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The Cuntz semigroup of unital commutative AI-algebras

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Abstract. We provide an abstract characterization for the Cuntz semigroup of unital commutative AI-algebras, as well as a characterization for abstract Cuntz semigroups of the form $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ for some T_1-space X. In our investigations, we also uncover new properties that the Cuntz semigroup of all AI-algebras satisfies.

1 Introduction

The celebrated Effros-Handelman-Shen theorem [11, Theorem 2.2] characterizes when a countable ordered abelian group G is order isomorphic to the ordered K_0-group of an approximately finite-dimensional (AF) C^*-algebra. More explicitly, it states that G is unperforated and has the Riesz interpolation property if and only if G is order isomorphic to the K_0-group of such a C^*-algebra.

In analogy to the definition of an AF-algebra, a C^*-algebra A is said to be an AI-algebra if A is $*$-isomorphic to an inductive limit whose building blocks have the form $C[0,1]\otimes F_n$ with F_n finite-dimensional for every n. In the unital commutative setting, an AI-algebra is of the form $C(X)$, with X homeomorphic to an inverse limit of (possibly increasing) finite disjoint unions of unit intervals.

In this paper, we use the Cuntz semigroup of a C^*-algebra, a refinement of K_0 introduced by Cuntz in [10] and used successfully in the classification of not necessarily simple C^*-algebras (see, for example, [7, 8, 19, 21]). We provide an abstract characterization for the Cuntz semigroup of unital commutative AI-algebras and introduce new properties that the Cuntz semigroup of every AI-algebra satisfies. This settles the range problem for the Cuntz semigroup of this class of commutative C^*-algebras; the corresponding problem in the setting of general AI-algebras was posed during the 2018 mini-workshop on the Cuntz semigroup in Houston, and is studied in [25].

Abstracting some of the properties that the Cuntz semigroup of a C^*-algebra always satisfies, the subcategory Cu of partially ordered monoids was introduced in [9]. This subcategory, whose objects are often called Cu-semigroups, contains the Cuntz

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semigroup of all C*-algebras, and has been studied extensively in [1, 3, 4, 5], among others. A relevant family in Cu is that of *countably based* Cu-semigroups satisfying (O5) (see Paragraph 2.3), which contains the Cuntz semigroups of all separable C*-algebras (see [2, 22]).

The range problem for this class of algebras consists of finding a list of properties that a Cu-semigroup S satisfies if and only if S is Cu-isomorphic to the Cuntz semigroup of a unital commutative AI-algebra. In order to do this, we first investigate when a compact metric space X is such that $C(X)$ is an AI-algebra, that is to say, we analyze when X is homeomorphic to an inverse limit of finite disjoint unions of unit intervals. To this end, and in analogy to the definition of a *chainable* continuum (see, for example, [17, Chapter 12]), we introduce *almost chainable* and *generalized arc-like* spaces (see Definitions 3.4 and 3.7, respectively). We prove the following theorem.

Theorem 1.1 (3.15) *Let X be a compact metric space. The following are equivalent:*

(i) *X is almost chainable.*
(ii) *X is a generalized arc-like space.*
(iii) *X is an inverse limit of finite disjoint copies of unit intervals.*
(iv) *$C(X)$ is an AI-algebra.*

The dimension of the spaces appearing in Theorem 1.1 is at most one, and in this case the Cuntz semigroup of $C(X)$ is isomorphic to the semigroup $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ of lower semicontinuous functions from X to $\{0,1,\dots,\infty\}$ (see, e.g., [20]). Thus, the next step in our approach is to characterize those Cu-semigroups of the form $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ for some T_1-space X. In Section 4, we define the notion of an Lsc-like Cu-semigroup. Given such a semigroup S, we prove in Section 5 that S has an associated T_1-topological space X_S (Paragraph 5.1) and that many properties defined for Cu-semigroups have a topological counterpart whenever the semigroup is Lsc-like (see Proposition 5.6). For example, an Lsc-like Cu-semigroup S has a *compact* order unit (in the sense of Paragraph 2.3) if and only if X_S is countably compact. In Theorem 6.4, we show that S is Lsc-like if and only if it is Cu-isomorphic to $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})$.

Using this characterization, together with the notion of covering dimension for Cu-semigroups introduced in [23], we obtain the following result.

Theorem 1.2 (6.5) *Let S be a* Cu*-semigroup satisfying (O5), and let $n\in\mathbb{N}\cup\{\infty\}$. Then, S is* Cu*-isomorphic to* $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ *with X a compact metric space such that* $\dim(X)=n$ *if and only if S is* Lsc*-like, countably based, has a compact order unit, and* $\dim(S)=n$.

In particular, a Cu*-semigroup S is* Cu*-isomorphic to the Cuntz semigroup of the* C**-algebra $C(X)$ with X compact metric and* $\dim(X)\leq 1$ *if and only if S is* Lsc*-like, countably based, satisfies (O5), has a compact order unit, and* $\dim(S)\leq 1$.

With these results at hand, we introduce in Section 7 notions of *chainability* for Cu-semigroups. These conditions aim at modeling, at the level of abstract Cuntz semigroups, the concepts of cover and chain for a topological space. We prove the following theorem.

Theorem 1.3 (7.11) *Let S be a* Cu*-semigroup. Then, S is* Cu*-isomorphic to the Cuntz semigroup of a unital commutative AI-algebra if and only if S is countably based,* Lsc*-like, weakly chainable, has a compact order unit, and satisfies (O5).*

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Finally, in Section 8, we generalize some of the properties used in Theorem 7.11 and show that these generalizations are satisfied for the Cuntz semigroup of every AI-algebra (see Theorem 8.7).

In [25], we continue the study of Cuntz semigroups of AI-algebras and develop tools toward a similar characterization of Theorem 7.11 in the general case.

2 Preliminaries

Given a C^*-algebra A and two positive elements $a, b \in A$, recall that a is *Cuntz subequivalent* to b, and write $a \precsim b$, if there exists a sequence $(r_n)_n$ in A such that $a = \lim_n r_n b r_n^*$. Moreover, if $a \precsim b$ and $b \precsim a$, we say that a and b are *Cuntz equivalent*, in symbols $a \sim b$.

The *Cuntz semigroup* of A, denoted by $\mathrm{Cu}(A)$, is defined as the quotient $(A \otimes \mathcal{K})_+/\sim$, where we denote by $[a]$ the class of an element $a \in (A \otimes \mathcal{K})_+$. Equipped with the order induced by $\precsim$ and the addition induced by $[a] + [b] = \left[\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\right]$, the Cuntz semigroup becomes a positively ordered monoid (see [9, 10]).

2.1 Given two elements x, y in a partially ordered set, we say that x is *way-below* y, and write $x \ll y$, if for every increasing sequence $(z_n)_n$ whose supremum exists and is greater than or equal to y there exists $n \in \mathbb{N}$ such that $x \leq z_n$.

In [9], it was shown that the Cuntz semigroup S of every C^*-algebra always satisfies the following properties:

(O1) Every increasing sequence in S has a supremum.
(O2) Every element in S can be written as the supremum of a $\ll$-increasing sequence.
(O3) For every $x' \ll x$ and $y' \ll y$, we have $x' + y' \ll x + y$.
(O4) For every pair of increasing sequences $(x_n)_n$ and $(y_n)_n$, we have $\sup_n x_n + \sup_n y_n = \sup_n (x_n + y_n)_n$.

2.2 In a more abstract setting, we say that a positively ordered monoid is a Cu-*semigroup* if it satisfies (O1)–(O4).

We also say that a map between two Cu-semigroups is a Cu-*morphism* if it is a positively ordered monoid morphism that preserves suprema of increasing sequences and the way-below relation. As proved in [9], every *-homomorphism φ between C^*-algebras induces a Cu-morphism $\mathrm{Cu}(\varphi)$ between their Cuntz semigroups.

Thus, one can consider the subcategory Cu of the category of positively ordered monoid as the category with Cu-semigroups and Cu-morphisms as its objects and morphisms, respectively. By the results from [9], the assignment $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$ is functorial.

Moreover, we know from [3, Corollary 3.2.9] that the category Cu has arbitrary inductive limits and that the functor Cu is arbitrarily continuous (see also [9]).

2.3 A Cu-semigroup is said to have *weak cancelation* if $x \ll y$ whenever $x + z \ll y + z$ for some element z. Given a *compact* element z, that is, an element such that $z \ll z$, weak cancelation implies that $x \leq y$ whenever $x + z \leq y + z$. Stable rank one C^*-algebras have weakly cancellative Cuntz semigroups by [22, Theorem 4.3].

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It was also proved in [22] that the Cuntz semigroup of a C*-algebra always satisfies the following property: for every $x' \ll x \leq z$, there exists c such that

$$x' + c \leq z \leq x + c.$$

Moreover, it was shown in [3, Proposition 4.6] that a stronger property termed (O5) was also satisfied in the Cuntz semigroup of any C*-algebra:

(O5) Given $x' \ll x$, $y' \ll y$ and $x + y \leq z$, there exists an element c such that $y' \ll c$ and $x' + c \leq z \leq x + c$.

Under the assumption of weak cancelation, (O5) and the weaker property from [22] coincide.

We will say that a Cu-semigroup is *countably based* if there exists a countable subset such that every element in the semigroup can be written as the supremum of an increasing sequence of elements in the subset. It follows from [2, Lemma 1.3] that Cuntz semigroups of separable C*-algebras are countably based.

2.4 Recall that a C*-algebra A is an AI-algebra if it is *-isomorphic to an inductive limit of the form $\lim_n C[0,1] \otimes F_n$ with F_n finite-dimensional for every n.

By [24], every unital commutative AI-algebra is *-isomorphic to an inductive limit of the form

$$C([0,1])^{n_1} \xrightarrow{\varphi_{2,1}} C([0,1])^{n_2} \xrightarrow{\varphi_{3,2}} C([0,1])^{n_3} \xrightarrow{\varphi_{4,3}} \cdots$$

with $n_i \leq n_{i+1}$ for each i and where all the homomorphisms are in standard form. That is to say, for every i, we have

$$\pi_{j,i+1}\varphi_{i+1,i}(f_1, \dots, f_{n_i}) = f_r \sigma_{j,i},$$

where $\pi_{j,i+1}\colon C([0,1])^{n_{i+1}} \to C([0,1])$ is the jth projection map, $\sigma_{j,i}\colon [0,1] \to [0,1]$ is a continuous function, and $r \leq n_i$.

Thus, every unital commutative AI-algebra is isomorphic to $C(X)$, where X is an inverse limit of (possibly increasing) finite disjoint unions of unit intervals. Conversely, it is easy to see that $C(X)$ is an AI-algebra for such an inverse limit X.

Since the space X above will always have dimension at most one, we know by [20, Theorem 1.1] that $\mathrm{Cu}(C(X)) \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$.

Moreover, note that, with the above notation, if there exists some i such that $n_i = n_j$ for every $j \geq i$, we can write $A \cong \oplus_{k=1}^{n_i} C(X_k)$ with X_k an inverse limit of unit intervals.

3 Chainable and almost chainable spaces

In this section, we will prove that a compact metric space X is homeomorphic to an inverse limit of finite disjoint copies of unit intervals (i.e., $C(X)$ is an AI-algebra) if and only if X satisfies an abstract property, which we call almost chainability (see Theorem 3.15). This is done in analogy to [17, Chapter 12], where it is shown that a continuum is homeomorphic to the inverse limit of unit intervals if and only if it is chainable, as defined below.

3.1 Recall that a *compactum* is a compact metric space, and that a *continuum* is a connected compactum. As in [17, Chapter 12], a *chain* in a continuum X will be a finite nonempty indexed collection $C = \{C_1, \dots, C_k\}$ of open subsets of X such that

$$C_i \cap C_j \neq \emptyset \text{ if and only if } |i - j| \leq 1.$$

The *mesh* of a chain $C = \{C_1, \dots, C_k\}$, in symbols $\mathrm{mesh}(C)$, is defined as

$$\mathrm{mesh}(C) = \max\{\mathrm{diam}(C_i)\}.$$

A chain of mesh less than ε is called an ε*-chain*.

Following [17, Chapter 12], one can now define chainable continua.

Definition 3.1 A continuum X is said to be *chainable* if for every positive ε there exists an ε-chain covering X.

We will say that a compactum is *piecewise chainable* if it can be written as the finite disjoint union of closed chainable subspaces.

Chainability may also be defined in a general setting, and one can check that the previous and following definitions coincide whenever X is a continuum (see Lemma 3.5).

Definition 3.2 A topological space X is said to be *topologically chainable* if any finite open cover of X can be refined by a chain, that is, an open cover $C_1, \dots, C_k$ such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Remark 3.2 Note that both of the abovementioned definitions imply that the dimension of the space is at most one, and that the space is connected whenever it is compact.

The following proposition shows why we consider chainable continua. More on chainable spaces can be found in [17, Chapter 12].

Proposition 3.3 *A compactum is an inverse limit of unit intervals if and only if it is chainable.*

Proof Let X be the inverse limit of unit intervals. Since the unit interval is compact and connected, X is connected and, consequently, a continuum.

If X is degenerate (i.e., a point), it is clearly chainable, so we may assume otherwise.

If X is nondegenerate, then [17, Theorems 12.11 and 12.19] and the comments following Theorem 12.19 of [17] imply that X is chainable.

Conversely, if X is chainable, it can either be degenerate (in which case we are done) or nondegenerate. By [17, Theorem 12.11], a nondegenerate chainable continuum is an inverse limit of unit intervals, so the result follows. ∎

Definition 3.3 A unital AI-algebra A will be said to be *block stable* if it is isomorphic to $C(X)$ with X a compact metric piecewise chainable space, that is, if $A \cong \oplus_{k=1}^{n} C(X_k)$ with X_k a chainable continuum for each k.

3.1 Generalized arc-like spaces

We now generalize the previous results and characterize the topological spaces arising from unital commutative AI-algebras in terms of a weaker property.

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3.4 Let X be a compactum. In analogy with the definition of chains, an *almost chain* in X will be a finite nonempty indexed collection $C = \{C_1, \dots, C_k\}$ of open subsets of X such that

$$C_i \cap C_j = \varnothing \text{ whenever } |i - j| \geq 2.$$

The *mesh* of an almost chain will be $\mathrm{mesh}(C) = \max\{\mathrm{diam}(C_i)\}$, and an almost chain of mesh less than ε will be called an *ε-almost chain*.

Definition 3.4 A compact metric space X will be said to be *almost chainable* if, for every $\varepsilon > 0$, there exists an ε-almost chain covering X.

Definition 3.5 A topological space X will be said to be *topologically almost chainable* if any finite open cover of X can be refined by an almost chain, that is, an open refinement $C_1, \dots, C_k$ such that $C_i \cap C_j = \varnothing$ whenever $|i - j| \geq 2$.

Lemma 3.5 *Definitions 3.4 and 3.5 coincide whenever X is a compact metric space. The same proof can be applied to chainability and topological chainability.*

Proof If X is topologically almost chainable, let $\varepsilon > 0$ and take an open cover of ε-balls in X. Since X is compact, there exist finitely many points $x_1, \dots, x_n$ such that their ε-balls cover X.

By topological almost chainability, this finite open cover can be refined by open subsets $C_1, \dots, C_k$ such that $C_i \cap C_j = \varnothing$ whenever $|i - j| \geq 2$. Since C_i is contained in some ε-ball, it follows that $C_1, \dots, C_k$ is an ε-almost chain.

Conversely, assume that X is almost chainable and take a (finite) open cover $U_1, \dots, U_n$.

Since X is a compact metric space, the cover has a nonzero Lebesgue number δ. That is, every subset of X having diameter less than δ is contained in some U_j.

Set $\varepsilon < \delta$ and consider an ε-almost chain $C = \{C_1, \dots, C_k\}$ covering X. Since $\mathrm{diam}(C_i) \leq \varepsilon < \delta$ for every i, it follows that each C_i is contained in some U_j. Consequently, $C_1, \dots, C_k$ is an open refinement of $U_1, \dots, U_n$ with the required property. ∎

We now define the notion of (ε, δ)-maps and generalized arc-like spaces. This is done in analogy with [17, Theorem 12.11].

Recall that a continuous map $f\colon X \to Y$ is an *ε-map* if $\mathrm{diam}(f^{-1}(y)) \leq \varepsilon$ for every $y \in Y$, where $\mathrm{diam}(\varnothing) = 0$ by definition.

Definition 3.6 Let X, Y be metric spaces. Given $\varepsilon, \delta > 0$, we will say that a continuous map $f\colon X \to Y$ is an *(ε, δ)-map* if $\mathrm{diam}(f^{-1}(Z)) < \varepsilon$ for every $Z \subseteq Y$ with $\mathrm{diam}(Z) < \delta$.

Recall also that a continuum X is *arc-like* if, for every $\varepsilon > 0$, there exists an ε-map from X onto $[0,1]$ (see [17, Definition 2.12]).

Definition 3.7 A compactum X is said to be a *generalized arc-like space* if, for every $\varepsilon > 0$, there exists $\delta > 0$ and an (ε, δ)-map $f\colon X \to I_1 \sqcup \cdots \sqcup I_n$ with $I_j = [0,1]$ for each j.

Remark 3.6 Given a finite disjoint union of unit intervals $I_1 \sqcup \cdots \sqcup I_n$ and some $\delta > 0$, one can clearly construct a (δ, δ')-map $r\colon I_1 \sqcup \cdots \sqcup I_n \to [0,1]$ (by simply rescaling $I_1 \sqcup \cdots \sqcup I_n$ until it fits in $[0,1]$).

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Thus, a compactum X is a generalized arc-like space if and only if, for every $\varepsilon > 0$, there exists an (ε, δ)-map $f\colon X \to [0,1]$ for some $\delta > 0$.

Lemma 3.7 *Inverse limits of finite disjoint unions of unit intervals are generalized arc-like spaces.*

Proof Let X be an inverse limit of finite disjoint unions of unit intervals, and let $([0,1] \sqcup \cdots \sqcup [0,1], f_{i,j})$ be its associated inverse system.

Recall that the metric on X is defined to be

$$\mathrm{d}((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\mathrm{d}_i(x_i, y_i)}{1 + \mathrm{d}_i(x_i, y_i)},$$

where d_i is the distance in the ith component of the inverse system.

Moreover, recall that the distance between two points x, y in $[0,1] \sqcup \cdots \sqcup [0,1]$ is either the usual distance if x, y belong to the same connected component or 2 if they do not.

Given $\varepsilon > 0$, let n be such that $\sum_{i>n} 2/2^i < \varepsilon/2$.

Then, since for every fixed n the maps $f_{i,n}$ are uniformly continuous for every $i \leq n$, there exists $\delta > 0$ such that

$$\mathrm{d}_i(f_{i,n}(x), f_{i,n}(y)) \leq \varepsilon/2n$$

whenever $\mathrm{d}_n(x, y) \leq \delta$.

Let $\pi_n\colon X \to [0,1] \sqcup \cdots \sqcup [0,1]$ be the nth canonical projection map, and take $Z \subseteq \pi_n(X)$ with $\mathrm{diam}_n(Z) \leq \delta$,

Take $x, y \in \pi_n^{-1}(Z)$. Then, we have

$$\mathrm{d}(x, y) = \sum_{i=1}^{n} \frac{1}{2^i} \frac{\mathrm{d}_i(f_{i,n}(x_n), f_{i,n}(y_n))}{1 + \mathrm{d}_i(f_{i,n}(x_n), f_{i,n}(y_n))} + \sum_{i>n} \frac{1}{2^i} \frac{\mathrm{d}_i(x_i, y_i)}{1 + \mathrm{d}_i(x_i, y_i)} \leq n\frac{\varepsilon}{2n} + \sum_{i>n} \frac{1}{2^i} 2 \leq \varepsilon,$$

since any pair of elements is at distance at most 2.

This shows that $\mathrm{diam}(\pi_n^{-1}(Z)) \leq \varepsilon$, and so π_n is an (ε, δ)-map, as required. ∎

Proposition 3.8 *A compactum X is a generalized arc-like space if and only if it is almost chainable.*

Proof We follow the proof of [17, Theorem 12.11] while making some minor adjustments.

Assume first that X is almost chainable and take $\varepsilon > 0$. Let C be an $\varepsilon/2$-almost chain covering X, and decompose it as $C = C_1 \sqcup \cdots \sqcup C_r$ with

$$C_1 = \{C_{i,1}\}_i, \ldots, C_r = \{C_{i,r}\}_i$$

$\varepsilon/2$-chains in X. Note that, for any i, l we have $C_{i,j} \cap C_{l,k} = \varnothing$ whenever $j \neq k$ and

$$X = \bigsqcup_j \bigcup_i C_{i,j},$$

because C covers X.

Take $s \leq r$. If $|C_s| \leq 2$, consider the map $f_s\colon \cup_i C_{i,s} \to [0,1]$ sending every element to 0. Note that $\mathrm{diam}(f_s^{-1}(Z)) < \varepsilon$ whenever $\mathrm{diam}(Z) < 1$.

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If $|C_s| \geq 3$, use the techniques in [17, Theorem 12.11] to obtain an (ε, δ_s)-map $f_s: \cup_i C_{i,s} \to [0,1]$.

Now, define $f = f_1 \sqcup \cdots \sqcup f_r: X \to I_1 \sqcup \cdots \sqcup I_r$, where $I_l = [0,1]$ for every l. Note that this is clearly continuous.

Setting $\delta < \min\{1, \delta_s\}$, one gets that, if the diameter of $Z \subseteq I_1 \sqcup \cdots \sqcup I_r$ is less than δ, Z is a subset of some I_l (since the distance between disjoint components is 2).

This implies that $f^{-1}(Z) = f_l^{-1}(Z)$ has diameter at most ε, as required.

Conversely, if X is a generalized arc-like, then given any ε there exists δ with $f: X \to I_1 \sqcup \cdots \sqcup I_r$ an (ε, δ)-map.

For each I_l, consider a δ-chain C_l. The inverse image of these chains through f gives the desired ε-almost chain for X. ■

Corollary 3.9 *Inverse limits of disjoint unions of unit intervals are almost chainable. In particular, each connected component is arc-like.*

Proof This follows from our previous two results. Note, however, that only one implication of Proposition 3.8 has been used. ■

We will now show that the converse of Corollary 3.9 also holds, thus obtaining a characterization of inverse limits of finite disjoint copies of unit intervals (see Theorem 3.15).

We first recall the following result from [13], even though this particular formulation is from [24, Lemma 2.14, p. 184].

Lemma 3.10 *Let C be a closed nonempty subset of $[0,1]$, and let $\varepsilon > 0$. Then, there exist a subset $Y \subseteq C$ which is a finite disjoint union of closed intervals (possibly degenerate) and a map $\alpha: C \to Y$ such that $\alpha(y) = y$ for every $y \in Y$ and $|\alpha(x) - x| < \varepsilon$ for every $x \in C$.*

Remark 3.11 With the previous notation, note that α is an onto 2ε-map from C to Y.

Indeed, for any $y \in Y$ and for any $x \in C$ with $\alpha(x) = y$, we must have $|x - y| < \varepsilon$. Since $y \in \alpha^{-1}(y)$, it follows that $\mathrm{diam}(\alpha^{-1}(y)) \leq \mathrm{diam}(B_\varepsilon(y)) < 2\varepsilon$.

From now on, by a *closed interval*, we will mean a possibly degenerate closed interval (that is to say, either a point or a nondegenerate closed interval).

3.12 Given any (ε, δ)-map $f: X \to [0,1]$, consider the induced onto ε-map $f: X \to \mathrm{Im}(f)$.

Since X is compact, so is $\mathrm{Im}(f)$. Using Lemma 3.10, we can find an onto δ-map $\alpha: \mathrm{Im}(f) \to Y$ with Y the finite disjoint union of closed intervals and, consequently, we get an onto ε-map $\alpha f: X \to Y$.

This shows that given any generalized arc-like compactum X (equivalently, any almost chainable compactum, by Proposition 3.8) and any $\varepsilon > 0$, there exists an onto ε-map $f: X \to Y$ with Y a finite disjoint union of closed intervals. Indeed, given any $\varepsilon > 0$, we know by Definition 3.7 and Remark 3.6 that there exists an (ε, δ)-map $f: X \to Y$. The conclusion now follows from the previous argument.

Using Paragraph 3.12, we will see that $C(X)$ is a commutative AI-algebra for any generalized arc-like compactum X. Since we already know that X is almost chainable whenever $C(X)$ is an AI-algebra (see Corollary 3.9), this will show that X is almost chainable if and only if $C(X)$ is a commutative AI-algebra.

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The following lemma is a natural generalization of [17, Lemma 12.17]. We follow both the structure and the notation of the proof of said lemma.

Lemma 3.13 *Let X be a compactum, let $g_1\colon X \to Y_1$ be an onto continuous map with Y_1 a finite disjoint union of closed intervals, and let $\eta > 0$.*

Then, there exists $\varepsilon > 0$ such that, for any onto ε-map $g_2\colon X \to Y_2$ with Y_2 a finite disjoint union of closed intervals, there exists a continuous map $\varphi\colon Y_2 \to Y_1$ such that $|g_1(x) - \varphi g_2(x)| < \eta$ for every $x \in X$.

Proof First, write

$$Y_1 = J_1 \sqcup \cdots \sqcup J_{n_1} \sqcup \{q_1\} \sqcup \cdots \sqcup \{q_{m_1}\}$$

with J_k closed nondegenerate intervals for every k.

Fix $m \in \mathbb{N}$ such that $1/m < \eta/2$ and define $s_i = i/m$ for every $0 \leq i \leq m$. Since g_1 is uniformly continuous, there exists some $\gamma > 0$ such that $\operatorname{diam}(g_1(A)) < 1/m$ whenever $\operatorname{diam}(A) < \gamma$.

Set $\varepsilon = \gamma/2$ and fix an onto ε-map $g_2\colon X \to Y_2$ as in the statement of the lemma. Recall that there exists $\delta > 0$ such that $\operatorname{diam}(g_2^{-1}(Z)) < 2\varepsilon = \gamma$ whenever $\operatorname{diam}(Z) < \delta$.

Now, fix $n \in \mathbb{N}$ such that $1/n < \delta/2$ and define $t_j = j/n$ for every $0 \leq j \leq n$.

As before, write Y_2 as

$$Y_2 = I_1 \sqcup \cdots \sqcup I_{n_2} \sqcup \{p_1\} \sqcup \cdots \sqcup \{p_{m_2}\}$$

with I_l closed nondegenerate intervals for every l.

For each l, k, consider the subsets

$$A_1^k = [s_0, s_1)\,,\ A_i^k = (s_{i-1}, s_{i+1})\,,\ A_m^k = (s_{m-1}, s_m] \subseteq J_k,$$
$$B_1^l = [t_0, t_1)\,,\ B_j^l = (t_{j-1}, t_{j+1})\,,\ B_n^l = (t_{n-1}, t_n] \subseteq I_l.$$

By construction, we know that $\operatorname{diam}(B_j^l) < \delta$ for every fixed j, l. This implies that $\operatorname{diam}(g_2^{-1}(B_j^l)) < \gamma$ and, consequently,

$$g_1(g_2^{-1}(B_j^l)) \subseteq A_i^k \text{ or } g_1(g_2^{-1}(B_j^l)) \subseteq \{q_{r'}\}$$

for some i, k, r'. Here, we have taken m large enough so that the distance between the connected components of Y_1 is greater than $1/m$.

Note that, by the same argument, we also have that for every $r \leq m_2$ there exist i, k with $g_1(g_2^{-1}(p_r)) \subseteq A_i^k$ or $g_1(g_2^{-1}(p_r)) \subseteq \{q_{r'}\}$ for some r'.

Moreover, since for every fixed l and every j we have $B_j^l \cap B_{j+1}^l \neq \varnothing$, we get

$$\varnothing \neq g_1(g_2^{-1}(B_j^l)) \cap g_1(g_2^{-1}(B_{j+1}^l))$$

because g_2 is onto.

It follows that, for every fixed l, the sets $g_1(g_2^{-1}(B_j^l))$ belong to the same connected component of Y_1 for every j. Thus, for every l, there either exists k such that $g_1(g_2^{-1}(I_l)) \subseteq J_k$ or there exists r' such that $g_1(g_2^{-1}(I_l)) = \{q_{r'}\}$.

Given a connected component Y of Y_2, we define the map $\varphi_Y\colon Y \to Y_1$ as follows:

If $g_1(g_2^{-1}(Y)) = \{q_{r'}\}$ for some r', define $\varphi_Y\colon Y \to \{q_{r'}\}$ as the constant map.

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Else, there exists some k such that $g_1(g_2^{-1}(Y)) \subseteq J_k$. If Y is degenerate, we can find $A_i^k \subseteq J_k$ such that $g_1(g_2^{-1}(Y)) \subseteq A_i^k$ for some i,k. Define $\varphi_Y\colon Y \to A_i^k \subseteq J_k$ as the constant map $\varphi_Y \equiv s_i$.

Finally, if Y is nondegenerate, it is of the form $Y = I_l$. Then, for every j, fix $i(j)$ such that $g_1(g_2^{-1}(B_j^l)) \subseteq A_{i(j)}^k$, and recall that

$$\varnothing \neq g_1(g_2^{-1}(B_j^l)) \cap g_1(g_2^{-1}(B_{j+1}^l))$$

for every j. This shows that $|i(j) - i(j+1)| \leq 1$.

Define $\varphi_{I_l}\colon I_l \to J_k$ as $\varphi_{I_l}(t_j) = s_{i(j)}$ and extend it linearly.

Let $\varphi := \varphi_{I_1} \sqcup \cdots \sqcup \varphi_{I_{n_2}} \sqcup \varphi_{p_1} \sqcup \cdots \sqcup \varphi_{p_{m_2}}\colon Y_2 \to Y_1$, which is clearly continuous. We will now see that $|g_1(x) - \varphi g_2(x)| < \eta$.

Thus, let $x \in X$ and let $B \subseteq Y_2$ such that $g_2(x) \in B$ with B being either B_j^l for some l,j or $\{p_r\}$ for some r. Note that $g_1(x) \in g_1(g_2^{-1}(B))$.

Thus, if $g_1(g_2^{-1}(B)) = \{q_{r'}\}$ for some r', we have $g_1(x) = q_{r'}$ and, consequently,

$$|g_1(x) - \varphi g_2(x)| = |q_{r'} - q_{r'}| = 0$$

by the definition of φ.

Finally, if $g_1(g_2^{-1}(B)) \subseteq A_i^k$, we have that $g_1(x) \in A_i^k$. Therefore, one gets $|g_1(x) - s_i| < 1/m$. There are now two different situations:

If $B = \{p_r\}$ for some r, we have defined φ_{p_r} as the constant map s_i. Thus, one gets

$$|g_1(x) - \varphi g_2(x)| = |g_1(x) - s_i| < 1/m < \eta/2.$$

Else, if $B = B_j^l$ for some l and j, let $i(j)$ be the previously fixed integer such that $g_1(g_2^{-1}(B)) \subseteq A_{i(j)}^k$.

Then, since $g_2(x) \in B$, we either have $t_{j-1} \leq g_2(x) \leq t_j$ or $t_j \leq g_2(x) \leq t_{j+1}$. This implies that $\varphi(g_2(x))$ is either between $s_{i(j-1)}$ and $s_{i(j)}$ or between $s_{i(j)}$ and $s_{i(j+1)}$.

Since $|i(j) - i(j+1)| \leq 1$, the triangle inequality implies that

$$|g_1(x) - \varphi g_2(x)| \leq |g_1(x) - s_{i(j)}| + |s_{i(j)} - \varphi g_2(x)| \leq 2/m < \eta,$$

as required. ∎

In order to prove our next result, we will need the following proposition, a proof of which can be found in [17, Proposition 12.18].

Proposition 3.14 *Let (X,d) be a compactum, and let $Y = \varprojlim(Y_i, f_i)$ be an inverse limit of compacta (Y_i, d_i) with $f_i\colon Y_{i+1} \to Y_i$.*

Assume that there exist two sequences of strictly positive real numbers (δ_i), (ε_i) with $\lim \varepsilon_i = 0$ and a family of onto ε_i-maps $g_i\colon X \to Y_i$ such that the following conditions hold:

(i) *For every pair $i < j$, we have $\operatorname{diam}(f_{i,j}(A)) \leq \delta_i/2^{j-i}$ for any $A \subseteq Y_j$ with $\operatorname{diam}(A) \leq \delta_j$.*
(ii) *$d_i(g_i(x), g_i(y)) > 2\delta_i$ whenever $d(x,y) \geq 2\varepsilon_i$.*
(iii) *$d_i(g_i, f_i g_{i+1}) \leq \delta_i/2$.*

Then, $X \cong Y$.

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We summarize our results in the following.

Theorem 3.15 *Let X be a compactum. The following are equivalent:*

(i) *X is almost chainable.*
(ii) *X is a generalized arc-like space.*
(iii) *X is homeomorphic to an inverse limit of finite disjoint copies of unit intervals.*
(iv) *$C(X)$ is an AI-algebra.*

Proof Conditions (i) and (ii) are equivalent by Proposition 3.8, whereas (iii) is equivalent to (iv) by the arguments in Paragraph 2.4. Furthermore, that (iii) implies (i) follows from Corollary 3.9. Thus, we are left to prove that (i) implies (iii).

Let X be an almost chainable compactum. Then, for every $\varepsilon > 0$, there exists an onto ε-map $f\colon X \to Y$ with Y a finite disjoint union of closed intervals.

As in [17, Theorem 12.19], we will inductively construct sequences of maps and real numbers satisfying the conditions of Proposition 3.14. We give the proof here for the sake of completeness, although the only difference with the original proof is that we replace $[0,1]$ with Y.

Let $0 < \varepsilon_1 \leq 1$, and consider an onto ε_1-map g_1 from X to a finite disjoint union of closed intervals Y_1.

Note that, in particular, we must have $g_1(x) \neq g_1(y)$ whenever $d(x,y) \geq 2\varepsilon_1$. By compactness of X, this implies that there exists some $\delta_1 > 0$ such that $|g_1(x) - g_1(y)| > 2\delta_1$ whenever $d(x,y) \geq 2\varepsilon_1$.

Setting $\eta = \delta_1/2$, let $\varepsilon > 0$ be as in our previous lemma and set $\varepsilon_2 = \min\{1/2, \varepsilon\}$. Then, let $g_2\colon X \to Y_2$ be an onto ε_2-map given by the almost chainability of X.

By our choice of ε_2, there exists some $f_1\colon Y_2 \to Y_1$ such that

$$|g_1(x) - f_1 g_2(x)| < \eta = \delta_1/2$$

for every $x \in X$.

As before, there exists $\delta_2 > 0$ such that $|g_2(x) - g_2(y)| > 2\delta_2$ whenever $d(x,y) \geq 2\varepsilon_2$. Furthermore, by the uniform continuity of f_1, we can choose $\delta_2 > 0$ so that $\operatorname{diam}(f_1(A)) \leq \eta$ whenever $\operatorname{diam}(A) \leq \delta_2$.

This shows that g_1, g_2 satisfy conditions (i)–(iii) of Proposition 3.14.

Proceeding as in [17, Theorem 12.19], one can now inductively find $\varepsilon_i, \delta_i > 0$ and maps $g_i\colon X \to Y_i$, $f_i\colon Y_{i+1} \to Y_i$ satisfying conditions (i)–(iii) of Proposition 3.14.

Applying Proposition 3.14, we have $X \cong \lim(Y_k, f_k)$ with Y_k finite disjoint unions of closed intervals. Clearly, this implies that $C(X)$ is an AI-algebra, as required. ∎

4 Lsc-like Cu-semigroups

In this section, we introduce Lsc-like Cu-semigroups (Definition 4.3) and prove some of their main properties. As we shall prove in Theorem 6.4, such Cu-semigroups are exactly those that are Cu-isomorphic to the Cu-semigroup of lower-semicontinuous functions $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ for some T_1 topological space.

Using Lemma 4.18, we also prove that the semigroup $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a Cu-semigroup whenever X is compact and metric (see Corollary 4.19).

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Definition 4.1 A Cu-semigroup S will be called *distributively lattice ordered* if S is a distributive lattice such that, given any pair of elements $x, y \in S$, we have $x + y = (x \vee y) + (x \wedge y)$.

We will say that a distributively lattice ordered Cu-semigroup is *complete* if suprema of arbitrary sets exist.

Remark 4.1 Given two increasing sequences $(x_n)_n$ and $(y_n)_n$ in a complete distributively lattice ordered Cu-semigroup, we have that $(\sup_n x_n) \vee (\sup_n y_n) = \sup_n (x_n \vee y_n)$.

Indeed, by definition, we know that $\sup_n x_n = \vee_{n=1}^{\infty} x_n$ and the equality

$$(\vee_{n=1}^{\infty} x_n) \vee (\vee_{n=1}^{\infty} y_n) = \vee_{n=1}^{\infty} (x_n \vee y_n)$$

holds in any complete lattice.

Throughout the paper, we will say that a sum of finitely many indexed elements $x_1 + \cdots + x_n$ is *ordered* if the sequence $(x_i)_{i=1}^{n}$ is increasing or decreasing.

Lemma 4.2 *Let S be a distributively lattice ordered* Cu*-semigroup. Given two finite decreasing sequences $(x_i)_{i=1}^{m}, (y_i)_{i=1}^{m}$, the following equality holds:*

$$\sum_{i=1}^{m} (x_i + y_i) = \sum_{i=1}^{2m} \vee_{j=0}^{m} (x_j \wedge y_{i-j}),$$

where, on the right-hand side, $x_i \wedge y_k = x_i$, and $x_k \wedge y_i = y_i$ whenever $k \leq m$ and $y_k = x_k = 0$ whenever $k > m$.

Note that the sum on the right-hand side is decreasingly ordered.

Proof This is a generalization of the equality $x + y = (x \vee y) + (x \wedge y)$ and is proved by induction. ∎

Remark 4.3 Given any finite sum $x_1 + \cdots + x_n$, we can apply Lemma 4.2 iteratively (first to $x_1 + x_2$, then to $((x_1 \vee x_2) + (x_1 \wedge x_2)) + x_3$, etc.) to obtain an ordered sum.

Therefore, every sum in a distributively lattice ordered semigroup can be written as an ordered sum.

Definition 4.2 Let S be a Cu-semigroup, and let H be a subset of S. We say that H is *topological* if, given two finite increasing sequences $(x_i)_{i=1}^{m}, (y_i)_{i=1}^{m}$ in H, we have

$$\sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} y_i$$

if and only if $x_i \leq y_i$ for every i.

Similar notions to that of topological order appear in other contexts, such as in [6, Definition 4.1].

Given any element r in partially ordered set P, we denote by $\downarrow r$ the set $\{s \in P \mid s \leq r\}$ (see, for example, [14, Definition O-1.3]).

Notation 4.4 Given a Cu-semigroup S and an element $y \in S$, we write $\infty y := \sup_n ny$. Furthermore, if S has a greatest element, we denote it by ∞.

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Definition 4.3 A Cu-semigroup S will be said to be *Lsc-like* if it is a complete distributively lattice ordered Cu-semigroup such that the following conditions hold:

(C1) For every pair of idempotent elements y, z in S, $y \geq z$ if and only if

$$\{x < \infty \mid x \text{ maximal idempotent}, x \geq y\}$$
$$\subseteq \{x < \infty \mid x \text{ maximal idempotent}, x \geq z\}.$$

(C2) There exists a topological subset of the form $\downarrow e$ such that the finite sums of elements in $\downarrow e$ are sup-dense in S.

The following example justifies our terminology.

Example 4.5 Any Cu-semigroup of the form $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ with X a T_1-space is Lsc-like. Indeed, $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is clearly a complete distributively lattice ordered semigroup.

Furthermore, the maximal idempotent elements $s < \infty$ are of the form $s = \infty\chi_{X\setminus\{x\}}$, and given any pair of elements $\infty f = \infty\chi_{\mathrm{supp}(f)}$ and $\infty g = \infty\chi_{\mathrm{supp}(g)}$, we know that $\infty\chi_{\mathrm{supp}(f)} \leq \infty\chi_{\mathrm{supp}(g)}$ if and only if $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$. That is, if and only if for every $x \in X$ such that $\mathrm{supp}(g) \subseteq X\setminus\{x\}$ we have $\mathrm{supp}(f) \subseteq X\setminus\{x\}$. This shows that $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ satisfies (C1).

To see (C2), consider the subset $\downarrow 1$. One can see that the order is topological, and it clearly generates a semigroup that is dense in $\mathrm{Lsc}(X, \overline{\mathbb{N}})$.

Recall that by an *order unit* we mean an element e such that $x \leq \infty e$ for every $x \in S$.

Remark 4.6 Let S be an Lsc-like Cu-semigroup. Since the semigroup generated by $\downarrow e$ is dense in S, e is an order unit.

Furthermore, let f be another order unit, and let $(f_m)_m$ be an increasing sequence of ordered finite sums of elements below e such that $f = \sup_m f_m$ (the sums can be assumed to be ordered by Remark 4.3). Write $f_m = \sum_{i=1}^{r_m} g_{i,m}$ with $e \geq g_{i,m} \geq g_{i+1,m}$ for each $i < r_m$.

Moreover, consider a $\ll$-increasing $(e_n)_n$ whose supremum is e.

Then, since $e \leq \infty f$, one has that for every $n \in \mathbb{N}$ there exist some $k, m \in \mathbb{N}$ with $e_n \leq k f_m = \sum_{i=1}^{r_m} k g_{i,m}$. Thus, since the order in $\downarrow e$ is topological and $\sum_{i=1}^{r_m} k g_{i,m}$ is an ordered sum of elements below e (the first k greatest elements are $g_{1,m}$, the next k are $g_{2,m}$, etc.), it follows that $e_n \leq g_{1,m} \leq f$.

This shows $e \leq f$ and, consequently, e is the least order unit of S.

4.7 Note that an Lsc-like Cu-semigroup S has no maximal idempotent elements $x < \infty$ if and only if $S = \{0\}$. Indeed, if S has no maximal idempotents, then every idempotent z satisfies $z \leq 0$ by condition (C1). This implies that every idempotent element is zero and, since for every $s \in S$ the element ∞s is idempotent and $\infty s \geq s$, we get $s = 0$ for every s, as desired. The converse is trivial.

Similarly, an element $s \in S$ satisfies $\infty s = \infty$ if and only if there are no maximal idempotents $x < \infty$ with $\infty s \leq x$.

Lemma 4.8 *Let S be an* Lsc*-like* Cu*-semigroup, and let $y \leq ne$, where e is the least order unit of S and $n \in \mathbb{N}$. Then, y can be written as an ordered sum of at most n nonzero terms in $\downarrow e$.*

Proof Write y as $y = \sup_r y_r$, where each y_r is a finite ordered sum of elements in $\downarrow e$ (see Lemma 4.2 and Remark 4.3).

As $y_r \le ne$ for each r and $\downarrow e$ is topological, each y_r has at most n nonzero summands.

Furthermore, since $y_r \le y_{r+1}$, the kth summands of $(y_r)_r$ form an increasing sequence for every k.

Taking their suprema, this shows that y is an ordered sum of at most n elements in $\downarrow e$. ■

Corollary 4.9 *Let S be an* Lsc-*like* Cu-*semigroup. Given $y_1, \dots, y_n \in S$ such that $y_1 + \cdots + y_n \ge s$ for some $s \le e$, we have $y_1 \vee \cdots \vee y_n \ge s$.*

Proof Take any $s' \ll s$, and let $y_i' \ll y_i$ be such that $y_1' + \cdots + y_n' \ge s'$. Since $\infty e \ge y_1 + \cdots + y_n \gg y_1' + \cdots + y_n'$, there exists some $k \in \mathbb{N}$ such that $ke \ge y_1' + \cdots + y_n'$.

By Lemma 4.8, each y_i' can be written as an ordered sum of at most k elements below e. That is, we can write $y_i' = \sum_{j=1}^{k} y_{j,i}$ with $e \ge y_{j,i} \ge y_{j+1,i}$ for each $j < k$.

Applying Lemma 4.2 iteratively, we have that $y_1' + \cdots + y_n'$ can be written as a finite ordered sum of elements below e, and that the greatest summand of the sum is $y_{1,1} \vee \cdots \vee y_{1,n}$.

Since the order in $\downarrow e$ is topological, we get $y_{1,1} \vee \cdots \vee y_{1,n} \ge s'$. It follows that

$$s' \le y_{1,1} \vee \cdots \vee y_{1,n} \le y_1' \vee \cdots \vee y_n' \le y_1 \vee \cdots \vee y_n.$$

Since this holds for every $s' \ll s$, we have $s \le y_1 \vee \cdots \vee y_n$ as required. ■

The next two lemmas will be of particular importance when working with the induced topology of an Lsc-like Cu-semigroup (see Paragraph 5.1). In particular, Lemma 4.11 gives an alternative version of (C1) in Definition 4.3.

Lemma 4.10 *Let S be an* Lsc-*like* Cu-*semigroup with least order unit e, and let $x \in S$. Then, $\infty x = \infty(x \wedge e)$ and $(\infty x) \wedge e = x \wedge e$.*

In particular, if $x < \infty$ is a maximal idempotent, we must have $x \wedge e \ne e$.

Proof By Lemma 4.8 and condition (C2) in Definition 4.3, x can be written as the supremum of finite ordered sums of elements in $\downarrow e$. Moreover, using that the order in $\downarrow e$ is topological, it follows that the sequence formed by the greatest element of each ordered sum is increasing. Let x' be the supremum of such a sequence.

Then, it is clear that $\infty x = \infty x'$ and $x \wedge e = x'$. Therefore, we have

$$\infty x = \infty x' = \infty(x \wedge e), \quad \text{and} \quad (\infty x) \wedge e = (\infty x') \wedge e = x' = x \wedge e,$$

as desired. ■

Lemma 4.11 *Let S be an* Lsc-*like* Cu-*semigroup with least order unit e. Let $y, z \le e$. If $z \le x$ for every maximal element $x < e$ such that $y \le x$, then $z \le y$.*

Proof If there are no maximal idempotent elements $x < \infty$, we know that $S = \{0\}$, so we may assume otherwise.

We claim that the maximal idempotent elements $x < \infty$ are precisely the elements ∞s with $s < e$ maximal.

To see this, let $x < \infty$ be a maximal idempotent and take $s = x \wedge e$. By Lemma 4.10, we have $\infty s = \infty(x \wedge e) = \infty x = x$. Let us check that s is maximal, and thus take $t \in S$ with $s \le t \le e$.

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By maximality of x, we either have $\infty t = \infty$ or $\infty t = x$. Using Lemma 4.10 once again, it follows that we have $t = (\infty t) \wedge e = \infty \wedge e = e$ or $t = x \wedge e = s$, as required.

Conversely, let $s < e$ be maximal and consider the element $\infty s = \sup_n ns$. Let $(s_k)_k$ be a $\ll$-increasing sequence with supremum s.

Given an idempotent element x such that $\infty s \leq x \leq \infty$, we know by Lemma 4.8 that there exists a $\ll$-increasing sequence $(x_m)_m$ with supremum x such that each x_m can be written as a finite increasing sum of elements in $\downarrow e$.

Let $(m_k)_k$ be an increasing sequence of integers such that $ks_k \leq x_{m_k}$ for every k.

Using that $\downarrow e$ is topological, each s_k is less than or equal to each of the first k summands of x_{m_k}. As in the proof of Lemma 4.10, note that the largest summands of each sum form an increasing sequence of elements below e. Letting x' be the supremum of this sequence, we get $s \leq x' \leq e$.

By maximality of s, we either have $s = x'$ or $x' = e$. Thus, it follows from Lemma 4.10 that $\infty s = x$ or $x = \infty x' = \infty$. This proves that ∞s is a maximal idempotent, as desired.

A similar argument shows that for any $y, z \leq e$ we have $y \leq z \leq e$ if and only if $\infty y \leq \infty z$. Consequently, if $z \leq s$ for every maximal element $s < e$ such that $y \leq s$, we know that $\infty z \leq \infty y$. Taking the infimum with e, one gets $z = (\infty z) \wedge e \leq (\infty y) \wedge e = y$, as required. ∎

Remark 4.12 Note that the previous lemma implies that $y < e$ if and only if there exists a maximal element $x < e$ with $y \leq x$.

Recall that a complete lattice $(P, \leq)$ is said to be a *complete Heyting algebra* if, for every element $s \in P$ and every subset $T \subseteq P$, the following holds:

$$s \wedge (\vee_{t \in T} t) = \vee_{t \in T} (s \wedge t).$$

Lemma 4.13 *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e. The subset $\downarrow e \subseteq S$ is a complete Heyting algebra.*

Proof By definition, we have to see that for every subset $T \subseteq \downarrow e$, one has $s \wedge (\vee_{t \in T} t) = \vee_{t \in T}(s \wedge t)$ for every $s \in \downarrow e$. Thus, let $x < e$ be maximal with $x \geq \vee_{t \in T}(s \wedge t)$, which happens if and only if $x \geq s \wedge t$ for every $t \in T$. Since $x \neq e$ and $x \vee x \geq x \vee (s \wedge t) = (x \vee s) \wedge (x \vee t)$, we either have $x \vee s = x$ or $x \vee t = x$ (since otherwise both of these unions would be equal to e and then $x \geq e$, a contradiction).

If $x = x \vee s \geq s$, we have $x \geq s \wedge (\vee_{t \in T} t)$. Else, $x \geq t$ for each $t \in T$, so we also get $x \geq s \wedge (\vee_{t \in T} t)$.

By Lemma 4.11, this shows that $s \wedge (\vee_{t \in T} t) \leq \vee_{t \in T}(s \wedge t)$.

The other inequality holds in any lattice. ∎

Let S be a Cu-semigroup, and let $y, z \in S$ be a pair of elements such that $y \leq z$. We say that an element u is the *almost complement of z by y* if, for every $x \in S$, we have $x + y \leq z$ if and only if $x \leq u$. If it exists, the almost complement is unique, and we denote by $y \backslash z$.

Given a Cu-semigroup S where every pair of elements has a supremum, we say that S is *sup-semilattice ordered* if for every $x, y, z \in S$ we have $x + (y \vee z) = (x + y) \vee (x + z)$.

A natural question to ask about Lsc-like Cu-semigroups is if they satisfy such a property. This is indeed the case.

Lemma 4.14 *Let S be an* Lsc-*like* Cu-*semigroup, and let e be its least order unit. Given* $y \leq z$ *with* $y \leq ne$ *for some n, the almost complement* $y\backslash z$ *exists.*

In particular, given $x, y, z \in S$*, we have* $x + (y \vee z) = (x + y) \vee (x + z)$.

Proof We will construct our almost complement $y\backslash z$ in three steps.

Step 1. Let e be the least order unit of S and assume that $y \leq z \leq e$. Then, consider the subset $T := \{x \in S \mid y + x \leq z\}$ and, since arbitrary suprema exist in S, we define

$$y\backslash z := \vee\{x \in S \mid y + x \leq z\} = \vee_{x\in T} x.$$

Furthermore, note that, for $x \in T$, $y + x = (y \vee x) + (y \wedge x)$ and, as $z \leq e$ and $\downarrow e$ is topological, we have $y \vee x \leq z$ and $y \wedge x = 0$.

Using that $\downarrow e$ is a complete Heyting algebra, we get $(\vee_{x\in T} x) \vee y = \vee_{x\in T}(x \vee y) \leq z$ and $(\vee_{x\in T} x) \wedge y = \vee_{x\in T}(x \wedge y) = 0$. Consequently, we get

$$y + (y\backslash z) = y \vee (y\backslash z) + y \wedge (y\backslash z) = y \vee (y\backslash z) \leq z.$$

This shows that $x \leq y\backslash z$ if and only if $y + x \leq z$.

Step 2. Now, given any pair of elements $y \leq z \leq ne$ for some $n \in \mathbb{N}$, write them as finite ordered sums of elements in $\downarrow e$, say $y = \sum_{i=1}^n y_i$ and $z = \sum_{j=1}^n z_j$.

Since $\downarrow e$ is topological, $y_i \leq z_i \leq e$ for every i, so we can define $y\backslash z := \sum_{i=1}^n y_i\backslash z_i$.

Note that, given any element x such that $y + x \leq z \leq ne$, we have $x \leq ne$ and so, by Lemma 4.8, x can be written as a finite ordered sum $\sum_{i=1}^n x_i$ with $x_i \leq e$ for every i. Moreover, we also have

$$\sum_{i=1}^n y_i \vee x_i \leq y + x \leq \sum_{i=1}^n z_i$$

and so $y_i \vee x_i \leq z_i$, which happens if and only if $x_i \leq y_i\backslash z_i$.

This shows that $x + y \leq z \leq ne$ for some n if and only if $x \leq y\backslash z$.

Step 3. Given $y \leq z \wedge ne$ for some n, we have that $y \leq z \wedge me$ for every $m \geq n$.

Thus, we can consider the element $y\backslash(z \wedge me)$. Furthermore, it is easy to check that $y\backslash(z \wedge me) \leq y\backslash(z \wedge (m+1)e)$ for every m. We define $y\backslash z := \sup_m y\backslash(z \wedge me)$, which has the required property.

Finally, to see that $x + (y \vee z) = (x + y) \vee (x + z)$ for any given x, y, z, note that "$\geq$" is clear. To prove "$\leq$," let $x' \ll x$ and let s be such that $x' + (y \vee z) \leq s$.

Since $x' \ll x \leq \infty e$, there exists some n with $x' \leq ne$, so we can consider the element $x'\backslash s$.

Thus, we know that $x' + (y \vee z) \leq s$ holds if and only if $y \vee z \leq x'\backslash s$, which in turn holds if and only if $x' + y, x' + z \leq s$. Consequently, we get

$$x' + (y \vee z) = (x' + y) \vee (x' + z).$$

Since the equality holds for every $x' \ll x$, it also holds for x. ∎

Lemma 4.15 *Let S be an* Lsc-*like* Cu-*semigroup with least order unit e, and let* $x, y, z \leq e$ *with* $x + y \leq x + z$*. Then,* $y \leq z$*.*

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Proof First, note that $y \le x \vee y \le x \vee z$. Indeed, since $x, y, z \le e$ and we know that $x + y = (x \vee y) + (x \wedge y)$ and $x + z = (x \vee z) + (x \wedge z)$, we have that $(x \vee y) + (x \wedge y) \le (x \vee z) + (x \wedge z)$.

Since the right- and left-hand sides of the previous inequality are ordered sums of elements below e, we can use that the order in $\downarrow e$ is topological to get $x \vee y \le x \vee z$, as desired.

Therefore, the sum $(z \vee x) + y$ is ordered, as we have $y \le z \vee x$.

Using Lemma 4.14 at the first and third steps, the inequality $x + y \le x + z$ at the second step, and that S is distributively lattice ordered at the last step, one obtains

$$\begin{aligned}(z \vee x) + y &= (y + z) \vee (y + x) \le (y + z) \vee (z + x) = z + (y \vee x) \\ &= (z \vee y \vee x) + z \wedge (y \vee x).\end{aligned}$$

Using once again that the order in $\downarrow e$ is topological, it follows that $y \le z \wedge (y \vee x) \le z$, as required. ∎

By extending the previous proof, one can check that whenever $y, z \in S$ and $x \le ne$ for some n, the same cancelation property holds. In particular, it follows that every Lsc-like Cu-semigroup has weak cancelation.

Remark 4.16 Using the previous form of cancelation and the equality

$$\begin{aligned}((x + y) \vee (x + z)) + (x + (y \wedge z)) &= x + y + x + z \\ &= ((x + y) \vee (x + z)) + ((x + y) \wedge (x + z)),\end{aligned}$$

one can check that S is inf-semilattice ordered, that is to say, $x + y \wedge z = (x + y) \wedge (x + z)$ for every x, y, z (see [1]). For elements $x, y, z \le ne$ for some n, note that this simply follows by canceling the term $((x + y) \vee (x + z))$.

Using that every element in S is the supremum of a $\ll$-increasing sequence of finite sums of elements below e, one can then prove that the equality $x + (y \wedge z) = (x + y) \wedge (x + z)$ is always satisfied.

As one might expect, having a topological order also affects the way below relation.

Lemma 4.17 *Let S be an* Lsc-*like* Cu-*semigroup with least order unit e. If $y, z, y', z' \le e$ are such that*

$$y + z \ll y' + z'$$

with $y \ge z$ and $y' \ge z'$, we have that $y \ll y'$ and $z \ll z'$.

The same holds for any pair of finite sums (i.e., with more than two summands).

Proof Write $y' = \sup_n y'_n$ and $z' = \sup_n z'_n$ with $(y'_n)_n$ and $(z'_n)_n$ $\ll$-increasing. Since $y + z \ll y' + z'$, we have $y + z \le y'_n + z'_n$ for some n.

Moreover, as we have that $z'_n \ll z' \le y'$, there exists a k such that $z'_n \le y'_k$. This implies that

$$y + z \le y'_n + z'_n \le y'_{\max\{n,k\}} + z'_n,$$

and, since the order is topological, we obtain $y \le y'_{\max\{n,k\}} \ll y'$ and $z \le z'_n \ll z'$. ∎

Even though the following lemma is probably well known, we prove it here for the sake of completeness. For second countable finite-dimensional compact Hausdorff

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spaces, it follows from a much more general result that $\mathrm{Lsc}(X, \overline{\mathbb{N}}) \in \mathrm{Cu}$ (see [2, Theorem 5.15]). As a consequence of Lemma 4.18, we will have that $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a Cu-semigroup whenever X is a compact metric space (see Corollary 4.19). Recently, this result has been further generalized by Elliott and Im (see [12, Proposition 1.16]).

Given $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$ and $n \in \mathbb{N}$, we write $\{f \geq n\}$ to denote the open set $f^{-1}([n, \infty])$. For an open set $U \subseteq X$, we denote by χ_U the indicator function of U.

Lemma 4.18 *For any topological space X and any pair $f, g \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$, one has $f \ll g$ if and only if*

$$\chi_{\{f\geq n\}} \ll \chi_{\{g\geq n\}} \text{ for every } n \text{ and } \sup(f) < \infty.$$

Proof First, let us assume that $f \ll g$. Fix $n \in \mathbb{N}$ and consider an increasing sequence $(h_k)_k$ such that

$$\chi_{\{g\geq n\}} \leq \sup_k h_k,$$

which happens if and only if $\chi_{\{g\geq n\}} \leq \chi_{\cup_k \mathrm{supp}(h_k)}$.

Define the increasing sequence of functions

$$G_k := (n-1) + \chi_{\mathrm{supp}(h_k)} \sum_{r=0}^{\infty} \chi_{\{g\geq n+r\}},$$

and note that $g \leq \sup_k G_k$.

Since $f \ll g$, we get that $f \leq G_k$ for some k and, consequently,

$$\{f \geq n\} \subseteq \{G_k \geq n\} = \mathrm{supp}(h_k) \cap \{g \geq n\} \subseteq \mathrm{supp}(h_k).$$

This in turn implies $\chi_{\{f\geq n\}} \leq \chi_{\mathrm{supp}(h_k)} \leq h_k$, so it follows that $\chi_{\{f\geq n\}} \ll \chi_{\{g\geq n\}}$. That $\sup(f) < \infty$ is clear.

Conversely, if $\sup(f) < \infty$, we know that $f = \sum_{i=1}^m \chi_{\{f\geq i\}}$ for some $m < \infty$. Furthermore, given an increasing sequence $(h_k)_k$ with $g \leq \sup_k h_k$, it follows that

$$\{g \geq n\} \subseteq \bigcup_k \{h_k \geq n\},$$

so $\chi_{\{g\geq n\}} \leq \sup_k \chi_{\{h_k\geq n\}}$. Since $\chi_{\{f\geq n\}} \ll \chi_{\{g\geq n\}}$ for every n, we get that for each i there exists an integer k_i with

$$\chi_{\{f\geq i\}} \leq \chi_{\{h_{k_i}\geq i\}}.$$

Taking $k = \max_{i=1,\dots,m}\{k_i\}$, we have

$$f \leq \sum_{i=1}^{m} \chi_{\{h_k\geq i\}} \leq h_k,$$

as desired. ∎

Corollary 4.19 *Let X be a compact metric space. Then $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a Cu-semigroup with pointwise order and addition.*

Proof Axioms (O1) and (O4) are always satisfied in $\mathrm{Lsc}(X, \overline{\mathbb{N}})$, so we only need to prove (O2) and (O3).

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Let U, V be open subsets of X, and note that $\chi_U \ll \chi_V$ if and only if U is compactly contained in V. Indeed, if $\chi_U \ll \chi_V$, we can write V as a countable increasing union of open sets V_n such that V_n is compactly contained in V_{n+1} for every n. Thus, one gets $\chi_U \ll \sup_n \chi_{V_n}$, which implies that U is contained in V_n for some n. Conversely, if $\overline{U} \subseteq V$ and $(W_n)_n$ is an increasing sequence of open sets with $V = \cup_n W_n$, it is clear that $U \subseteq W_n$ for some n. This shows $\chi_U \ll \chi_V$, as required.

In particular, it follows that every indicator function can be written as the supremum of a $\ll$-increasing sequence. Since every element in $S = \mathrm{Lsc}(X, \overline{\mathbb{N}})$ is the supremum of finite sums of indicator functions, one can check that S satisfies (O2).

Now, let $f' \ll f$ and $g' \ll g$ in S, which by Lemma 4.18 implies that $\sup(f), \sup(g) < m \leq \infty$ and $\{f' \geq n\}, \{g' \geq n\}$ are compactly contained in $\{f \geq n\}, \{g \geq n\}$, respectively. Thus, we have

$$\bigcup_{k=0}^{m} \overline{(\{f' \geq k\} \cap \{g' \geq n-k\})} \subseteq \bigcup_{k=0}^{m} (\{f \geq k\} \cap \{g \geq n-k\})$$

for every $n \leq \sup(f) + \sup(g)$, where note that the left-hand side is equal to $\overline{\{f' + g' \geq n\}}$ and the right-hand side is contained in $\{f + g \geq n\}$. By Lemma 4.18, we have $f' + g' \ll f + g$, which shows that S satisfies (O3). ■

5 The topological space of an Lsc-like Cu-semigroup

In this section, we associate with each Lsc-like Cu-semigroup S a topological space X_S. In Proposition 5.3, we prove some of the properties that such a topological space must satisfy and, using these, we show that $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ is always a Cu-semigroup (see Theorem 5.9). In Theorem 6.4, we will see that S and $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ are in fact Cu-isomorphic.

We also introduce notions for Cu-semigroups that have a topological equivalent when the semigroup is Lsc-like. More explicitly, given an Lsc-like Cu-semigroup S, we characterize when X_S is second countable, normal, and metric in terms of algebraic properties of S (see Proposition 5.6).

5.1 Let S be an Lsc-like Cu-semigroup with least order unit e. The *topological space* X_S of S is defined as

$$X_S := \{x \in S \mid x < e \text{ maximal}\},$$

with closed subsets

$$C_y := \{x \in X_S \mid x \geq y\}, \quad y \leq e.$$

We check that this is indeed a topology for X_S.

Lemma 5.2 *Let S be an* Lsc-*like* Cu-*semigroup. Then,* $\{X_S \backslash C_y \mid y \leq e\}$ *is a* T_1-*topology for* X_S.

Proof First, note that $C_0 = X_S$ and that $C_e = \varnothing$. Moreover, $C_x = \{x\}$ for every $x \in X_S$. Thus, our topology is T_1.

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To see that arbitrary intersections of C_y's are of the form C_z for some $z \leq e$, simply note that

$$\bigcap_i C_{y_i} = C_{\vee_i y_i}.$$

Furthermore, one also has that

$$\bigcup_{i=1}^n C_{y_i} = C_{\wedge_{i=1}^n y_i}.$$

Indeed, given $x \in X_S$ with $x \geq \wedge_{i=1}^n y_i$, we have

$$x = x \vee x \geq x \vee (\wedge_{i=1}^n y_i) = (x \vee y_1) \wedge \cdots \wedge (x \vee y_n).$$

Since x is maximal, for each i, we either have $x \vee y_i = x$ or $x \vee y_i = e$.

However, note that the previous inequality implies that we cannot have $x \vee y_i = e$ for every i, so $x = x \vee y_j \geq y_j$ for some j.

The other inclusion is clear. ∎

Retaining the above notation, for every $y \leq e$, we will denote by U_y its *associated open subset*. That is, $U_y = X \backslash C_y$.

We list some properties of these sets. Recall that, for every pair of elements $y \leq z \leq e$, the element $y \backslash z$ denotes the almost complement of z by y, as constructed in Lemma 4.14.

Proposition 5.3 *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e. Then:*

(i) *For every $y, z \leq e$, $C_y \subseteq C_z$ if and only if $y \geq z$.*
(ii) *For every $y \leq e$, $U_y = \{x \in X_S \mid y \vee x = e\}$.*
(iii) *Given $y, z \leq e$ such that $U_y \subseteq C_z$, we have $y \wedge z = 0$.*
(iv) *The closure of U_y, denoted by $\overline{U_y}$, is $C_{y\backslash e}$ for every $y \leq e$.*
(v) *Given $y \leq e$, we have $\mathrm{Int}(C_y) = X_S \backslash \overline{(X_S \backslash C_y)} = U_{y\backslash e}$, where $\mathrm{Int}(C_y)$ stands for the interior of C_y.*
(vi) *For every $y, z \leq e$, $C_y \subseteq U_z$ if and only if $y \vee z = e$.*

Proof To see (i), recall that, by definition, $C_y \subseteq C_z$ if and only if $x \geq z$ for every $x \geq y$ with $x < e$ maximal. Using Lemma 4.11, we see that this is equivalent to $y \geq z$.

For (ii), let $y \leq e$ and take $x < e$ be maximal. Thus, we have $x \leq y \vee x \leq e$. Since x is maximal, we either have $x = y \vee x \geq y$ (i.e., $x \in C_y$) or $y \vee x = e$. Thus, $U_y = \{x \in X_S : y \vee x = e\}$.

To prove (iii), let us assume, for the sake of contradiction, that $y \wedge z \neq 0$. Then, $U_{y\wedge z}$ is nonempty and we can consider a maximal element $x \in U_{y\wedge z}$.

By (ii), we have $x \vee (y \wedge z) = e$. Thus, one gets $x \vee y = e$ and, consequently, $x \geq z$ from our assumption that $U_y \subseteq C_z$. However, we also have

$$e = x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) = e \wedge x = x,$$

which is a contradiction, as required.

Let us now prove (iv) and, consequently, (v). First, note that, if $y \vee x = e$, we have

$$y + x = (y \vee x) + (y \wedge x) \geq e \geq y\backslash e + y.$$

Canceling y (see Lemma 4.15), we have that $x \geq y\backslash e$. This shows $U_y \subseteq C_{y\backslash e}$.

Conversely, let z be such that $U_y \subseteq C_z$. By (iii), we know that this implies $y \wedge z = 0$ and, consequently, $y + z \leq e$. Thus, $z \leq y\backslash e$ or, equivalently, $C_{y\backslash e} \subseteq C_z$.

Finally, for (vi), assume first that $C_y \subseteq U_z$. Furthermore, assume for the sake of contradiction that $y \vee z \neq e$. Then, there exists $x \in X_S$ with $x \geq y \vee z$. This implies $x \geq y$ and, consequently, $x \vee z = e$ from $C_y \subseteq U_z$. However, we have

$$x = x \vee y \vee z \geq x \vee z = e,$$

which is a contradiction.

Conversely, if $y \vee z = e$, take $x \geq y$, which implies that $x \vee z \geq y \vee z = e$. In particular, $x \in U_z$. ∎

Example 5.4 Let X be a T_1 topological space. Recall from Example 4.5 that $S = \mathrm{Lsc}(X, \overline{\mathbb{N}})$ is an Lsc-like Cu-semigroup with least order unit 1. Then, the topological space of S is homeomorphic to X.

Indeed, note that the maximal elements below 1 are the characteristic functions $\chi_{X\backslash\{x\}}$. Thus, we have

$$X_S = \{\chi_{X\backslash\{x\}} \mid x \in X\}$$

and

$$\begin{aligned} U_{\chi_{\mathcal{U}}} &= X_S\backslash C_{\chi_{\mathcal{U}}} = X_S\backslash\{\chi_{X\backslash\{x\}} \mid \chi_{X\backslash\{x\}} \geq \chi_{\mathcal{U}}\} \\ &= X_S\backslash\{\chi_{X\backslash\{x\}} \mid x \in X\backslash\mathcal{U}\} = \{\chi_{X\backslash\{x\}} \mid x \in \mathcal{U}\} \end{aligned}$$

for every open subset $\mathcal{U} \subseteq X$.

It should now be clear that $\varphi\colon X_S \to X$ defined as $\chi_{X\backslash\{x\}} \mapsto x$ is a homeomorphism between X and X_S.

The following characterizes compact containment under certain conditions.

Lemma 5.5 *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e. Assume that e is compact and that S satisfies (O5). Then X_S is normal.*

Furthermore, given $y, z \leq e$, we have $\overline{U_y} \subseteq U_z$ if and only if $y \ll z$.

Proof Let $x, y < e$ be two elements such that $C_x \cap C_y = C_e = \varnothing$. In terms of the elements in S, this is equivalent to $x + y \geq e \gg e$. Then, we can take $x' \ll x$ and $y' \ll y$ such that $x' + y' \gg e$.

Using (O5), there exist $c, d \leq e$ such that

$$x' + c \leq e \leq x + c, \quad y' + d \leq e \leq y + d.$$

Consequently, we also have $x' + c + y' + d \leq e + e$ with $x' + y' \gg e$. Since every Lsc-like Cu-semigroup has weak cancelation, it follows that $c + d \leq e$.

Since our order is topological, we get $c \wedge d = 0$, $x \vee c = e$, and $y \vee d = e$.

Using the properties listed in Proposition 5.3, the previous inequalities imply that $C_x \subseteq U_c$, $C_y \subseteq U_d$, and $U_c \cap U_d = U_{c\wedge d} = \varnothing$. Thus, X_S is normal.

Now, let $y, z \leq e$ and assume that $\overline{U_y} \subseteq U_z$, which by (iv) in Proposition 5.3 happens if and only if $(y\backslash e) \vee z = e$. Furthermore, since $y\backslash e + z \geq e$, we have $y\backslash e + z \geq e \gg e \geq y\backslash e + y$. As elements below e have cancelation (see Lemma 4.15), one gets $z \gg y$.

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Conversely, assume that $e \geq z \gg y$. Then, by (O5), there exists an element x such that $y + x \leq e$ and $e \leq z + x$.

From the first inequality, it follows that $x \leq y \backslash e$, so $e \leq z + x \leq z + y \backslash e$ as required. ■

In Proposition 5.6, we study notions for Cu-semigroups that have a topological equivalent when the semigroup is Lsc-like. Recall from Paragraph 2.3 that a Cu-semigroup is said to be *countably based* if it has a countable sup-dense subset.

Definition 5.1 We say that an inf-semilattice ordered Cu-semigroup S is *normal* if there exists an order unit $z \in S$ such that, whenever $x + y \geq z$ for some $x, y \in S$, there exist $s, t \in S$ with

$$x + s \geq z, \quad y + t \geq z, \quad s \wedge t = 0.$$

Proposition 5.6 *Let S be an* Lsc-*like* Cu-*semigroup, and let X_S be its associated topological space. Then:*

(i) *X_S is second countable if and only if S is countably based.*
(ii) *X_S is countably compact if and only if S has a compact order unit.*
(iii) *X_S is normal if and only if S is normal.*
(iv) *X_S is a metric space whenever S is countably based and normal.*
(v) *X_S is a compact metric space whenever S is countably based, has a compact order unit, and satisfies (O5).*

Proof Let $e \in S$ be the least order unit of S. To show (i), assume first that S is countably based with a countable basis B, and let $\sum'(\downarrow e)$ denote the set of finite sums of elements in $\downarrow e$. Naturally, the set $B' := B \cap \sum'(\downarrow e)$ is also a countable basis for S.

Given an open set U_y with $y \leq e$, write $y = \sup_n y_n$ with $y_n \in B'$. We have $\cup_n U_{y_n} = U_y$, and so X_S is second countable.

Conversely, assume that X_S is second countable with basis $C = \{U_{z_n}\}_n$. Note that the family C' consisting of all the finite unions of sets in C is also countable.

Then, any open subset U_y can be written as the countable union of increasing open subsets $U_{z'_n}$ of C'. We know that this is equivalent to $y = \sup_n z'_n$. This implies that the set $\sum'\{z'_n\}_n$ of finite sums from $\{z'_n\}_n$ is a countable basis for $\sum'(\downarrow e)$ and, since $\sum'(\downarrow e)$ is dense in S, $\sum'\{z'_n\}_n$ is a countable basis for S.

To prove (ii), note that it is easy to check that $e \in S$ is compact if and only if X_S is countably compact. Therefore, we are left to prove that, if there exists a compact order unit in S, then e must also be compact.

To see this, let p be a compact order unit in S, which implies that $p \leq ne$ for some $n \in \mathbb{N}$. Since we know that p can be written as a finite ordered sum of elements below e, there exist $m \in \mathbb{N}$ and elements $q_m \leq \cdots \leq q_1 \leq e$ such that $p = q_1 + \cdots + q_m$.

By weak cancelation applied to $q_1 + \cdots + q_m \ll q_1 + \cdots + q_m$, the element q_1 is compact and satisfies $\infty q_1 = \infty p = \infty$. Thus, q_1 is a compact order unit with $e \geq q_1$. By minimality of e, one gets $e = q_1$ compact as required.

Let us now show (iii). First, assume that S is normal, and let z be the associated order unit. Let C_x, C_y be closed subsets of X_S with $x, y \leq e$, and recall that C_x, C_y are disjoint if and only if $x \vee y = e$.

Since e is an order unit, we have $\infty x + \infty y = \infty \geq z$. Thus, we get s, t such that $\infty x + s \geq z$, $\infty y + t \geq z$, and $s \wedge t = 0$. As $z \geq e$, we know by Corollary 4.9 that $\infty x \vee s \geq e$ and $\infty y \vee t \geq e$.

Since $x, y \leq e$, taking the infimum with e and using Lemma 4.10, we have $e = (\infty x \wedge e) \vee (s \wedge e) = x \vee (s \wedge e)$, $y \vee (t \wedge e) = e$, and $(s \wedge e) \wedge (t \wedge e) = 0$. By (vi) in Proposition 5.3, it follows that $C_x \subseteq U_{s\wedge e}$, $C_y \subseteq U_{t\wedge e}$, and $U_{s\wedge e} \cap U_{t\wedge e} = \varnothing$. This implies that X_S is normal.

Conversely, if X_S is normal, it is easy to see that S is normal by setting $z = e$ in the definition of normality.

To prove (iv), we have that X_S is second countable and normal by (i) and (iii), and that $C_x = \{x\}$ for any $x \in X_S$. Thus, points are closed in our topology, so X_S is Hausdorff. We can now use Urysohn's metrization theorem to conclude that X_S is metric (see, e.g., [16, Theorem 34.1]).

For (v), note that $e \ll e$. Thus, Lemma 5.5 implies that X_S is normal. Following the arguments above, one gets that X_S is metric. Moreover, we also know that X_S is second countable and countably compact by (i) and (ii) above. Thus, X_S is compact. ∎

We will now show that $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ is a Cu-semigroup with the usual way-below relation for every Lsc-like Cu-semigroup S. Note that (O1) and (O4) are always satisfied, so we are left to prove (O2) and (O3).

Lemma 5.7 *Let S be an* Lsc-*like* Cu-*semigroup with least order unit e. Given $y, z \leq e$, we have $\chi_{U_y} \ll \chi_{U_z}$ in* $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ *if and only if $y \ll z$ in S.*

Proof Assume $y \ll z$, and let $(f_n)_n$ be an increasing sequence in $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ such that $\chi_{U_z} \leq \sup_n f_n$. In particular, note that this holds if and only if $\chi_{U_z} \leq \chi_{\cup_n \mathrm{supp}(f_n)}$ or, equivalently, if

$$\bigcap_n (X_S \backslash \mathrm{supp}(f_n)) \subseteq C_z.$$

Denote by z_n the elements in $\downarrow e$ with $C_{z_n} = X_S \backslash \mathrm{supp}(f_n)$. Given that $\mathrm{supp}(f_n) \subseteq \mathrm{supp}(f_{n+1})$, (i) in Proposition 5.3 implies that z_n is increasing.

Using the proof of Lemma 5.2 in the first step, we can rewrite the previous inclusion as

$$C_{\sup_n (z_n)} = \bigcap_n C_{z_n} \subseteq C_z.$$

Applying (i) in Proposition 5.3 once again, one gets $z \leq \sup_n (z_n)$ and, consequently, $y \leq z_n$ for some n. This implies that $U_y \subseteq U_{z_n}$ or, equivalently, that $\chi_{U_y} \leq \chi_{U_{z_n}} = \chi_{\mathrm{supp}(f_n)} \leq f_n$. This shows $\chi_{U_y} \ll \chi_{U_z}$.

Now, let $y, z \leq e$ be such that $\chi_{U_y} \ll \chi_{U_z}$, and consider an increasing sequence $(h_n)_n$ in S such that $z \leq \sup_n (h_n)$. Note that, by taking $z \wedge h_n$ instead of h_n, we can assume $h_n \leq e$ for every n.

Applying again (the proof of) Lemma 5.2, one gets

$$\bigcap_n C_{h_n} = C_{\sup_n (h_n)} \subseteq C_z,$$

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and, consequently, we have $\sup_n \chi_{U_{h_n}} \geq \chi_{U_z}$ since $\chi_{U_y} \ll \chi_{U_z}$, there exists some n with $\chi_{U_y} \leq \chi_{U_{h_n}}$; i.e., $C_{h_n} \subseteq C_y$.

Using (i) in Proposition 5.3 one last time, one sees that $y \leq h_n$ as required. ∎

Proposition 5.8 *Let S be a* Cu-*semigroup. If S is* Lsc-*like, then* $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ *satisfies (O2).*

Proof Take $f \in \mathrm{Lsc}(X_S, \overline{\mathbb{N}})$, and let $(y_i)_i$ be the sequence on $\downarrow e$ such that

$$\{f \geq i\} = U_{y_i},$$

where recall that the sequence is decreasing as a consequence of (i) in Proposition 5.3.

Since S satisfies (O2), we have $y_i = \sup_n y_{i,n}$ with $y_{i,n} \ll y_{i,n+1}$ for every n.

For every fixed k, we have

$$y_1 \geq \cdots \geq y_k \gg y_{k,n} \text{ for all } n.$$

Thus, for every i, one can choose inductively $n_{i,k}$ with $k \geq i$ such that

$$y_{i,n_{i,k}} \ll y_{i,n_{i,k+1}}, \quad \text{and} \quad y_{1,n_{1,k}} \geq \cdots \geq y_{k,n_{k,k}}.$$

Indeed, we begin by setting $n_{1,1} = 1$ (i.e., $y_{1,n_{1,1}} = y_{1,1}$). Then, assuming that we have defined $n_{i,k}$ for every $i, k \leq m-1$ (and $k \geq i$) for some fixed m, we set $n_{m,m} = 1$, that is, $y_{m,n_{m,m}} = y_{m,1}$. We then set $n_{m-1,m}$ large enough so that $y_{m-1,n_{m-1,m}} \geq y_{m,n_{m,m}}$ and $n_{m-1,m} \geq n_{m-1,m-1}$. Similarly, we set $n_{m-2,m} \geq n_{m-2,m-1}$ such that $y_{m-2,n_{m-2,m}} \geq y_{m-1,n_{m-1,m}}$ and define $n_{i,m}$ for every $i \leq m-2$ in the same fashion.

Now, consider the sums $f_k = \sum_{i=1}^{k} \chi_{U_{y_{i,n_{i,k}}}}$, which are ordered by construction. Thus, one has $U_{y_{i,n_{i,k}}} = \{f_k \geq i\}$ for every i. Since $y_{i,n_{i,k}} \ll y_{i,n_{i,k+1}}$ for every i, it follows from Lemmas 4.18 and 5.7 that

$$f_k = \sum_{i=1}^{k} \chi_{U_{y_{i,n_{i,k}}}} \ll \sum_{i=1}^{k} \chi_{U_{y_{i,n_{i,k+1}}}} \leq f_{k+1}.$$

It is now easy to check that $\sup_k f_k = f$. ∎

Theorem 5.9 *Let S be a* Cu-*semigroup. If S is* Lsc-*like, the monoid* $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ *is a* Cu-*semigroup.*

Proof Note that the semigroup $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ always satisfies (O1) and (O4). Moreover, we already know that (O2) is also satisfied by Proposition 5.8. Thus, we are left to prove (O3).

Let $f \ll f'$ and $g \ll g'$. By Lemma 4.18, this implies that

$$\chi_{\{f \geq i\}} \ll \chi_{\{f' \geq i\}} \quad \text{and} \quad \chi_{\{g \geq i\}} \ll \chi_{\{g' \geq i\}}$$

for every i. We also know that there exists $m < \infty$ such that $\sup(f), \sup(g) \leq m$.

Let y_i, y_i', z_i, and z_i' be elements in $\downarrow e$ such that

$$U_{y_i} = \{f \geq i\}, \quad U_{y_i'} = \{f' \geq i\}, \quad U_{z_i} = \{g \geq i\}, \quad \text{and} \quad U_{z_i'} = \{g' \geq i\}.$$

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By Lemma 5.7, we have $y_i \ll y'_i$ and $z_i \ll z'_i$ for every i and, since S satisfies (O3), one gets

$$\sum_{i=1}^{m}(y_i + z_i) \ll \sum_{i=1}^{m}(y'_i + z'_i).$$

By Lemma 4.2, these sums can be rewritten as

$$\sum_{i=1}^{2m} \vee_{j=0}^{m}(y_j \wedge z_{i-j}) \ll \sum_{i=1}^{2m} \vee_{j=0}^{m}(y'_j \wedge z'_{i-j})$$

and, since both the right- and left-hand sides of the previous inequality are ordered, we can use Lemma 4.17 to obtain

$$\vee_{j=0}^{m}(y_j \wedge z_{i-j}) \ll \vee_{j=0}^{m}(y'_j \wedge z'_{i-j})$$

for every i.

Note that

$$\{f + g \geq i\} = \bigcup_{j=0}^{m}(\{f \geq j\} \cap \{g \geq i - j\}),$$

so, by the equalities in the proof of Lemma 5.2, one gets

$$X_S \backslash \{f + g \geq i\} = \bigcap_{j=0}^{m}((X_S \backslash \{f \geq j\}) \cup (X_S \backslash \{g \geq i - j\})) = \bigcap_{j=0}^{m}(C_{y_j \wedge z_{i-j}})$$
$$= C_{\vee_{j=0}^{m}(y_j \wedge z_{i-j})}$$

and, consequently, $\chi_{\{f+g \geq i\}} = \chi_{U_{\vee_{j=0}^{m}(y_j \wedge z_{i-j})}}$.

The same argument also shows that $\chi_{\{f'+g' \geq i\}} \geq \chi_{U_{\vee_{j=0}^{m}(y'_j \wedge z'_{i-j})}}$.

By using Lemma 5.7, we get $\chi_{\{f+g \geq i\}} \ll \chi_{\{f'+g' \geq i\}}$ for every i. Lemma 4.18 then implies that $f + g \ll f' + g'$, as desired. ∎

6 An abstract characterization of Lsc(X,N)

In this section, we prove that every Lsc-like Cu-semigroup S is Cu-isomorphic to the semigroup of lower-semicontinuous functions $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ (see Theorem 6.4). To do so, we first define a map $\varphi' \colon \mathrm{Lsc}(X_S, \overline{\mathbb{N}})_{fs} \to S$, where $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})_{fs}$ is the subsemigroup of functions with finite supremum.

We then extend this map to a Cu-morphism $\varphi \colon \mathrm{Lsc}(X_S, \overline{\mathbb{N}}) \to S$. Finally, the following (probably well-known) lemma will be used to complete the proof.

Lemma 6.1 *Let S, H be* Cu*-semigroups, and let $\varphi \colon S \to H$ be a* Cu*-morphism such that:*

(i) *φ is an order embedding on a basis of S.*
(ii) *$\varphi(S)$ is a basis for H.*

Then, φ is a Cu*-isomorphism*

Proof It is easy to see that φ is a global order embedding.

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To prove surjectivity, let $h \in H$. Since $\varphi(S)$ is a basis for H, we can write $h = \sup_n \varphi(s_n)$ for some $s_n \in S$. Furthermore, as we know that φ is an order embedding, the sequence $(s_n)_n$ is increasing in S, so $\sup_n \varphi(s_n) = \varphi(s)$ for $s = \sup_n s_n$. ∎

Definition 6.1 Let S be an Lsc-like Cu-semigroup. Given $f \in \mathrm{Lsc}(X_S, \overline{\mathbb{N}})_{fs}$, there exists $m < \infty$ such that we can write

$$f = \sum_{i=1}^{m} \chi_{\{f \geq i\}}.$$

We define the map $\varphi' \colon \mathrm{Lsc}(X_S, \overline{\mathbb{N}})_{fs} \to S$ as $\varphi'(f) = \sum_{i=1}^{m} z_i$, where $\{f \geq i\} = U_{z_i}$.

Lemma 6.2 *φ' is a positively ordered monoid morphism and an order embedding.*

Proof Let $f = \sum_{i=1}^{m} \chi_{U_{z_i}}$ as above and take $g = \sum_{j=1}^{n} \chi_{U_{y_j}}$. We will first prove by induction on m that $\varphi'(f+g) = \varphi'(f) + \varphi'(g)$.

For $m = 1$, f is simply χ_{U_z} for some open subset U_z. Since $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ is distributively lattice ordered and $0 \leq 0 \leq \cdots \leq 0 \leq f$ is an increasing sequence, we can apply Lemma 4.2 to get

$$f + \sum_{j=1}^{n} \chi_{U_{y_j}} = \chi_{U_z \cup U_{y_1}} + \chi_{(U_z \cap U_{y_1}) \cup U_{y_2}} + \cdots + \chi_{U_z \cap (\cap_j U_{y_j})}.$$

Applying φ' and the equalities in the proof of Lemma 5.2 at the first step, and Lemma 4.2 at the second step, one gets

$$\begin{aligned}\varphi'(f+g) &= (z \vee y_1) + ((z \wedge y_1) \vee y_2) + \cdots + (z \wedge y_1 \wedge \cdots \wedge y_n) \\ &= z + (y_1 + \cdots + y_n) = \varphi'(f) + \varphi'(g).\end{aligned}$$

Now, fix any finite m and assume that the result has been proved for any $k \leq m-1$. Then, using the induction hypothesis at the second step, and the case $m = 1$ at the third and fourth steps, we have

$$\begin{aligned}\varphi'(f+g) &= \varphi'((f - \chi_{U_{z_m}}) + (g + \chi_{U_{z_m}})) = \varphi'(f - \chi_{U_{z_m}}) + \varphi'(g + \chi_{U_{z_m}}) \\ &= \varphi'(f - \chi_{U_{z_m}}) + \varphi'(g) + \varphi'(\chi_{U_{z_m}}) = \varphi'(f - \chi_{U_{z_m}} + \chi_{U_{z_m}}) + \varphi'(g) \\ &= \varphi'(f) + \varphi'(g),\end{aligned}$$

as desired.

Note that this could have also been proved using Lemma 4.2.

To see that φ' is an order embedding, let $f \leq g$ in $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})_{fs}$ and note that $f \leq g$ if and only if $\{f \geq i\} \subseteq \{g \geq i\}$ for every i. Let $z_i, y_i \in S$ be such that $U_{z_i} = \{f \geq i\}$ and $U_{y_i} = \{g \geq i\}$

By (i) in Proposition 5.3, $U_{z_i} \subseteq U_{y_i}$ if and only if $z_i \leq y_i$ for every i. Furthermore, note that the sequences $(z_i)_{i=1}^m, (y_i)_{i=1}^m$ are both decreasing. Since we have a topological order, $z_i \leq y_i$ for every i if and only if

$$\sum_{i=1}^{m} z_i \leq \sum_{i=1}^{m} y_i,$$

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where note that the right- and left-hand sides correspond to $\varphi'(f)$ and $\varphi'(g)$, respectively. ∎

Using φ', one can now construct a Cu-isomorphism.

Theorem 6.3 *Let S be an* Lsc*-like* Cu*-semigroup. Then, the* Cu*-morphism φ' extends to a* Cu*-isomorphism.*

Proof We will need the following claims.

Claim 1 Let $(f_n)_n$ and $(g_n)_n$ be two $\ll$-increasing sequences with the same supremum in $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})$. Then, $\sup_n \varphi'(f_n) = \sup_n \varphi'(g_n)$.

Since $(f_n)_n$ and $(g_n)_n$ have the same supremum, we know that for every n there exist k, m such that $f_n \leq g_m$ and $g_n \leq f_k$.

Applying φ', we get

$$\varphi'(f_n) \leq \varphi'(g_m) \leq \sup_m \varphi'(g_m), \quad \text{and} \quad \varphi'(g_n) \leq \varphi'(f_k) \leq \sup_k \varphi'(f_k),$$

and so $\sup_k \varphi'(f_k) = \sup_m \varphi'(g_m)$ as desired.

By Claim 1, we can define the map $\varphi\colon \mathrm{Lsc}(X_S,\overline{\mathbb{N}}) \to S$ as $\varphi(f) = \sup_n \varphi'(f_n)$, where $(f_n)_n$ is a $\ll$-increasing sequence with supremum f. We will see that φ is a Cu-morphism that extends φ' and that it satisfies the conditions in Lemma 6.1 (i.e., φ is a Cu-isomorphism).

Claim 2 Let $(f_n)_n$ be an increasing sequence in $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})_{fs}$ with supremum $f = \sup f_n \in \mathrm{Lsc}(X_S,\overline{\mathbb{N}})_{fs}$. Then, we have $\varphi'(f) = \sup_n \varphi'(f_n)$.

To prove the claim, let $f_n = \chi_{U_{z_n}}$ for every n, and recall that $\sup_n \chi_{U_{z_n}} = \chi_{U_{\sup_n(z_n)}}$. This is equivalent to

$$\varphi'(\sup_n f_n) = \sup_n(z_n) = \sup_n \varphi'(f_n).$$

Now, given any increasing sequence as in the statement of the lemma with supremum f, we know that $\sup(f) < \infty$, say $\sup(f) = m \in \mathbb{N}$.

Thus, given $U_{i,n} = \{f_n \geq i\}$ for $1 \leq i \leq m$, we can write

$$f_n = \sum_{i=1}^{m} \chi_{U_{i,n}}$$

with some possibly empty $U_{i,n}$'s.

We have

$$f = \sup_n(f_n) = \sum_{i=1}^{m} \chi_{\cup_n U_{i,n}},$$

where $\cup_n U_{i,n} = \{f \geq i\}$.

Using that φ' preserves suprema of indicator functions, we have

$$\varphi'(f) = \sum_{i=1}^{m} \varphi'(\chi_{\cup_n U_{i,n}}) = \sup_n \varphi'(\sum_{i=1}^{m} \chi_{U_{i,n}}) = \sup_n \varphi'(f_n),$$

as required.

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Since φ' preserves addition and S is a Cu-semigroup, it is clear that φ also preserves addition. Note that the proof of Claim 2 also shows that φ extends φ' and that φ is order-preserving.

To see that φ preserves suprema, let $\varphi(f) = \sup_n \varphi'(f_n)$ with $(f_n)_n$ $\ll$-increasing with supremum f and consider an increasing sequence $(g_n)_n$ whose supremum is also f.

Then, for every n, there exists an m with $f_n \leq g_m$ and, consequently, $\varphi'(f_n) = \varphi(f_n) \leq \varphi(g_m)$. It follows that $\varphi(f) = \sup_n \varphi'(f_n) \leq \sup_m \varphi(g_m)$.

On the other hand, $f \geq g_m$ for every m, so $\varphi(f) = \sup_m \varphi(g_m)$.

Now, let $f, g \in \mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ be such that $f \ll g$. Then, we know by Lemma 4.18 that this happens if and only if

$$\chi_{\{f\geq i\}} \ll \chi_{\{g\geq i\}} \text{ for every } i \quad \text{and} \quad \sup(f) = m < \infty.$$

Letting $y_i, z_i \in S$ such that $U_{y_i} = \{f \geq i\}$ and $U_{z_i} = \{g \geq i\}$, we know that

$$\varphi(f) = \varphi'(f) = y_1 + \cdots + y_m \ll z_1 + \cdots + z_m \leq \sup_n \sum_{i=1}^{n} z_i = \varphi(g).$$

Finally, note that the image of φ is clearly dense in S, since φ' is surjective on $\downarrow e$. Furthermore, φ is an order embedding in $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})_{fs}$, since φ coincides with φ' in this basis of $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$.

Thus, since the conditions in Lemma 6.1 are satisfied, it follows that φ is a Cu-isomorphism. ∎

Theorem 6.4 *Let S be a* Cu*-semigroup. Then, S is* Lsc*-like if and only if S is* Cu*-isomorphic to* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ *for a T_1 topological space X.*

Proof We already know that $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is Lsc-like whenever it is a Cu-semigroup (see Example 4.5), and the converse follows from Theorem 6.3. ∎

In [23], a notion of covering dimension for Cu-semigroups is introduced. This dimension satisfies many of the expected permanence properties [23, Proposition 3.10], and is related to other dimensions, such as the nuclear dimension of C*-algebras [23, Theorem 4.1] and the Lebesgue covering dimension (see [23, Proposition 4.3 and Corollary 4.4]).

Using such a notion, one can prove the following.

Theorem 6.5 *Let S be a* Cu*-semigroup satisfying (O5), and let $n \in \mathbb{N} \cup \{\infty\}$. Then, S is* Cu*-isomorphic to* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ *with X a compact metric space such that* $\dim(X) = n$ *if and only if S is* Lsc*-like, countably based, has a compact order unit, and* $\dim(S) = n$.

In particular, a Cu*-semigroup S is* Cu*-isomorphic to the Cuntz semigroup of* $C(X)$ *with X compact metric and* $\dim(X) \leq 1$ *if and only if S is* Lsc*-like, countably based, satisfies (O5), has a compact order unit, and* $\dim(S) \leq 1$.

Proof The forward implication follows from Examples 4.5 and 5.4, Proposition 5.6, and [23, Corollary 4.4].

To prove the converse, use Theorem 6.4 and (v) in Proposition 5.6 to deduce that $S \cong \mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ with X_S compact metric. Then, it follows from [23, Corollary 4.4] that $\dim(X_S) = \dim(S) = n$, as required.

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Now, assume that S is Cu-isomorphic to the Cuntz semigroup of $C(X)$ with X compact metric and $\dim(X) \leq 1$. By [20, Theorem 1.1] we know that $\mathrm{Cu}(C(X)) \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$. In particular, S satisfies (O5) (e.g., [22]).

Thus, it now follows from our previous argument that S is Lsc-like, countably based, satisfies (O5), has a compact order unit, and $\dim(S) \leq 1$.

Conversely, if S satisfies the list of properties in the second part of the statement, note that $S \cong \mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ with $\dim(X_S) \leq 1$ again by Theorem 6.4. Using [20, Theorem 1.1] a second time, it follows that $\mathrm{Lsc}(X_S, \overline{\mathbb{N}}) \cong \mathrm{Cu}(C(X_S))$, as desired. ∎

7 Chain conditions and the Cuntz semigroup of commutative AI-algebras

In this section, we introduce the notions of piecewise chainable and weakly chainable Cu-semigroups and prove that, together with some additional properties, these notions give a characterization of when S is Cu-isomorphic to the Cuntz semigroup of a unital commutative block stable AI-algebra and a unital commutative AI-algebra, respectively (see Theorems 7.4 and 7.11).

We also show that the Cuntz semigroup of any AI-algebra is weakly chainable, thus uncovering a new property that the Cuntz semigroup of any AI-algebra satisfies (see Corollary 7.9).

We first prove the following categorical proposition, which summarizes the results of the above sections. We denote by Top the category of topological spaces, and by $\mathcal{T}_1^{\mathrm{Cu}}$ the subcategory of Top whose objects X are the T_1 spaces such that $\mathrm{Lsc}(X, \overline{\mathbb{N}}) \in \mathrm{Cu}$. Note that, by Corollary 4.19, this includes all compact, metric spaces.

Proposition 7.1 *Let* Lsc *be the subcategory of* Cu *consisting of* Lsc*-like* Cu*-semigroups.*
There exists a faithful and essentially surjective contravariant functor $T\colon \mathcal{T}_1^{\mathrm{Cu}} \to \mathrm{Lsc}$ *that is full on isomorphisms.*

Proof For every topological space $X \in \mathcal{T}_1^{\mathrm{Cu}}$, define $T(X) = \mathrm{Lsc}(X, \overline{\mathbb{N}})$.

Furthermore, given any continuous map $f\colon X \to Y$, set $T(f)\colon \mathrm{Lsc}(Y, \overline{\mathbb{N}}) \to \mathrm{Lsc}(X, \overline{\mathbb{N}})$ as the unique Cu-morphism such that $T(f)(\chi_U) = \chi_{f^{-1}(U)}$ for every open subset U of Y.

Note that, given $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{T}_1^{\mathrm{Cu}}$, we have

$$T(g \circ f)(\chi_U) = \chi_{(g\circ f)^{-1}(U)} = \chi_{f^{-1}g^{-1}(U)} = (T(f) \circ T(g))(\chi_U).$$

Thus, T is a contravariant functor, which is clearly faithful by construction.

Moreover, we know by Theorem 6.4 that for every Lsc-like Cu-semigroup S there exists a T_1-space X_S with $S \cong \mathrm{Lsc}(X_S, \overline{\mathbb{N}})$. Therefore, T is essentially surjective.

Now, let $\varphi\colon S \to T$ be a Cu-isomorphism of Lsc-like Cu-semigroups. Using Theorem 6.4, we get a Cu-isomorphism of the form $\phi\colon \mathrm{Lsc}(X_S, \overline{\mathbb{N}}) \to \mathrm{Lsc}(X_T, \overline{\mathbb{N}})$.

Since $\phi(1) = 1$, indicator functions must map to indicator functions. Since ϕ is a Cu-isomorphism, maximal elements below 1 must map to maximal elements below 1. More explicitly, for every $x \in X_S$, there exists $y \in X_T$ such that $\phi(\chi_{X_S\setminus\{x\}}) = \chi_{X_T\setminus\{y\}}$.

We define the map $f\colon X_T \to X_S$ as $y \mapsto x$, which is bijective because ϕ is a Cu-isomorphism.

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To see that it is continuous, let U be an open subset of X_S and let $V \subseteq X_T$ be such that $\phi(\chi_U) = \chi_V$. Then, given $y \in X_T$, we have that $y \in V$ if and only if

$$1 \leq \chi_{X_T \setminus \{y\}} + \chi_V = \phi(\chi_{X_S \setminus \{f(y)\}} + \chi_U).$$

Since ϕ is a Cu-isomorphism, this in turn holds if and only if $(X_S \setminus \{f(y)\}) \cup U = X_S$ or, equivalently, if $f(y) \in U$.

This shows that $f^{-1}(U) = V$ and, consequently, that f is continuous.

Finally, let $V \subseteq X_T$ be open. Since ϕ is an isomorphism, there exists some open subset $U \subseteq X_S$ such that $\chi_V = \phi(\chi_U)$.

By the argument above, one has $V = f^{-1}(U)$ and, since f is bijective, it follows that $f(V) = f(f^{-1}(U)) = U$. This shows that f is open.

Thus, f is a homeomorphism between X_S and X_T, as required. ■

We now introduce chainable and piecewise chainable inf-semilattice ordered Cu-semigroups.

Definition 7.1 Let S be an inf-semilattice ordered Cu-semigroup. An element $x \in S$ is said to be *chainable* if for every sum $y_1 + \cdots + y_n \geq x$, there exist elements $z_1, \cdots, z_m$ such that:

(i) For every i, there exists some k with $z_i \leq y_k$.
(ii) $z_i \wedge z_j \neq 0$ if and only if $|i - j| \leq 1$.
(iii) $z_1 + \cdots + z_m \geq x$.

S will be called *chainable* if it has a chainable order unit.

Moreover, we will say that S is *piecewise chainable* if there exist chainable elements $s_1, \ldots, s_n$ such that $s_1 + \cdots + s_n$ is an order unit and $s_i \wedge s_j = 0$ whenever $i \neq j$.

Lemma 7.2 *Given an* Lsc*-like* Cu*-semigroup S with least order unit e and an element $y \leq e$, U_y is topologically chainable if and only if y is chainable.*

In particular, S is chainable if and only if X_S is topologically chainable.

Proof If y is chainable, take a finite cover $U_{y_1} \cup \cdots \cup U_{y_n} = U_y$. We have that $y = y_1 \vee \cdots \vee y_n \leq y_1 + \cdots + y_n$. Thus, applying the chainability of y, one gets elements $z_1, \ldots, z_m$ such that for every i there exists k with $z_i \leq y_k \leq y$. This shows that $z_1 \vee \cdots \vee z_m \leq y$.

By Corollary 4.9 and (iii) in Definition 7.1, we have $z_1 \vee \cdots \vee z_m \geq y$ and, consequently, $z_1 \vee \cdots \vee z_m = y$. This shows that $U_{z_1}, \ldots, U_{z_m}$ is a cover for U_y.

Using the equalities in the proof of Lemma 5.2 and conditions (i)–(iii) in Definition 7.1, one sees that $U_{z_1}, \ldots, U_{z_m}$ is a chain that refines our original cover in the sense of Definition 3.1.

Conversely, if U_y is topologically chainable and we have a sum $y_1 + \cdots + y_n \geq y$, we can apply Corollary 4.9 once again to obtain

$$(y_1 \wedge y) \vee \cdots \vee (y_n \wedge y) = y.$$

This shows that $U_{y_1 \wedge y} \cup \cdots \cup U_{y_n \wedge y} = U_y$, and we can use the chainability of U_y to obtain a chain refining this cover. Using Lemma 5.2, it is easy to check that the elements below e corresponding to the open subsets of the chain satisfy conditions (i)–(iii) in Definition 7.1.

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In particular, the previous argument shows that e is chainable whenever X_S is topologically chainable. By definition, this implies that S is chainable.

Conversely, if S is chainable, we have a chainable order unit s. Let us now show that e is also chainable, which by the above arguments will imply that $U_e = X_S$ is topologically chainable.

Thus, let $y_1 + \cdots + y_n \geq e$, which by Corollary 4.9 implies that $y_1 \vee \cdots \vee y_n \geq e$. Since s is an order unit, one has

$$\infty y_1 \vee \cdots \vee \infty y_n \geq \infty e = \infty = \infty s \geq s.$$

Using that s is chainable, we obtain elements $z_1, \ldots, z_m$ satisfying (i)–(iii) in Definition 7.1. In particular, since for every i there exists k with $z_i \leq \infty y_k$, one can use Lemma 4.10 in the second step to get

$$z_i \wedge e \leq (\infty y_k) \wedge e = y_k \wedge e \leq y_k.$$

Furthermore, since e is the least order unit in S and $z_1 + \cdots + z_m \geq s \geq e$, it follows from Corollary 4.9 that $z_1 \vee \cdots \vee z_m \geq e$. Taking the infimum by e and using Corollary 4.9 once again, we get $z_1 \wedge e + \cdots + z_m \wedge e \geq e$.

This shows that the elements $z_i \wedge e$ satisfy conditions (i)–(iii) in Definition 7.1 for $y_1 \vee \cdots \vee y_n \geq e$, as desired. ∎

Lemma 7.3 *A countably based* Lsc*-like* Cu*-semigroup S with a compact order unit is piecewise chainable if and only if X_S is.*

Proof If X_S is piecewise chainable, there exist chainable components $Y_1, \ldots, Y_n$ such that $X_S = Y_1 \sqcup \cdots \sqcup Y_n$. Since chainability implies connectedness (whenever the space is compact), there is a finite number of connected components, and so these are clopen.

By Lemma 7.2, the disjoint chainable components correspond to disjoint chainable elements, so S is piecewise chainable by definition.

Conversely, if S is piecewise chainable, each element s_i in the definition of chainable corresponds to a chainable open subset of X_S, which is disjoint from the other chainable open subsets by construction. ∎

Theorem 7.4 *Let S be a* Cu*-semigroup. Then, S is* Cu*-isomorphic to the Cuntz semigroup of a unital block stable commutative AI-algebra if and only if S is countably based,* Lsc*-like, piecewise chainable, has a compact order unit, and satisfies (O5).*

Proof Let S be Cu-isomorphic to the Cuntz semigroup of a unital commutative block stable AI-algebra. Then, we know from [20, Theorem 1.1] and Definition 3.3 that $S \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$ with X a compact, metric, piecewise chainable space. In particular, S satisfies (O5), has a compact order unit, is countably based, and is Lsc-like. Using Lemma 7.3, it also follows that S is piecewise chainable.

Conversely, assume that S satisfies all the conditions in the list. By Theorem 6.4 and (v) in Proposition 5.6, we have $S \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$ with X a compact metric space.

Then, it follows from Example 5.4 and Lemma 7.3 that X is piecewise chainable. In particular, it has dimension less than or equal to one by Remark 3.2. Thus, $\mathrm{Cu}(C(X)) \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$ by [20, Theorem 1.1], so S is isomorphic to the Cuntz semigroup of a unital commutative block stable AI-algebra. ∎

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We now define weak chainability for any Cu-semigroup and prove that every Cu-semigroup of an AI-algebra satisfies such a condition. Moreover, we also show that an Lsc-like Cu-semigroup is weakly chainable if and only if its associated space is almost chainable.

Given two elements x, y in a Cu-semigroup, we write $x \propto y$ if there exists $n \in \mathbb{N}$ with $x \leq ny$.

Definition 7.2 We will say that a Cu-semigroup S is *weakly chainable*, or that it satisfies the *weak chainability condition* if, for any $x, y, y_1, \ldots, y_n$ such that

$$x \ll y \ll y_1 + \cdots + y_n,$$

there exist $x', z_1, \ldots, z_m \in S$ such that $x' \leq y$, $x \propto x'$ and:

(i) For any i, there exists j such that $z_i \leq y_j$.
(ii) $z_i + z_j \leq x'$ whenever $|i - j| \geq 2$.
(iii) $z_1 + \cdots + z_m \geq x'$.

Lemma 7.5 *Let X be a compact metric space. Then,* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ *is weakly chainable if and only if X is almost chainable.*

Proof First, recall that $S = \mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a Cu-semigroup whenever X is compact and metric by Corollary 4.19. Furthermore, also recall from Example 4.5 that S is an Lsc-like Cu-semigroup with least order unit 1, and that $X \cong X_S$ by Example 5.4.

Now, assume that S is weakly chainable, and let $U_{y_1}, \ldots, U_{y_n}$ be a cover of X_S. Then, the elements $y_1, \ldots, y_n \leq 1$ satisfy $y_1 \vee \cdots \vee y_n \geq 1$ and, consequently, $y_1 + \cdots + y_n \geq 1$.

Set $x = y = 1$, and apply Definition 7.2 to obtain elements $x', z_1, \ldots, z_m$ satisfying the conditions in the definition. Note that, since x' satisfies $x' \leq 1 \leq kx'$ for some $k \in \mathbb{N}$, it follows from the second inequality that $x' \geq 1$ and, therefore, $x' = 1$. Let $U_{z_1}, \ldots, U_{z_m}$ be the open subsets of X_S corresponding to $z_1, \ldots, z_m$, respectively. Using (i)–(iii) in Definition 7.2, it is easy to see that such sets form an almost chain refining the original cover. This implies that X_S is almost chainable and, since $X \cong X_S$, so is X.

Conversely, assume that X is almost chainable and let $x, y, y_1, \ldots, y_n \in S$ be as in Definition 7.2. Set $x' = x \wedge 1$. Then, we know by Corollary 4.9 that $y_1 \vee \cdots \vee y_n \geq y \gg x'$. Taking the infimum with 1, one has $(y_1 \wedge 1) \vee \cdots \vee (y_n \wedge 1) \gg x'$.

Moreover, note that x' satisfies $x' \leq y$ and $x \propto x'$. Since X is compact, metric, and almost chainable, we know by Remark 3.2 that its dimension is less than 2. This implies by [20, Theorem 1.1] that $\mathrm{Cu}(C(X)) \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$ and, in particular, that S satisfies (O5).

Thus, by the proof of Lemmas 5.2 and 5.5, $(y_1 \wedge 1) \vee \cdots \vee (y_n \wedge 1) \gg x'$ corresponds to a cover $\overline{U_{x'}} \subseteq U_{y_1} \cup \cdots \cup U_{y_n}$. In particular, the open sets $X \backslash \overline{U_{x'}}, U_{y_1}, \ldots, U_{y_n}$ form a cover of X and, since X is almost chainable, there exists an almost chain $C_1, \ldots, C_m$ covering X and refining $X \backslash \overline{U_{x'}}, U_{y_1}, \ldots, U_{y_n}$.

Now, take the almost chain $C_1 \cap U_{x'}, \ldots, C_m \cap U_{x'}$, which clearly covers $U_{x'}$. For each i, let $z_i \in S$ be the associated element to $C_i \cap U_{x'}$. These elements satisfy the desired conditions in Definition 7.2.

Indeed, to see condition (ii), note that $z_i \leq x'$ for every i, so it follows that $z_i + z_j \leq x'$ if and only if $z_i \wedge z_j = 0$. By the proof of Lemma 5.2, this is equivalent to

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$(C_i \cap U_{x'}) \cap (C_i \cap U_{x'}) = \emptyset$. Since $\{C_i \cap U_{x'}\}_i$ is an almost chain, this condition is satisfied.

Conditions (i) and (iii) follow similarly using that $\{C_i \cap U_{x'}\}_i$ is a cover of $U_{x'}$ refining $\{U_{y_j}\}_j$. ∎

Using Remark 3.2, one gets the following result.

Corollary 7.6 *Given X compact, metric, and connected, the* Cu*-semigroup* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ *is weakly chainable if and only if X is chainable.*

It would be interesting to know whether a general Lsc-like Cu-semigroup S is weakly chainable if X_S is almost chainable.

Lemma 7.7 *Given two weakly chainable* Cu*-semigroups S and T, their direct sum* $S \oplus T$ *is also weakly chainable.*

Proof Take $x, y, y_1, \dots, y_n \in S \oplus T$ as in Definition 7.2. Write $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $y_j = (y_{j,1}, y_{j,2})$ with $x_1, y_1, y_{j,1} \in S$ and $x_2, y_2, y_{j,2} \in T$.

Since S and T are weakly chainable, one gets elements x_1', $z_{1,1}, \dots, z_{m,1} \in S$ and x_2', $z_{1,2}, \dots, z_{m',2} \in T$ satisfying the conditions in Definition 7.2. Define $x' = (x_1', x_2')$ and note that $x' \leq y$ and that there exists some $k \in \mathbb{N}$ with $x \leq kx'$.

Now, set $z_i = (z_{i,1}, 0)$ for $i \leq m$ and $z_i = (0, z_{i-m+1,2})$ for $i > m$. We have that

$$\begin{aligned} z_1 + \cdots + z_{m+m'-1} &= ((z_{1,1}, 0) + \cdots + (z_{m,1}, 0)) + ((0, z_{1,2}) + \cdots + (0, z_{m',2})) \\ &\geq (x_1', 0) + (0, x_2') = x'. \end{aligned}$$

As expected, we also get that for every i there exists a j such that $z_i \leq (y_{j,1}, 0) \leq y_j$ or $z_i \leq (0, y_{j,2}) \leq y_j$.

Now, take z_i, z_j with $|i - j| \geq 2$. If $i, j \leq m$, we have $z_i + z_j \leq (x_1', 0) \leq x'$. Similarly, $z_i + z_j \leq (0, x_2') \leq x'$ whenever $i, j > m$.

Moreover, if $i \leq m$ and $j > m$, we know that $z_i = (z_{i,1}, 0) \leq (x_1', 0)$ and $z_j = (0, z_{j-m+1,2}) \leq (0, x_2')$. This implies $z_i + z_j \leq (x_1', 0) + (0, x_2') = x'$.

Since $z_1, \dots, z_{m+m'-1}$ satisfy all the required properties, S satisfies the weak chainability condition. ∎

Proposition 7.8 *Let* $S = \lim_n S_n$ *be a sequential inductive limit of* Cu*-semigroups. Assume that* S_n *is weakly chainable for each n. Then, S is also weakly chainable.*

Proof Let $S = \lim S_n$ with S_n weakly chainable for every n. Given an element $x \in S_n$, let us denote its image through the canonical map $S_n \to S$ by $[x]$.

Let $x, y, y_1, \dots, y_n \in S$ be as in Definition 7.2. Then, let $m \in \mathbb{N}$ be such that there exist elements u, v, and v_j in S_m with $[v_j] \ll y_j$,

$$x \ll [u] \ll [v] \ll y \ll [v_1] + \cdots + [v_n] \ll y_1 + \cdots + y_n,$$

and $u \ll v \ll v_1 + \cdots + v_n$.

Since S_m is weakly chainable, we obtain elements $u', z_1, \dots, z_m \in S_m$ satisfying the conditions in Definition 7.2. We have:

(i) $u' \leq v, u \leq ku'$ for some $k \in \mathbb{N}$. This implies $[u'] \leq [v] \leq y$ and $x \leq [u] \leq k[u']$.

(ii) For any i, there exists j such that $z_i \leq v_j$, which shows that $[z_i] \leq [v_j] \leq y_j$.

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(iii) $z_i + z_j \leq u'$ whenever $|i - j| \geq 2$. Consequently, $[z_i] + [z_j] \leq [u']$ whenever $|i - j| \geq 2$.

Since $z_1 + \cdots + z_m \geq u'$, one also gets $[u'] \leq [z_1] + \cdots + [z_m]$. Thus, S is weakly chainable, as desired. ∎

Corollary 7.9 *The Cuntz semigroup of any AI-algebra is weakly chainable.*

Example 7.10 The Cu-semigroups $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}})$ and $\mathrm{Lsc}([0,1]^2, \overline{\mathbb{N}})$ do not satisfy the weak chainability condition.

Indeed, this follows clearly from Corollary 7.6, as $\mathbb{T}$ and $[0,1]^2$ are not chainable continua.

Using the results developed thus far, one can now use an analogous proof to that of Theorem 7.4 to prove the following theorem.

Theorem 7.11 *Let S be a* Cu*-semigroup. Then, S is* Cu*-isomorphic to the Cuntz semigroup of a unital commutative AI-algebra if and only if S is countably based,* Lsc*-like, weakly chainable, has a compact order unit, and satisfies (O5).*

8 New properties of the Cuntz semigroup of an AI-algebra

Inspired by the abstract characterization obtained above, in this section, we introduce properties that are satisfied by the Cuntz semigroups of all AI-algebras and that are not satisfied by other well-known Cu-semigroups. In Definition 7.2, we have already introduced one such property, which is not satisfied by $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}})$ (see Example 7.10). We now introduce the conditions of Cu-semigroups with refinable sums and almost ordered sums, and show that the Cuntz semigroups of all AI-algebras satisfy these properties. We also prove that Z, the Cuntz semigroup of the Jiang–Su algebra $\mathcal{Z}$, does not have refinable sums (see Example 8.2).

Definition 8.1 We say that a Cu-semigroup S has *refinable sums* if, given a finite $\ll$-increasing sequence

$$x_1 \ll \cdots \ll x_n$$

and elements $x_1', \ldots, x_n'$ such that $x_i \propto x_i'$ for every i, there exist finite decreasing sequences $(y_j^i)_{j=1}^l$ such that:

(i) $x_{i+1}' \geq y_1^i$ for every i.
(ii) $y_j^i \ll y_j^{i+1}$ for every i and j.
(iii) $x_i \ll y_1^i + \cdots + y_l^i \ll x_{i+1}$.

Example 8.1 Any Cu-semigroup S of the form $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ has refinable sums.

To see this, let x_i, x_i' as in Definition 8.1, and let $\tilde{x}_i$ be such that

$$x_1 \ll \tilde{x}_1 \ll x_2 \ll \tilde{x}_2 \ll x_3 \ll \cdots \ll x_n.$$

Since $\tilde{x}_i \in S_{\ll}$ for each i, they can all be written as ordered finite sum of elements below one. Furthermore, by possibly adding some zeros, we may assume that all $\tilde{x}_i$'s have the same amount of summands. That is to say, we have

$$x_1 \ll \tilde{x}_1 = y_1^1 + \cdots + y_l^1 \ll x_2 \ll \tilde{x}_2 = y_1^2 + \cdots + y_l^2 \ll x_3 \ll \cdots \ll x_n.$$

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Thus, we know by Lemma 4.17 that $y^i_j \ll y^{i+1}_j$ for every i, j. Moreover, since we have $\tilde{x}_i \ll x_{i+1} \propto x'_{i+1}$, one can find $x''_{i+1} \ll x'_{i+1}$ such that $\tilde{x}_i \propto x''_{i+1}$.

Since $x''_{i+1} \in S_{\ll}$, we have that $\tilde{x}_i \propto x''_{i+1} \propto x''_{i+1} \wedge 1$. Applying the topological order in S, we obtain $y^i_1 \leq x''_{i+1} \wedge 1 \leq x'_{i+1}$, as required.

Example 8.2 Let $Z = (0, \infty] \sqcup \mathbb{N}$, and denote by $n^\natural \in (0, \infty]$ the associated element to $n \in \mathbb{N}$. Order and addition in Z are defined normally in each component, and given $x \in (0, \infty]$ and $n \in \mathbb{N}$, we set $x \leq n$ if and only if $x \leq n^\natural$ in $(0, \infty]$; $n \leq x$ if and only if $n^\natural < x$; and $x + n = x + n^\natural$. It was proved in [18, Theorem 3.1] that Z is Cu-isomorphic to the Cuntz semigroup of the Jiang–Su algebra $\mathcal{Z}$, as defined in [15].

We claim that Z does not have refinable sums. Indeed, assume for the sake of contradiction that it does, and consider the elements $x_1 = x'_1 = x_2 = 1$ and $x'_2 = 0.5$ in Z. These satisfy

$$x_1 \ll x_2, \quad x_1 \propto x'_1, \quad \text{and} \quad x_2 \propto x'_2.$$

Thus, applying Definition 8.1, we obtain a finite decreasing sequence $(y^1_j)^l_{j=1}$ such that

$$1 = x_1 \ll y^1_1 + \cdots + y^1_l \ll x_2 = 1.$$

Since $(y^1_j)^l_{j=1}$ is decreasing, one must have $y^1_1 = 1$ and $y^1_i = 0$ whenever $i \geq 2$. However, it follows from (i) in Definition 8.1 that

$$1 = y^1_1 \leq x'_2 = 0.5,$$

which is a clear contradiction.

Proposition 8.3 *Let S be a* Cu*-semigroup that can be written as a sequential inductive limit* $\lim S_k$ *of* Cu*-semigroups S_k that have refinable sums. Then, S also has refinable sums.*

Proof Let $S = \lim S_k$ where each S_k has refinable sums. As in the proof of Proposition 7.8, let us denote the image through the canonical map $S_k \to S$ of an element $x \in S_k$ by $[x]$.

Let $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$ be elements in S as in Definition 8.1. Let $k \in \mathbb{N}$ such that, for every $i \leq n - 1$, there exist elements $u_{2i-1}, u_{2i}, v_{2i-1}, v_{2i} \in S_k$ satisfying

$$x_i \ll [u_{2i-1}] \ll [u_{2i}] \ll x_{i+1}, \quad [v_{2i}] \leq x'_{i+1},$$

and

$$u_1 \ll \cdots \ll u_{2n-2}, \quad \text{and} \quad u_i \propto v_i.$$

Since S_k has refinable sums, we obtain decreasing sequences $(y^i_j)^l_{j=1}$ for $i \leq 2n - 2$ satisfying the properties of Definition 8.1. In particular, we get

$$x_i \ll [u_{2i-1}] \ll [y^{2i-1}_1] + \cdots + [y^{2i-1}_1] \ll [u_{2i}] \ll x_{i+1},$$

and $[y^{2i-1}_1] \leq [v_{i+1}] \leq x'_{i+1}$.

It follows that S has refinable sums. ∎

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Definition 8.2 A Cu-semigroup S is said to have *almost ordered sums* if for any finite set of elements $x_1, \dots, x_n$ in S there exist elements $y_{j,i}$ in S such that

$$x_1 + \cdots + x_n = \sup_i (y_{1,i} + \cdots + y_{n,i})$$

and such that:

(i) $y_{1,i} \geq \cdots \geq y_{n,i}$.
(ii) $(y_{n,i})_i$ is increasing and bounded by $x_1, \dots, x_n$.
(iii) If $x' \ll x_{j_1}, \dots, x_{j_r} \leq z$ for $j_1, \dots, j_r$ pairwise different, we have $x' \leq y_{r,i}$ and $y_{n+1-r,i} \leq z$ for every sufficiently large i.

Example 8.4 If S is a distributively lattice ordered Cu-semigroup, S has almost ordered sums. This applies, in particular, to Cu-semigroups such that $S \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$ for some X.

Indeed, given $x_1, \dots, x_n$, set

$$y_{1,i} = x_1 \vee \cdots \vee x_n, \quad y_{2,i} = (x_1 \wedge x_2) \vee \cdots \vee (x_{n-1} \wedge x_n), \quad \dots, \quad \text{and}$$
$$y_{n,i} = x_1 \wedge \cdots \wedge x_n,$$

for every i, and note that by Lemma 4.2 we have

$$x_1 + \cdots + x_n = y_{1,i} + \cdots + y_{n,i}.$$

This implies that S has almost ordered sums.

Example 8.5 Let $Z' = Z \cup \{1''\}$ with $1''$ a compact element not comparable with 1 such that $1 + x = 1'' + x$ for every $x \in Z \backslash \{0\}$ and $k1'' = k$ for every $k \in \mathbb{N}$. Then, Z' does not have almost ordered sums.

To see this, consider the sum $1 + 1''$ and assume, for the sake of contradiction, that Z' has refinable sums. Then, there exist elements $y_{1,i}, y_{2,i}$ such that $1 + 1'' = \sup_i y_{1,i} + y_{2,i}$.

Since $1 + 1'' = 2$ is compact, for every sufficiently large i, we have $1 + 1'' = y_{1,i} + y_{2,i}$.

This implies that $y_{1,i} = 2$ and $y_{2,i} = 0$, since we know that $1, 1'' \leq y_{1,i}$ and that $1, 1''$ are not comparable.

However, we also have $1, 1'' \leq 1.5$, so we get $2 = y_{1,i} \leq 1.5$, a contradiction.

Proposition 8.6 *Sequential inductive limits of distributively lattice ordered* Cu-*semigroups have almost ordered sums.*

Proof Let $S = \lim_k (S_k, \varphi_{k+1,k})$ be the inductive limit of distributively lattice ordered Cu-semigroups S_k. As before, given an element $x \in S_k$, let us denote its image through the canonical map $S_k \to S$ by $[x]$.

Let $x_1, \dots, x_n$ be elements in S. One can check that there exists an increasing sequence of integers (k_l) and elements take $x_1^l, \dots, x_n^l \in S_{k_l}$ such that $([x_j^l])_l$ are $\ll$-increasing sequences in S with suprema x_j for every $j \leq n$, in such a way that $\varphi_{k_{l+1},k_l}(x_j^l) \ll x_j^{l+1}$ for every j and l.

Since each S_k is distributively lattice ordered, for every l, there exist elements $y_1^l, \dots, y_n^l$ in S_{k_l} with

$$x_1^l + \cdots + x_n^l = y_1^l + \cdots + y_n^l,$$

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satisfying the properties of Definition 8.2 (see Example 8.4). This implies, in particular, $\sup_l([y_1^l] + \cdots + [y_n^l]) = x_1 + \cdots + x_n$.

We will now check that the elements $[y_1^l], \ldots, [y_n^l]$ satisfy conditions (i)–(iii) in Definition 8.2.

By construction, one has $y_1^l \geq \cdots \geq y_n^l$ for every l, so condition (i) is satisfied. For condition (ii), let $l \in \mathbb{N}$. Then, applying condition (ii) in S_{k_l}, we have

$$\varphi_{k_{l+1},k_l}(y_n^l) \leq \varphi_{k_{l+1},k_l}(x_1^l), \ldots, \varphi_{k_{l+1},k_l}(x_n^l) \ll x_1^{l+1}, \ldots, x_n^{l+1},$$

and, by condition (iii) in $S_{k_{l+1}}$, we get $\varphi_{k_{l+1},k_l}(y_n^l) \leq y_n^{l+1}$. It follows that condition (ii) is satisfied.

To prove (iii), take $x', z \in S$ such that $x' \ll x_{j_1}, \ldots, x_{j_r} \leq z$ for some pairwise different $j_1, \ldots, j_r \leq n$. For a large enough l, there exist u such that

$$u \ll x_{j_1}^l, \ldots, x_{j_r}^l$$

and $x' \ll [u]$ in S.

This implies that, for every $l' \geq l$, one has

$$\varphi_{k_{l'},k_l}(u) \ll \varphi_{k_{l'},k_l}(x_{j_1}^l) + \cdots + \varphi_{k_{l'},k_l}(x_{j_r}^l)$$

in $S_{k_{l'}}$.

Consequently, we have $\varphi_{k_{l'},k_l}(u) \leq d_r^{l'}$ and so $x \ll [u] \ll [d_r^{l'}]$ in S for every $l' \geq l$. Furthermore, since $x_{j_1}, \ldots, x_{j_r} \leq z$, for every l, there exists some $z_l \in S_{k_l}$ with $[z_l] \ll z$ and

$$x_{j_1}^l, \ldots, x_{j_r}^l \leq z_l.$$

Therefore, one gets $y_{n+1-r}^l \leq z_l$. This implies $[y_{n+1-r}^l] \leq [z_l] \ll z$ for every l, as required. ■

Theorem 8.7 *Let A be an AI-algebra. Then, its Cuntz semigroup* $\mathrm{Cu}(A)$ *is weakly chainable and has refinable sums and almost ordered sums.*

Proof The Cuntz semigroup $\mathrm{Cu}(A)$ is weakly chainable by Corollary 7.9. Furthermore, using the same arguments as in Lemma 7.7, it is easy to see that finite direct sums of Cu-semigroups having refinable sums or almost ordered sums have refinable sums or almost ordered sums, respectively. Thus, it follows from Example 8.1 and Proposition 8.3 that S has refinable sums.

By Example 8.4 and Proposition 8.6, $\mathrm{Cu}(A)$ also has almost ordered sums. ■

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