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THE HYPERCORE OF A SEMIGROUP

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In this paper the "hypercore" of a semigroup S is defined to be the subsemigroup generated by the union of all the subsemigroups of S without non-universal cancellative congruences, provided that at least one such subsemigroup exists: otherwise it is taken to be the empty set. It is shown first that if the hypercore of S is nonempty (which holds, for example, when S contains an idempotent) then it is the largest subsemigroup of S with no non-universal cancellative congruence, is full and unitary in S, and is contained in the identity class of every group congruence on S (Theorem 1).

A semigroup S is E-inversive if and only if, for all $x \in S$, there exists $y \in S$ such that $(xy)^2 = xy$. If S is E-inversive [in particular, regular] then the hypercore of S is the largest E-inversive [regular] subsemigroup of S with no non-universal group congruence (Theorem 2). Another description of the hypercore in the E-inversive case, this time in terms of a descending sequence of full unitary subsemigroups, is provided by Theorem 3. To conclude, there is a discussion of some particular cases.

The concept of the hypercore plays a crucial part in a companion paper, by one of us, on congruence-free regular semigroups [**8**].

1. Definitions and properties

The notation and terminology, with few exceptions, will be that of [**1**]. The set of idempotents (possibly empty) of a semigroup S will be denoted by $E(S)$ and the subsemigroup of S generated by a nonempty subset A of S will be denoted by $\langle A \rangle$. We say that a subsemigroup T of S is

(i) *full* if and only if $E(S) \subseteq T$,
(ii) *unitary* if and only if

$$(\forall t \in T)(\forall x \in S) \quad [tx \in T \Rightarrow x \in T] \text{ and } [xt \in T \Rightarrow x \in T].$$

A congruence ρ on S is termed a *group congruence* [*cancellative congruence*] if and only if S/ρ is a group [cancellative semigroup]; further, the ρ-class containing $x \in S$ is written as $x\rho$. It is clear that every group congruence is a cancellative congruence and that the universal congruence, $S \times S$, is a group congruence on S.

The following result is elementary and well-known.

Lemma 1. *Let ρ be a group congruence on a semigroup S. Then the identity ρ-class is a full unitary subsemigroup of S.* □

For a semigroup S, let $\mathscr{S}_S$ denote the set of all subsemigroups A of S such that A has no cancellative congruence except the universal congruence $A \times A$. We define the *hypercore* of S, denoted by $\operatorname{hyp}(S)$, as follows:

$$\operatorname{hyp}(S) = \begin{cases} \langle \cup_{A \in \mathscr{S}_S} A \rangle & \text{if } \mathscr{S}_S \neq \varnothing, \\ \varnothing & \text{otherwise.} \end{cases}$$

Note that if $E(S) \neq \varnothing$ then $\operatorname{hyp}(S) \neq \varnothing$, since $\{e\} \in \mathscr{S}_S$ for all $e \in E(S)$. On the other hand, if S is a cancellative semigroup with no identity element then clearly $\operatorname{hyp}(S) = \varnothing$. An example of a semigroup S with $E(S) = \varnothing$ and $\operatorname{hyp}(S) = S$ was given by McAlister and O'Carroll [**6**, Ex. 1.5].

Theorem 1. *Let S be a semigroup with* $\operatorname{hyp}(S) \neq \varnothing$. *Then*

(i) $\operatorname{hyp}(S) \in \mathscr{S}_S$;
(ii) $\operatorname{hyp}(S)$ *is full and unitary in S;*
(iii) *if ρ is a group congruence on S then each $A \in \mathscr{S}_S$ is contained in the identity ρ-class and so* $\operatorname{hyp}(S)$ *is contained in the identity ρ-class.*

Proof. For brevity, write $\mathscr{S} = \mathscr{S}_S$ and $T = \operatorname{hyp}(S)$.

(i) Let ρ be any cancellative congruence on T. Then, for each $\in \mathscr{S}$, $\rho \cap (A \times A)$ is a cancellative congruence on A and so $\rho \cap (A \times A) = A \times A$; that is, $A \times A \subseteq \rho$. Thus, for each $x \in \cup_{A \in \mathscr{S}} A$, $x\rho \in E(T/\rho)$. But a cancellative semigroup contains at most one idempotent and so $x\rho = y\rho$ for all $x, y \in \cup_{A \in \mathscr{S}} A$. It follows that $t\rho = u\rho$ for all $t, u \in T$; that is, $\rho = T \times T$. Hence $T \in \mathscr{S}$.

(ii) If $e \in E(S)$ then $\{e\} \in \mathscr{S}$ and so $e \in T$. Thus T is full.

Next, we show that T is unitary. Take any $t \in T$ and $x \in S$. Suppose that $tx \in T$. Let ρ be a cancellative congruence on $\langle T \cup \{x\} \rangle$. Since $\rho \cap (T \times T)$ is a cancellative congruence on T, we see from (i) that $T \times T \subseteq \rho$. Thus, in $\langle T \cup \{x\} \rangle / \rho$,

$$(t\rho)(x\rho) = (tx)\rho = t\rho = (t\rho)^2.$$

Hence $x\rho = t\rho$, by cancellation. This shows that ρ is the universal congruence on $\langle T \cup \{x\} \rangle$. Consequently, $\langle T \cup \{x\} \rangle \subseteq T$ and so $x \in T$. A similar argument shows that if $xt \in T$ then $x \in T$. Thus T is unitary.

(iii) Let ρ be a group congruence on S. Take $A \in \mathscr{S}$. Then, as in the proof of (i), $A \times A \subseteq \rho$. Hence A is contained in an idempotent element of S/ρ; that is, A is contained in the identity ρ-class. Thus the same is true for $\operatorname{hyp}(S) = \langle \cup_{A \in \mathscr{S}} A \rangle$. □

Following Clifford and Preston [**1**, Section 3.2, Ex. 8], we say that a semigroup S is *E-inversive* if and only if for all $x \in S$ there exists $y \in S$ such that $xy \in E(S)$. This property can be shown to have left-right symmetry: indeed S is E-inversive if and only if for all $x \in S$ there exists $z \in S$ such that $xz \in E(S)$ and $zx \in E(S)$. The class of E-inversive semigroups is extensive: besides containing all semigroups with a zero it contains the class of all eventually regular semigroups [**2**], which, in turn, contains all regular semigroups and all group-bound semigroups (see [**4**]).

Characterisations of the hypercore of an E-inversive (in particular, regular) semigroup, in terms of group congruences, are provided by Theorems 2 and 3 below.

Lemma 2. *Every full unitary subsemigroup of an E-inversive [regular] semigroup is E-inversive [regular].*

Proof. Let S be an E-inversive semigroup and T a full unitary subsemigroup of S. Take $x \in T$. Then there exist $y \in S$ and $e \in E(S)$ such that $xy = e$. But $e \in T$, since T is full. Hence, since T is unitary, $y \in T$. Thus T is E-inversive. A similar argument gives the result for the regular case. □

The next lemma was essentially noted by McAlister and O'Carroll [**6**, p. 13]: we omit the proof.

Lemma 3. *Every cancellative congruence on an E-inversive semigroup is a group congruence.* □

Since every semigroup has a least cancellative congruence it follows at once from Lemma 3 that an E-inversive semigroup S always possesses a least group congruence. We denote this congruence by $\sigma(S)$. The identity $\sigma(S)$-class will be called the *core* of S and designated by $\operatorname{core}(S)$. By Lemma 1, $\operatorname{core}(S)$ is a full unitary subsemigroup of S; hence, by Lemma 2, $\operatorname{core}(S)$ is E-inversive and is regular if S is regular. Note that S has no non-universal group congruence if and only if $\operatorname{core}(S) = S$.

Theorem 2. *Let S be an E-inversive [regular] semigroup. Then* $\operatorname{hyp}(S)$ *is the greatest E-inversive [regular] subsemigroup of S with no non-universal group congruence; that is, the greatest E-inversive [regular] subsemigroup A of S with* $\operatorname{core}(A) = A$.

Proof. By Theorem 1(ii), $\operatorname{hyp}(S)$ is a full unitary subsemigroup of S. Hence, by Lemma 2, $\operatorname{hyp}(S)$ is E-inversive [regular]. Also, by Theorem 1(i), $\operatorname{hyp}(S)$ has no non-universal group congruence. Let T be any E-inversive [regular] subsemigroup of S with no non-universal group congruence. Then, by Lemma 3, $T \in \mathscr{S}_S$. Thus $T \subseteq \operatorname{hyp}(S)$. □

For an arbitrary semigroup S with $E(S) \neq \varnothing$ the following two statements are readily verified: (a) if T is a full unitary subsemigroup of S and if U is a full unitary subsemigroup of T then U is a full unitary subsemigroup of S, (b) the intersection of any nonvacuous family of full unitary subsemigroups of S is again a full unitary subsemigroup of S.

Now suppose again that S is E-inversive [regular]. In view of (a) and (b) above, together with Lemmas 1 and 2, we can define a family (S_α) of full unitary (and therefore E-inversive [regular]) subsemigroups of S, indexed by the ordinals, inductively by the rule:

$$S_0 = S, \qquad S_\alpha = \begin{cases} \operatorname{core}(S_\beta) & \text{if } \alpha = \beta + 1, \\ \cap_{\beta < \alpha} S_\beta & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Evidently if $\beta \leqq \alpha$ then $S_\beta \supseteq S_\alpha$. We call (S_α) the *core series* of S. From cardinality considerations it can be seen that there exists a unique ordinal τ such that

$$S_\tau = S_{\tau+1} = S_{\tau+2} = \dots, \qquad S_\alpha \neq S_{\alpha+1} \text{ if } \alpha < \tau.$$

The subsemigroup S_τ is a full unitary E-inversive [regular] subsemigroup of S with the additional property that $\operatorname{core}(S_\tau) = S_\tau$. We call S_τ the *limit* of the core series of S.

Our final result relates the hypercore to the core series of S—and provides motivation for the terminology.

Theorem 3. *Let S be an E-inversive semigroup. Then* $\operatorname{hyp}(S)$ *is the limit of the core series of S.*

Proof. Let (S_α) be the core series of S, with limit S_τ. As remarked above, S_τ is an E-inversive subsemigroup of S with $\operatorname{core}(S_\tau) = S_\tau$. Hence, by Theorem 2, $S_\tau \subseteq \operatorname{hyp}(S)$.

It remains to prove that $\operatorname{hyp}(S) \subseteq S_\tau$. First, $\operatorname{hyp}(S) \subseteq S = S_0$. Assume, inductively, that $\operatorname{hyp}(S) \subseteq S_\gamma$ for all $\gamma < \alpha$. We show that $\operatorname{hyp}(S) \subseteq S_\alpha$. There are two cases.

Case (i): $\alpha = \beta + 1$ for some ordinal β. By Theorem 1(i), $\operatorname{hyp}(S) \in \mathscr{S}_S$. Hence, since $\operatorname{hyp}(S) \subseteq S_\beta$, we have that $\operatorname{hyp}(S) \subseteq \operatorname{core}(S_\beta)$, by Theorem 1(iii) (with S_β replacing S); that is, $\operatorname{hyp}(S) \subseteq S_\alpha$.

Case (ii): α is a limit ordinal. For all $\beta < \alpha$, $\operatorname{hyp}(S) \subseteq S_\beta$. Hence $\operatorname{hyp}(S) \subseteq \cap_{\beta < \alpha} S_\beta = S_\alpha$.

Consequently, by transfinite induction, $\operatorname{hyp}(S) \subseteq S_\tau$. $\square$

2. Remarks and examples

Every semigroup S with $E(S) \neq \varnothing$ contains a least full unitary subsemigroup, namely the intersection of all full unitary subsemigroups of S. We shall denote this by $U(S)$. From Theorem 1(ii), (iii) we see that, for an arbitrary E-inversive semigroup S,

$$U(S) \subseteq \operatorname{hyp}(S) \subseteq \operatorname{core}(S). \tag{1}$$

If S is a semigroup with a zero or if S is an idempotent-generated regular semigroup then clearly $U(S) = \operatorname{hyp}(S) = \operatorname{core}(S) = S$.

Now consider the case of an inverse semigroup S. By [7, Theorem 1], for all $a, b \in S$ we have that

$$(a, b) \in \sigma(S) \quad \Leftrightarrow \quad (\exists e \in E(S))\, ea = eb.$$

Thus $\operatorname{core}(S) = \{a \in S : (\exists e \in E(S)) ea = e\}$ $(= E(S)\omega$, in the notation of [5]), from which it follows that $U(S) = \operatorname{hyp}(S) = \operatorname{core}(S)$.

For a regular semigroup, however, the inequalities in (1) may be strict, as will be demonstrated below. Before proceeding to an example we mention a characterisation, due to Feigenbaum [3], of the core of such a semigroup. A subsemigroup A of a regular semigroup S is termed *self-conjugate* if and only if, for all $a \in A$, for all $x \in S$ and for all inverses x' of x, $x'ax \in A$. The intersection of all the full unitary self-conjugate subsemigroups of S is itself such a subsemigroup and is just core(S).

Example. Let S denote the Rees matrix semigroup $\mathcal{M}(G; I, \Lambda; P)$, where G is a group, I and Λ are nonempty sets and $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix over G [**1**, Section 3.1]. We assume, without loss of generality, that $1 \in I \cap \Lambda$ and that P is normalised so that $p_{1i} = p_{\lambda 1} = e$, the identity of G, for all $i \in I$ and all $\lambda \in \Lambda$. Let H denote the subgroup of G generated by $\{p_{\lambda i}: \lambda \in \Lambda, i \in I\}$. Then, as can readily be verified,

$$U(S) = \mathcal{M}(H; I, \Lambda; P). \tag{2}$$

For any subgroup K of G with $H \subseteq K \subseteq G$ let H^K denote the normal closure of H in K. From the characterisation of core(S) as the least full unitary self-conjugate subsemigroup of S it is straightforward to prove that

$$\text{core}(S) = \mathcal{M}(H^G; I, \Lambda; P). \tag{3}$$

(Furthermore, $S/\sigma(S) \cong G/H^G$, as noted by Stoll [**9**]).

Now define a sequence (N_α) of subgroups of G, indexed by the ordinals, inductively as follows:

$$N_0 = G, \qquad N_\alpha = \begin{cases} H^{N_\beta} & \text{if } \alpha = \beta + 1, \\ \bigcap_{\beta < \alpha} N_\beta & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Thus if $\beta \leqq \alpha$ then $N_\beta \supseteq N_\alpha$. By considering cardinality we see that there is a unique ordinal τ such that

$$N_\tau = N_{\tau+1} = N_{\tau+2} = \dots, \qquad N_\alpha \neq N_{\alpha+1} \text{ if } \alpha < \tau.$$

Let (S_α) denote the core series of S. Then, using (3), we can easily show, by transfinite induction, that

$$(\forall \alpha \leqq \tau)\ S_\alpha = \mathcal{M}(N_\alpha; I, \Lambda; P).$$

In particular, S_τ is the limit of the core series and so, by Theorem 3,

$$\text{hyp}(S) = \mathcal{M}(N_\tau; I, \Lambda; P). \tag{4}$$

Finally, we mention two (finite) special cases.

(a) Take G to be the alternating group of degree 4, $I = \Lambda = \{1, 2\}$ and

$$P = \begin{bmatrix} e & e \\ e & g \end{bmatrix},$$

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where g is an element of G of period 2. Then, from (2), (3) and (4), $U(S)=\text{hyp}(S)=\text{core}(\text{core}(S))\neq\text{core}(S)$.

(b) Replace G in (a) by the alternating group of degree 5 and choose I, Λ, g, P as before. Then $U(S)\neq\text{hyp}(S)=\text{core}(S)=S$.

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