), \dots, P_m(z))$ be a system constructed in Lemma 1. Let $k \in N$ and assume that $M \geq k(s+1)$. Then the system $(Q^{\langle k \rangle}(z); P_1^{\langle k \rangle}(z), \dots, P_m^{\langle k \rangle}(z))$ defined in Lemma 2 has the following properties:*

$$|r_{D+ks} Q^{\langle k \rangle}|_v \leq (C(k,D)(2\gamma)^S C_0^{(1+(\log S)^u)(D+ks)})^{\beta_v} |Q|_v |T|_v^k,$$
$$|r_{D+ks} P_i^{\langle k \rangle}|_v \leq (C(k,D)^2 (2\gamma)^{2S} (2CC_0^{1+(\log S)^u})^{D+ks})^{\beta_v} |Q|_v |T|_v^k, \quad i = 1, \dots, m,$$

where $C(k,D) = (k+1)(s+1)^k(D+1)2^D$.

Let us denote, for all $k = 0, 1, \dots, Q_0^{\langle k \rangle} = r_{D+ks} Q^{\langle k \rangle}$, $Q_i^{\langle k \rangle} = r_{D+ks} P_i^{\langle k \rangle}$, $i = 1, \dots, m$. We then have the following lemma, analogous to Lemma 2 of Väänänen and Xu [14].

Lemma 4. *Let δ, $0 < \delta < 1/(m + m^2(s+1))$, be given, and let $\theta \in K$ satisfy $\theta T(\theta) \neq 0$. There exists a positive constant c_0, depending only on the system* (2), *such that, for all*

$$D > N = c_0(m^{-1} - \delta)^{-1} S m^2,$$

there exist integers $k_0, k_1, \dots, k_m$,

$$0 \leq k_0 < k_1 < \cdots < k_m \leq J = D - mM + m(m+1)(s+1)/2,$$

satisfying

$$\Delta(\theta) = \begin{vmatrix} Q_0^{\langle k_0 \rangle}(\theta) & Q_1^{\langle k_0 \rangle}(\theta) & \dots & Q_m^{\langle k_0 \rangle}(\theta) \\ \vdots & \vdots & & \vdots \\ Q_0^{\langle k_m \rangle}(\theta) & Q_1^{\langle k_m \rangle}(\theta) & \dots & Q_m^{\langle k_m \rangle}(\theta) \end{vmatrix}.$$

PROOF. We prove here that the determinant

$$\nabla(x) = \begin{vmatrix} Q^{\langle 0\rangle}(x) & \dots & Q^{\langle m\rangle}(x) \\ P_1^{\langle 0\rangle}(x) & \dots & P_1^{\langle m\rangle}(x) \\ \cdot & \dots & \cdot \\ P_m^{\langle 0\rangle}(x) & \dots & P_m^{\langle m\rangle}(x) \end{vmatrix}$$

is not identically zero for $D > N$. Then the proof follows immediately from the important Theorem 1.2 of Chudnovsky [3]. In our proof we follow Chudnovsky [3], Section 3.

Suppose the $\nabla(x) \equiv 0$. Let $1 \le m$ be the integer such that the first l columns of ∇ are linearly independent over $\mathbb{C}(x)$, but the $(l+1)$st column is linearly dependent on them. Let F denote the matrix formed by these l columns, and let R and U denote the matrices formed by the first l rows and last $m-l+1$ rows of F, respectively. We may assume, without loss of generality, that $\det R \not\equiv 0$.

Following Nesterenko [7], Section 3, we see that rational functions elements of the matrix

$$UR^{-1} = (e_{ij}(x)/e(x)), \qquad e_{ij}, e \in \mathbb{C}(x),$$

satisfy $\max(\deg e_{ij}, \deg e) \le c(1)Sm$, where the constant $c(1) > 0$ depends only on the system (2). Denote by G the $l \times (m+1)$ matrix with l rows $(g_i(x), 0, \dots, -\delta_{i+1,j}, \dots, 0)$, $i = 1, \dots, l$, and let G_0 and G_1 denote the matrices formed by the first l column and the last $m-l+1$ last columns of G, respectively. Denoting $T = GF$ we see, as in [3], Section 3, that

$$\operatorname{ord}_{x=0} \det(e(x)TR^{-1}) \ge l(M - (l-1)(s+1)).$$

On the other hand,

$$e(x)TR^{-1} = e(x)G_0 + G_1E,$$

where $E = (e_{ij}(x))$. Thus $\det(e(x)TR^{-1})$ is a polynomial in x, $y_1(x), \dots, y_n(x)$, say $P(x, y_1(x), \dots, y_n(x))$, satisfying $\deg_x P \le c(1)Sml$, $\deg_y P \le S$. By the algebraic independence of $y_1, \dots, y_n$ we know that P is not identically zero in x. Using the result of Bertrand and Beukers [1] we obtain the estimate

$$\operatorname{ord}_{x=0} \det(e(x)TR^{-1}) \le c(1)Slm^2 + c(2)m^2$$

with positive constant $c(2)$ depending only on (2). Thus

$$l(M - (l-1)(s+1)) \le c(1)Slm^2 + c(2)m^2.$$

The above inequality is impossible for all $(m^{-1} - \delta)D > c_0Sm^2$ with some positive constant c_0 depending only on (2). This proves our Lemma 4.

We next define rational functions $L_{t,j} = L_{t,j}(\theta)$, t, $j = 0, 1, \dots, m$, as the solutions of the system

$$\sum_{t=0}^{m} L_{t,j} Q_t^{\langle k_i\rangle}(\theta) = \delta_{i,j}, \qquad i, j = 0, 1, \dots, m,$$

of linear equations. By Cramer's rule,

$$L_{t,j}(\theta) = R_{t,j}(\theta)/\Delta(\theta), t, \qquad j = 0, 1, \dots, m,$$

where $R_{t,j}(\theta)$ is the t, j-cofactor of the matrix corresponding to $\Delta(\theta)$. We now define linear forms F_j in $g_0(\theta), g_1(\theta), \dots, g_m(\theta)$ by the formulae

$$F_j(\theta) = \sum_{t=0}^{m} M_{t,j}(\theta) g_t(\theta), \qquad j = 0, 1, \dots, m,$$

where $M_{t,j}(\theta) = R_{t,j}(\theta)\theta^{-\omega}$, $\omega = (m-1)(M+D) = (m-1)J$. Using Theorem 4.1 of Chudnovsky [3] we immediately obtain the following important result.

LEMMA 5. *Let the hypothesis of Lemma 4 be satisfied. For all $D > N$, the linear forms $F_0(\theta), \dots, F_m(\theta)$ in $g_0(\theta), \dots, g_m(\theta)$ are linearly independent and have polynomial coefficients $M_{t,j} = M_{t,j}(\theta)$ satisfying*

$$\deg_\theta M_{t,j} \le D - (m-1)M + J(ms + m - 1), \qquad t, j = 0, 1, \dots, m.$$

Further, we have

$$\operatorname{ord}_{\theta=0} F_j(\theta) \ge D + M - J, \qquad j = 0, 1, \dots, m.$$

LEMMA 6. *The polynomials $M_{t,j}$ appearing in Lemma 5 satisfy the estimate*

$$|M_{t,j}|_v \le (m!)^{\beta_v}((D + Js + 1)C(J, D)^2(2\gamma)^{2S} \cdot (2CC_0^{(1+(\log S)^u)})^{(D+Js)})^{m\beta_v} |Q|_v^m |T|_v^{mJ}, \qquad t, j = 0, 1, \dots, m.$$

PROOF. The result follows immediately from Lemma 3 and the definition of the polynomials $M_{t,j}$.

LEMMA 7. *Let δ, $0 < \delta < 1/(3m^2(s+1))$, be given. Assume that*

$$D > \max\{\delta^{-1}(1 + (m+1)(s+1)/2), m/(1 - 3\delta m^2(s+1)), N\}.$$

If $\theta \in K$, then

$$|M_{t,j}(\theta)|_v \le (D(m^{-1} + \delta(m + 2m^2(s+1))) + 1)^{\beta_v} |M_{t,j}|_v \cdot \max(1, |\theta|_v^{D(m^{-1}+\delta(m+2m^2(s+1)))}), \qquad t, j = 0, 1, \dots, m.$$

Further, if $|\theta|_v < (4C)^{-\beta_v} C_0^{-\alpha_v(1+(\log S)^u)}$, then we have the estimates

$$|F_j(\theta)|_v \le (2\gamma)^{S(\beta_v+\alpha_v)}(2(m+1)(D(m^{-1} + \delta(m + 2m^2(s+1))) + 1)^{\beta_v} \cdot \max_{t,j} |M_{t,j}|_v ((2C)^{\beta_v} C_0^{\alpha_v(1+(\log S)^u)} |\theta|_v)^{D(1+m^{-1}-3\delta m)}, \qquad j = 0, 1, \dots, m.$$

PROOF. It follows from the hypothesis that the hypotheses of Lemmas 4, 5 and 6 are valid, and $J \le 2\delta mD$. In addition, we obviously have

$$D - (m-1)M + J(ms + m - 1) \le D(m^{-1} + \delta(m + 2m^2(s+1)))$$

and

$$D + M - J \ge (1 + m^{-1} - 3\delta m)D.$$

The estimate for $|M_{t,j}(\theta)|_v$ now follows from Lemma 5. To prove the second estimate, we write

$$M_{t,j}(\theta) = \sum_{l=0}^{R} m_{t,j,l}\theta^l, \quad g_t(\theta) = \sum_{i=0}^{\infty} g_{t,i}\theta^i,$$

where $R \le D(m^{-1} + \delta(m + 2m^2(s+1)))$, by Lemma 5. Lemma 5 also implies

$$F_j(\theta) = \sum_{t=0}^{m} M_{t,j}(\theta) g_t(\theta) = \sum_{i \ge D+M-J} \left(\sum_{t=0}^{m} \sum_{l=0}^{\min(i,R)} m_{t,j,l} g_{t,i-l} \right) \theta^i.$$

$$\begin{aligned} |F_j(\theta)_v &\le \sum_{i \ge D+M-J} \left| \sum_{t=0}^{m} \sum_{l=0}^{\min(i,R)} m_{t,j,l} g_{t,i-l} \right|_v |\theta|_v^i \\ &\le ((m+1)(R+1))^{\beta_v} \max_{t,j} |M_{t,j}|_v (2\gamma)^{S\beta_v} \sum_{i \ge D+M-J} ((2C)^{\beta_v} |\theta|_v)^i \\ &\le (2^{S+1}\gamma^S (m+1)(R+1))^{\beta_v} \max_{t,j} |M_{t,j}|_v ((2C)^{\beta_v} |\theta|_v)^{(1+m^{-1}-3\delta m)D}. \end{aligned}$$

In the case $v|p$ we have $|r_i g_{t,i-l}|_v \le 1$, which implies

$$|g_{t,i-l}|_v \le 1/|r_i|_v \le r_i \le (2\gamma)^S C_0^{(1+(\log S)^u)i}.$$

Thus

$$|F_j(\theta)|_v \le \max_{t,j} |M_{t,j}|_v (2\gamma)^S (C_0^{1+(\log S)^u} |\theta|_v)^{(1+m^{-1}-3\delta m)D}.$$

This proves Lemma 7.

4. Proof of the Theorem

The main lines of the proof follow the work Väänänen [12]. Let $\theta \in K$ satisfy $h(\theta) \le h \ge e^e$ and

$$(3') \qquad \begin{cases} \theta T(\theta) \ne 0, & \log h \ge (1 + \max(3,\lambda))^{4n} (\log\log h)^u, \\ |\theta|_v < e^{-c\lambda(\log h)^{(4n-1)/4n} (\log\log h)^{u/4n}}, \end{cases}$$

where c will be given in (6). We define the natural number S by $S = [l(h)^{1/4n}]$, where $l(h) = (\log h)/(\log\log h)^u$, and denote

$$t = \binom{n+S-\lambda}{n}, \qquad w = m + 1 = t.$$

As in Väänänen [12], we obtain the estimates

$$(5) \qquad S^n/n! \le m \le c_1 S^n, \qquad t < m,\ w \le c_2 \lambda S^{n-1},$$

with positive constants c_1 and c_2 depending only on n. For the constant c appearing in (3) or (3)$'$ we take the value

$$(6) \qquad c = c_1^3 c_2 A/(4n)^u + 3c_2(n!) + 1,$$

where

$$A = 4(s+3)(\log 4\gamma\Gamma C C_0^2 + 1) + \log h(T).$$

We now choose a natural number D in such a way that

$$(D-1)\lambda(\log h)^{(4n-1)/4n}(\log\log h)^{u/4n} \le t \log h(P) < D\lambda(\log h)^{(4n-1)/4n}(\log\log h)^{u/4n}.$$

Here we assume $h(P)$ to be large enough, say $h(P) \ge H_0$, that D satisfies the conditions of Lemma 7:

$$D > \max\{\delta^{-1}(1 + (m+1)(s+1)/2), m/(1 - 3\delta m^2(s+1)), N\},$$

where we choose

$$\delta = 1/(2m(m + 2m^2(s+1))).$$

By the definitions of D and N it follows that we may choose

$$\log H_0 = \bar{c}\lambda(\log h)^{2-1/4n}(\log\log h)^{u(1-4n)/4n},$$

where $\bar{c} > 0$ is a constant depending only on (2).

By multiplying the polynomial $P(y_1(z), \dots, y_n(z))$ by the power-products

$$y_1^{k_1}(z)\cdots y_n^{k_n}(z), 0 \le k_1 + \cdots + k_n \le S - \lambda,$$

we obtain linear forms in $g_0(z), \dots, g_m(z)$, say

$$\psi_i(z) = \sum_{j=0}^{m} a_{j,i} g_j(z), \qquad i = 1, \dots, t,$$

where the $a_{j,i}$ are the coefficients of P or zero.

We now use Lemma 5 to find w linear forms, say $F_{j_k}(\theta)$, $k = 1, \dots, w$, such that these forms together with the forms $\psi_i(\theta)$, $i = 1, \dots, t$, are linearly independent (by (3)$'$, we have $|\theta|_v < (4C)^{-\beta_v} C_0^{-\alpha_v(1+(\log S)^u)}$). Then the

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determinant of these forms, say

$$\Delta_1(\theta) = \begin{vmatrix} a_{01} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{0t} & \dots & a_{mt} \\ M_{0,j_1}(\theta) & \dots & M_{m,j_1}(\theta) \\ \vdots & & \vdots \\ M_{0,j_w}(\theta) & \dots & M_{m,j_w}(\theta) \end{vmatrix}$$

must differ from zero. This determinant $\Delta_1(\theta) \in K$, and thus the product formula gives $\prod_v |\Delta_1(\theta)|_v = 1$.

By the above formula, we first obtain a lower bound

$$(7)\quad \log|\Delta_1(\theta)|_v = -\sum_{v_1 \neq v} \log|\Delta_1(\theta)|_{v_1} \geq -\sum_{v_1\neq v} (\beta_{v_1}(m+1)\log(m+1) + t\log|P|_{v_1} + w\log\max_{i,k}|M_{i,j_k}(\theta)|_{v_1}).$$

On the other hand,

$$(8)\quad |\Delta_1(\theta)|_v \leq (m+1)^{\beta_v} \max\left\{\max_{1\leq j\leq t} |\text{cofactor } (1,j)|_v |\Psi_j(\theta)|_v\,,\ \cdot \max_{1\leq i\leq w} |\text{cofactor } (1,t+i)|_v |F_{j_i}(\theta)|_v\right\},$$

and here cofactor $(1,j)$ means the $1,j$-cofactor of the matrix corresponding to $\Delta_1(\theta)$.

Since

$$\log((m+1)^{\beta_v} \max_{1\leq i\leq w} |\text{cofactor } (1,t+i)|_v |F_{j_i}(\theta)|_v) \leq \beta_v(m+1)\log(m+1) + t\log|P|_v + (w-1)\log\max_{i,k}|M_{i,j_k}(\theta)|_v + \log\max_i |F_{j_i}(\theta)|_v,$$

we have, by Lemma 7, the upper estimate

$$(9)\ \log\left((m+1)^{\beta_v}\max_{1\leq i\leq w}|\text{cofactor } (1,t+i)|_v|F_{j_i}(\theta)|_v\right) - \log|\Delta_1(\theta)|_v \leq (m+1)\log(m+1) + t\log h(P) + w\sum_{v_1\neq v}\log^+\max_{i,k}|M_{i,j_k}(\theta)|_{v_1} + (w-1)\log^+\max_{i,k}|M_{i,j_k}(\theta)|_v + \log\{(2\gamma)^{S(\beta_v+\alpha_v)}(2(m_1)D(m^{-1}+\delta(m+2m^2(s+1)))+1)^{\beta_v} \cdot \max_{t,j}|M_{t,j}|_v((2C)^{\beta_v}C_0^{\alpha_v(1+(\log S)^u)}|\theta|_v)^{D(1+m^{-1}-3\delta m)}\}.$$

Using Lemma 1, 6 and 7 we see that (9) is smaller than

$$
\begin{aligned}
&(m+1)\log(m+1)+t\log h(P)\\
&\quad+w\sum_{v_1}\{\beta_{v_1}\log(2D/m)+D(m^{-1}+\delta(m+2m^2(s+1)))\log^+|\theta|_{v_1}\\
&\qquad+\beta_{v_1}m\log m+m\beta_{v_1}(\log(D+Js+1)+2\log C(J,D)\\
&\qquad+2S\log 2\gamma+(D+Js)(\log 2C+(1+(\log S)^u)\log C_0)\\
&\qquad\qquad+m\log|Q|_{v_1}+mJ\log|T|_{v_1}\}\\
&\quad+S(\beta_v+\alpha_v)\log 2\gamma+\beta_v\log 2(m+1)\\
&\quad+D(1+m^{-1}-3\delta m)(\beta_v\log 2C+\alpha_v(1+(\log S)^u)\log C_0)\\
&\quad+D\log|\theta|_v\le wmD(\log h(T)+2\log 2\gamma+10+3\log 2C)\\
&\quad+3wmD(1+(\log S)^u)\log C_0+wm\log h(Q)\\
&\quad+2wm^{-1}D\log h(\theta)+t\log h(P)+D\log|\theta|_v\\
&\le Awm^3D(\log S)^u+2wm^{-1}D\log h(\theta)+t\log h(P)+D\log|\theta|_v\\
&\le Ac_1^3c_2\lambda S^{4n-1}(\log S)^uD+2c_2(n!)\lambda S^{-1}D\log h(\theta)+t\log h(P)+D\log|\theta|_v\\
&\le(c-1)\lambda(\log h)^{(4n-1)/4n}(\log\log h)^{u/4n}D+t\log h(P)+D\log|\theta|_v\\
&\le t\log h(P)-D\lambda(\log h)^{(4n-1)/4n}(\log\log h)^{u/4n}<0.
\end{aligned}
$$

It thus follows from (7) and (8) that

$$
\begin{aligned}
&-\sum_{v_1\ne v}\left(\beta_{v_1}(m+1)\log(m+1)+t\log|P|_{v+1}+w\log\max_{i,k}|M_{i,j_k}(\theta)|_{v_1}\right)\\
&\qquad\le\log\left((m+1)^{\beta_v}\max_{1\le j\le t}|\text{cofactor }(1,j)|_v|\Psi_j(\theta)|_v\right).
\end{aligned}
$$

Completely analogously to the above deduction we now obtain

$$
\begin{aligned}
\log\max_{1\le j\le t}|\Psi_j(\theta)|_v&>-(t\log h(P)+(c-1)\lambda(\log h)^{(4n-1)/4n}(\log\log h)^{u/4n}D)\\
&>-(c+1)t\log h(P)>-(c+1)c_1l(h)^{1/4}\log h(P).
\end{aligned}
$$

Since

$$
\max_{1\le j\le t}|\Psi_j(\theta)|_v\le\max_i(1,|y_i(\theta)|_v)^S|P(y_1(\theta),\dots,y_n(\theta))|_v,
$$

the truth of the Theorem follows.

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