Canad. Math. Bull. Vol. **64** (4), 2023, pp. 1368–1375
http://dx.doi.org/10.4153/S0008439523000425

From the Ideal Theorem to the class number

Olivier Bordellès

Abstract. In this article, we provide an explicit upper bound for $h_K \mathcal{R}_K d_K^{-1/2}$ which depends on an effective constant in the error term of the Ideal Theorem.

1 Introduction

Let K be a number field of degree $n \geqslant 3$, signature (r_1, r_2), discriminant $(-1)^{r_2} d_K$, class number h_K, regulator $\mathcal{R}_K$, and let w_K be the number of roots of unity in K. Let κ_K be the residue at $s = 1$ of the Dedekind zeta-function $\zeta_K(s)$ attached to K. Estimating h_K is a long-standing problem in algebraic number theory. One of the classic way is the use of the so-called *analytic class number formula* stating that

$$h_K \mathcal{R}_K = \frac{w_K}{2^n} \left(\frac{2}{\pi} \right)^{r_2} d_K^{1/2} \kappa_K, \tag{1.1}$$

and to use Hecke's integral representation and the functional equation of the Dedekind zeta-function to majorize κ_K. This is done in [7, 8] with additional properties of log-convexity of some functions related to ζ_K which enables Louboutin to reach the following bound:

$$h_K \mathcal{R}_K \leqslant \frac{w_K}{2} \left(\frac{2}{\pi} \right)^{r_2} \left(\frac{e \log d_K}{4n - 4} \right)^{n-1} d_K^{1/2}. \tag{1.2}$$

Let $r_K(m)$ be the mth coefficient of ζ_K, i.e., the number of nonzero integral ideals of $\mathcal{O}_K$ of norm m, and denote $\Delta_K(x)$ to be the error term in the Ideal Theorem, i.e.,

$$\Delta_K(x) = \sum_{m \leqslant x} r_K(m) - \kappa_K x. \tag{1.3}$$

The aim of this work is to prove the following result.

Theorem 1.1 *Let K be an algebraic number field of degree $n \geqslant 3$, and set $\gamma_3 = 214$ and $\gamma_n = 10$ if $n \geqslant 4$. Assume that there exist $\alpha \in \left(0, \frac{2}{n}\right)$ and a constant*

$$C_K \geqslant \exp\left(\max\left(\gamma_n \, , \, \alpha n + \frac{1}{4\alpha^2} \right) \right)$$

Received by the editors March 16, 2023; revised April 26, 2023; accepted May 23, 2023.
Published online on Cambridge Core May 26, 2023.
AMS subject classification: **11R29**, **11R42**.
Keywords: Class number, Dedekind zeta-function, Ideal Theorem.

such that, for $x \geqslant 1$,

$$|\Delta_K(x)| \leqslant C_K x^{1-\alpha}, \tag{1.4}$$

where $\Delta_K(x)$ is given in (1.3). Then

$$h_K \mathcal{R}_K < \frac{3w_K}{2}\left(\frac{2}{\pi}\right)^{r_2}\left(\frac{\left(\frac{1}{2\alpha}\log C_K\right)^{n-1}}{(n-1)!} - \frac{\left(\frac{1}{2\alpha}\log C_K\right)^{n-2}}{(n-2)!}\right) d_K^{1/2}.$$

2 Tools

The first lemma is a Titchmarsh-like generalization of [12, Theorem 12.5] to number fields established by Ayoub [1]. The result is stated for the quadratic case, but as it can be seen in the proof and as the author points it out, it is still true for the general case (see also [5, Lemma 13.3]).

Lemma 2.1 *Let $n \geqslant 3$ and μ_K be the infimum of the real numbers σ for which the integral*

$$\int_{-\infty}^{\infty} \frac{|\zeta_K(\sigma+it)|^2}{|\sigma+it|^2}\,\mathrm{d}t$$

converges. Then $\mu_K \leqslant 1 - \frac{2}{n}$ and, for all $\mu_K < \sigma < 1$, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\zeta_K(\sigma+it)|^2}{|\sigma+it|^2}\,\mathrm{d}t = \int_0^{\infty} \Delta_K(x)^2 x^{-1-2\sigma}\,\mathrm{d}x.$$

Proof Assume $\frac{1}{n} \leqslant \sigma \leqslant 1 - \frac{1}{n}$. By [3, Theorem 4], we have, for all $\varepsilon > 0$,

$$\int_1^T |\zeta_K(\sigma+it)|^2\,\mathrm{d}t \ll T^{n(1-\sigma)+\varepsilon}.$$

The exponent of T is less than 2 for $\sigma > 1 - \frac{2-\varepsilon}{n}$ and note that $1 - \frac{2}{n} \geqslant \frac{1}{n}$ since $n \geqslant 3$. This implies that

$$\int_{T/2}^{T} \frac{|\zeta_K(\sigma+it)|^2}{|\sigma+it|^2}\,\mathrm{d}t \ll T^{-\eta}$$

for some $\eta = \eta(\varepsilon) > 0$. Now replacing T by $2^{-j}T$ and summing over $j \geqslant 1$ yields $\mu_K \leqslant 1 - \frac{2}{n}$. For the second part of the lemma, we start by using Perron's formula [11, (2.3) p. 217] yielding

$$\sum\nolimits'_{n \leqslant x} r_K(n) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\zeta_K(s)}{s} x^s\,\mathrm{d}s,$$

where $\sum'$ means that if x is a positive integer, then $\frac{1}{2} r_K(x)$ comes instead of $r_K(x)$. Moving the line of integration to some $1 - (600)^{-2/3} n^{-7/3} \leqslant c < 1$ sufficiently close to 1, we get

$$\sum\nolimits'_{n \leqslant x} r_K(n) = \kappa_K x + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta_K(s)}{s} x^s\,\mathrm{d}s,$$

https://doi.org/10.4153/S0008439523000425 Published online by Cambridge University Press

so that

$$\Delta_K(x) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{\zeta_K(s)}{s} x^s \, \mathrm{d}s. \tag{2.1}$$

Using [9] (see also [2, Theorem 7.18]), there exists an absolute constant $c_0 > 0$ such that, for $1 - (600)^{-2/3} n^{-7/3} \leqslant c < \sigma < 1$ and $t \geqslant e$,

$$\left| \frac{\zeta_K(\sigma + it)}{\sigma + it} \right| \leqslant e^{c_0 n^8 d_K^2} t^{600 n^{7/2} (1-\sigma)^{3/2} - 1} (\log t)^{2/3} \leqslant e^{c_0 n^8 d_K^2} t^{-1} (\log t)^{2/3},$$

so that $\zeta_K(s)s^{-1} \to 0$ uniformly in the strip $1 - (600)^{-2/3} n^{-7/3} \leqslant c < \sigma < 1$ as $|t| \to \infty$. Hence, on integrating over the rectangle $d \pm iT$, $c \pm iT$, with $\gamma_K < d < c < 1$, we infer that (2.1) holds for any $\gamma_K < c < 1$. Replacing in (2.1) x by $1/x$, taking $\gamma_K < c < 1$, and using Parseval's formula [5, Identity (A5)] yields

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta_K(c+it)|^2}{|c+it|^2} \, \mathrm{d}t &= \int_0^\infty \Delta_K(1/x)^2 x^{2c-1} \, \mathrm{d}x \\ &= \int_0^\infty \Delta_K(u)^2 u^{-2c-1} \, \mathrm{d}u \end{aligned}$$

as required. ∎

Corollary 2.2 *Assume hypothesis (1.4), and let* $0 < \delta < \alpha < \frac{2}{n}$. *Then*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta_K(1-\delta+it)|}{|1-\delta+it|^2} \, \mathrm{d}t \leqslant \frac{1}{2\sqrt{1-\delta}} \left(\frac{\kappa_K}{\sqrt{\delta}} + \frac{C_K}{\sqrt{\alpha - \delta}} \right).$$

Proof Using Lemma 2.1, (1.4) and the trivial bound $|\Delta_K(x)| \leqslant \kappa_K x$ when $x \in [0, 1)$

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta_K(1-\delta+it)|^2}{|1-\delta+it|^2} \, \mathrm{d}t &\leqslant \kappa_K^2 \int_0^1 x^{2\delta-1} \mathrm{d}x + C_K^2 \int_1^\infty x^{2(\delta-\alpha)-1} \mathrm{d}x \\ &= \frac{1}{2} \left(\frac{\kappa_K^2}{\delta} + \frac{C_K^2}{\alpha - \delta} \right), \end{aligned}$$

and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta_K(1-\delta+it)|}{|1-\delta+it|^2} \, \mathrm{d}t \\ &\leqslant \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \frac{|\zeta_K(1-\delta+it)|^2}{|1-\delta+it|^2} \, \mathrm{d}t \right)^{1/2} \left(\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{|1-\delta+it|^2} \right)^{1/2} \\ &\leqslant \frac{1}{2\sqrt{1-\delta}} \left(\frac{\kappa_K^2}{\delta} + \frac{C_K^2}{\alpha-\delta} \right)^{1/2} \leqslant \frac{1}{2\sqrt{1-\delta}} \left(\frac{\kappa_K}{\sqrt{\delta}} + \frac{C_K}{\sqrt{\alpha-\delta}} \right) \end{aligned}$$

as asserted. ∎

Lemma 2.3 *Uniformly for all* $x \geqslant 1$ *and all* $n \in \mathbb{Z}_{\geqslant 1}$, *we have*

$$\sum_{m \leqslant x} r_K(m) \log \frac{x}{m} \leqslant \frac{x \log x}{(n-1)!} (\log x + n - 1)^{n-2}.$$

https://doi.org/10.4153/S0008439523000425 Published online by Cambridge University Press

Proof Let τ_n be the nth Piltz–Dirichlet divisor function. We have $r_K(m) \leqslant \tau_n(m)$ and from the bound [2, Exercise 78]

$$\sum_{m \leqslant t} \tau_n(m) \leqslant t \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(\log t)^j}{j!} \qquad (t \geqslant 1),$$

so that

$$\begin{aligned}
\sum_{m \leqslant x} r_K(m) \log \frac{x}{m} &= \int_1^x \frac{1}{t} \left(\sum_{m \leqslant t} r_K(m) \right) \mathrm{d}t \\
&\leqslant \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{1}{j!} \int_1^x (\log t)^j \,\mathrm{d}t \\
&= x \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \left\{ \sum_{k=0}^{j} \frac{(-\log x)^k}{k!} - \frac{1}{x} \right\} \\
&= -x \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \sum_{k=j+1}^{\infty} \frac{(-\log x)^k}{k!} \\
&= -x \sum_{k=1}^{\infty} \frac{(-\log x)^k}{k!} \sum_{j=0}^{\min(n-1,k-1)} (-1)^j \binom{n-1}{j} \\
&= -x \sum_{k=1}^{\infty} (-1)^{k+\min(n-1,k-1)} \frac{(\log x)^k}{k!} \binom{n-2}{\min(n-1,k-1)} \\
&= x \sum_{k=1}^{n-1} \frac{(\log x)^k}{k!} \binom{n-2}{k-1} \\
&\leqslant \frac{x \log x}{(n-1)!} (\log x + n - 1)^{n-2},
\end{aligned}$$

where we used [4, Identity (1.5)] in the 6th line, and the fact that, for $0 \leqslant k \leqslant n-3$,

$$\begin{aligned}
\frac{1}{(k+1)!} &= \frac{1}{(n-1)!} \prod_{i=2}^{n-k-1} (i+k) \\
&\leqslant \frac{1}{(n-1)!} \left(\frac{1}{n-k-2} \sum_{i=2}^{n-k-1} (i+k) \right)^{n-k-2} \\
&\leqslant \frac{(n-1)^{n-k-2}}{(n-1)!}
\end{aligned}$$

by the GM–AM inequality stating that $(a_1 \dots a_N)^{1/N} \leqslant N^{-1}(a_1 + \dots + a_N)$, where $a_k > 0$, and this bound also holds when $k = n-2$. ∎

Lemma 2.4 *If $n \geqslant 3$, $0 < \alpha < \frac{2}{n}$ and $\log C_K \geqslant \gamma_n$, then*

$$\frac{\left(\alpha^{-1} \log C_K\right)^{n-2}}{(n-2)!} \geqslant \frac{e^{\alpha(n-1)+1} \sqrt{\log C_K}}{\sqrt{\alpha(1-\alpha)}}.$$

https://doi.org/10.4153/S0008439523000425 Published online by Cambridge University Press

Proof Squaring the inequality of the lemma, it is equivalent to show

$$(\log C_K)^{2n-5} \geqslant ((n-2)!)^2 \times \frac{\alpha^{2n-5} e^{2\alpha(n-1)+2}}{1-\alpha}.$$

The function $\alpha \in \left(0, \frac{2}{n}\right) \mapsto \frac{\alpha^{2n-5} e^{2\alpha(n-1)+2}}{1-\alpha}$ is nondecreasing, so that

$$\frac{\alpha^{2n-5} e^{2\alpha(n-1)+2}}{1-\alpha} \leqslant \frac{2^{2n-5} e^{6-4/n} n^{6-2n}}{n-2}$$

and, therefore, it suffices to show

$$\log C_K \geqslant 2 \left(\frac{e^{6-4/n} n^{6-2n}}{n-2} \right)^{\frac{1}{2n-5}} ((n-2)!)^{\frac{2}{2n-5}}.$$

Using Stirling's bound, the inequality of the lemma is guaranteed as soon as

$$\log C_K \geqslant 2(2\pi)^{\frac{1}{2n-5}} e^{-\frac{12n^3-84n^2+143n-48}{6n(n-2)(2n-5)}} \left(\frac{(n-2)^{2n-4}}{n^{2n-6}} \right)^{\frac{1}{2n-5}},$$

and since the right-hand side is a nonincreasing function in $n \geqslant 3$, it then suffices that $\log C_K \geqslant s_n$, where $s_3 := 4\pi e^{17/6} \approx 213{,}7$ and $s_n = 2^{4/3}\pi^{1/3}e^{13/36} \approx 5{,}3$ whenever $n \geqslant 4$. ∎

3 Proof of the main result

By (1.1), it is sufficient to show that

$$\kappa_K < 3 \left(\frac{\left(\alpha^{-1} \log C_K\right)^{n-1}}{(n-1)!} - \frac{2\left(\alpha^{-1} \log C_K\right)^{n-2}}{(n-2)!} \right). \tag{3.1}$$

Assume $n \geqslant 3$, and let $0 < \delta < \alpha < \frac{2}{n}$ and $x \geqslant 1$ satisfying

$$\delta(1-\delta) \geqslant x^{-2\delta} \tag{3.2}$$

and

$$x \geqslant e^{1+\frac{1}{4\alpha^3}}. \tag{3.3}$$

By another Perron's formula (see, for instance, [11, (2.9) p. 220]), we have

$$\sum_{m \leqslant x} r_K(m) \log \frac{x}{m} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta_K(s)}{s^2} x^s \, \mathrm{d}s.$$

Shifting the contour integration to the line $\sigma = 1 - \delta$ and picking up the residue of the integrand at the unique simple pole $s = 1$, we obtain by Cauchy's theorem

$$\begin{aligned} \sum_{m \leqslant x} r_K(m) \log \frac{x}{m} &= \kappa_K x + \frac{1}{2\pi i} \int_{1-\delta-i\infty}^{1-\delta+i\infty} \frac{\zeta_K(s)}{s^2} x^s \, \mathrm{d}s \\ &:= \kappa_K x + I_\delta(x), \end{aligned}$$

https://doi.org/10.4153/S0008439523000425 Published online by Cambridge University Press

and using Corollary 2.2, we get

$$\begin{aligned} |I_\delta(x)| &\leqslant \frac{x^{1-\delta}}{2\pi}\int_{-\infty}^{\infty}\frac{|\zeta_K(1-\delta+it)|}{|1-\delta+it|^2}\,\mathrm{d}t \\ &\leqslant \frac{x^{1-\delta}}{2\sqrt{1-\delta}}\left(\frac{\kappa_K}{\sqrt{\delta}}+\frac{C_K}{\sqrt{\alpha-\delta}}\right). \end{aligned}$$

Therefore, using (3.2), we derive

$$\begin{aligned} \sum_{m\leqslant x} r_K(m)\log\frac{x}{m} &\geqslant \kappa_K x\left(1-\frac{x^{-\delta}}{2\sqrt{\delta(1-\delta)}}\right)-\frac{x^{1-\delta}}{2}\frac{C_K}{\sqrt{(1-\delta)(\alpha-\delta)}} \\ &\geqslant \frac{\kappa_K x}{2}-\frac{x^{1-\delta}}{2}\frac{C_K}{\sqrt{(1-\delta)(\alpha-\delta)}}, \end{aligned}$$

and Lemma 2.3 yields

$$\begin{aligned} \kappa_K &\leqslant \frac{2}{x}\sum_{m\leqslant x} r_K(m)\log\frac{x}{m}+\frac{x^{-\delta}C_K}{\sqrt{(1-\delta)(\alpha-\delta)}} \\ &\leqslant \frac{2\log x}{(n-1)!}(\log x+n-1)^{n-2}+\frac{x^{-\delta}C_K}{\sqrt{(1-\delta)(\alpha-\delta)}}, \end{aligned}$$

whenever x and δ satisfy (3.2). Now choose

$$\delta=\alpha-\frac{1}{\log x}. \tag{3.4}$$

Note that (3.2) is satisfied if $(\alpha\log x-1)(1+(1-\alpha)\log x)\geqslant e^2(\log x)^2x^{-2\alpha}$. But using (3.3), we get

$$(\alpha\log x-1)(1+(1-\alpha)\log x)\geqslant\frac{1}{16\alpha^2}\geqslant e^2(\log x)^2x^{-2\alpha}.$$

Therefore, with the choice (3.4), we derive

$$\begin{aligned} \kappa_K &\leqslant \frac{2\log x}{(n-1)!}(\log x+n-1)^{n-2}+\frac{e\,C_Kx^{-\alpha}\log x}{\sqrt{(1-\alpha)\log x+1}} \\ &< \frac{2\log x}{(n-1)!}(\log x+n-1)^{n-2}+\frac{e\,C_Kx^{-\alpha}}{\sqrt{1-\alpha}}\sqrt{\log x} \end{aligned}$$

provided that (3.3) is fulfilled. We next choose

$$x=C_K^{1/\alpha}e^{1-n}.$$

This yields

$$\begin{aligned} \kappa_K &< 2\left(\frac{\left(\alpha^{-1}\log C_K\right)^{n-1}}{(n-1)!}-\frac{\left(\alpha^{-1}\log C_K\right)^{n-2}}{(n-2)!}\right)+\frac{e^{\alpha(n-1)+1}\sqrt{\log C_K+\alpha(1-n)}}{\sqrt{\alpha(1-\alpha)}} \\ &< 2\left(\frac{\left(\alpha^{-1}\log C_K\right)^{n-1}}{(n-1)!}-\frac{\left(\alpha^{-1}\log C_K\right)^{n-2}}{(n-2)!}\right)+\frac{e^{\alpha(n-1)+1}\sqrt{\log C_K}}{\sqrt{\alpha(1-\alpha)}}. \end{aligned}$$

Now, since $\log C_K \geqslant \gamma_n \geqslant 10 > 10 - \frac{10}{n} \geqslant 5\alpha(n-1)$, we derive using Lemma 2.4

$$\begin{aligned}\frac{\left(\alpha^{-1}\log C_K\right)^{n-1}}{(n-1)!} - \frac{4\left(\alpha^{-1}\log C_K\right)^{n-2}}{(n-2)!} &= \frac{\left(\alpha^{-1}\log C_K\right)^{n-2}}{(n-2)!}\left(\frac{\alpha^{-1}\log C_K}{n-1} - 4\right) \\ &> \frac{\left(\alpha^{-1}\log C_K\right)^{n-2}}{(n-2)!} \\ &\geqslant \frac{e^{\alpha(n-1)+1}\sqrt{\log C_K}}{\sqrt{\alpha(1-\alpha)}}\end{aligned}$$

and, therefore,

$$\kappa_K < 3\left(\frac{\left(\alpha^{-1}\log C_K\right)^{n-1}}{(n-1)!} - \frac{2\left(\alpha^{-1}\log C_K\right)^{n-2}}{(n-2)!}\right)$$

which is (3.1). Now substituting this bound into (1.1) yields the desired result. □

4 Example

Improving a result in Sunley's thesis [10], Lee [13] proved that, for all $x > 0$,

$$|\Delta_K(x)| \leqslant \Theta_K\, d_K^{\frac{1}{n+1}} (\log d_K)^{n-1} x^{1-\frac{2}{n+1}}$$

with $\Theta_K := 0{,}17\left(\frac{6n-2}{n-1}\right) 2{,}26^n\, e^{4n+26/n}\, n^{n+1/2}\left(44{,}39 \times 0{,}082^n\, n! + \frac{13}{n-1}\right)$. Hence, one can take $\alpha = \frac{2}{n+1}$ and

$$C_K := \Theta_K\, d_K^{\frac{1}{n+1}} (\log d_K)^{n-1}.$$

The hypothesis $\log C_K \geqslant \max\left(\gamma_n\,,\ \alpha n + \frac{1}{4\alpha^2}\right)$ is easily fulfilled, and noticing that, for any $n \geqslant 3$,

$$\tfrac{1}{2} n^2 \log n < \tfrac{n+1}{4} \log \Theta_K < 3n^2 \log n,$$

we infer that Theorem 1.1 yields the following result.

Corollary 4.1 *Let K be an algebraic number field of degree $n \geqslant 3$. Then*

$$h_K \mathcal{R}_K < \frac{3w_K}{2}\left(\frac{2}{\pi}\right)^{r_2}\left(\frac{\left(\frac14\log d_K + L_K\right)^{n-1}}{(n-1)!} - \frac{\left(\frac14\log d_K + \ell_K\right)^{n-2}}{(n-2)!}\right) d_K^{1/2},$$

where $\left.\begin{matrix} L_K \\ \ell_K \end{matrix}\right\} := \frac14 (n^2-1)\log\log d_K + \left\{\begin{matrix} 3n^2\log n \\ \frac12 n^2 \log n. \end{matrix}\right.$

https://doi.org/10.4153/S0008439523000425 Published online by Cambridge University Press

Note that Stirling's bound yields

$$h_K \mathcal{R}_K < \frac{3w_K}{2\sqrt{2\pi}} \left(\frac{2}{\pi}\right)^{r_2} \left\{ \frac{1}{\sqrt{n-1}} \left(\frac{e \log d_K}{4n-4} + \frac{eL_K}{n-1} \right)^{n-1} \right. \left. - \frac{e^{-\frac{1}{12(n-2)}}}{\sqrt{n-2}} \left(\frac{e \log d_K}{4n-8} + \frac{e\ell_K}{n-2} \right)^{n-2} \right\} d_K^{1/2}$$

which may be more easily compared to (1.2). It finally should be pointed out that Lee slightly improved in [6] the value of C_K, showing that

$$C_K := \frac{0.54\,(3n-1)\rho_K(n)}{(n-1)^2(\log m_K)^{n-1}} \times n^{3/2} \times n!$$

with $m_K = (n^2\pi/4)^n (n!)^{-2}$ and, if $n > 13$,

$$\rho_K(n) := (n+1)^{n-\frac{1}{2}-\frac{1}{2n}} \left(\tfrac{5}{8} + \tfrac{\pi}{2} + \tfrac{1}{n} + \tfrac{3}{8n^2} \right)^{1/2} e^{4.13n + \frac{0.02}{n}}.$$

Note that the author also gives the value of $\rho_K(n)$ in the range $2 \leqslant n \leqslant 13$. Although an improvement over the previous value of C_K, this result is somewhat irrelevant on ours, while the calculations of L_K and ℓ_K are more tedious.

Acknowledgment The author deeply acknowledges the anonymous referee for some corrections and remarks that have significantly enhanced the paper. The author also warmly thanks Stephan R. Garcia and Ethan S. Lee for sending him Sunley's thesis [10] and Lee's preprint [13].

References

[1] R. G. Ayoub, *A mean value theorem for quadratic fields*. Pacific J. Math. **8**(1958), 23–27.
[2] O. Bordellès, *Arithmetic tales*, Advanced Edition, Universitext, Springer, Basel, 2020.
[3] K. Chandrasekharan and R. Narasimhan, *The approximate functional equation for a class of zeta-functions*. Math. Ann. **152**(1963), 30–64.
[4] H. W. Gould, *Combinatorial identities. A standardized set of tables listing 500 binomial coefficient summations*, Henry E. Gould. VIII, Morgantown, WV, 1972.
[5] A. Ivić, *The Riemann zeta-function. Theory and applications*, republication of the John Wiler & Sons Ed., Dover, Mineola, NY, 2003.
[6] E. S. Lee, *On the number of integral ideals in a number field*. J. Math. Anal. Appl. **517**(2023), Article ID 126585, 25 pp.
[7] S. Louboutin, *Explicit bounds for residues of Dedekind zeta functions, values of L-functions at $s = 1$, and relative class numbers*. J. Number Theory **85**(2000), 263–282.
[8] S. Louboutin, *Explicit upper bounds for residues of Dedekind zeta functions and values of L-functions at $s = 1$, and explicit lower bounds for relative class numbers of CM-fields*. Canad. J. Math. **53**(2001), 1194–1222.
[9] W. Stas, *On the order of the Dedekind zeta-function near the line $\sigma = 1$*. Acta Arith. **35**(1979), 195—202.
[10] J. E. Sunley, *On the class numbers of totally imaginary quadratic extensions of totally real fields*. Ph.D. thesis, University of Maryland, College Park, MD, 1971.
[11] G. Tenenbaum, *Introduction à la Théorie Analytique et Probabiliste des Nombres*, Belin, 2008 (French).
[12] E. M. Titchmarsh, *The theory of the Riemann zeta-function*, Clarendon Press, Oxford, 1986. Notes by D. R. Heath-Brown.
[13] E. S. Lee, On the number of ideals in a number fields, preprint, 23 *pp*., 2022.

Lycée Dupuy, Le Puy-en-Velay, 2, allée de la Combe, Aiguilhe 43000, France
e-mail: borde43@wanadoo.fr