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A CHARACTERIZATION OF UNIVERSAL LOEB MEASURABILITY FOR COMPLETELY REGULAR HAUSDORFF SPACES

J. M. ALDAZ

ABSTRACT. In this paper it is shown that the construction of measures on standard spaces via Loeb measures and the standard part map does not depend on the full structure of the internal algebra being used. A characterization of universal Loeb measurability is given for completely regular Hausdorff spaces, and the behavior of this property under various topological operations is investigated.

1. **Introduction.** Since the time of their discovery ([10]), Loeb measures have often been used to define or represent measures in standard topological spaces by means of the standard part map st (see [13] for the first such application). It is thus natural to ask when the standard part map is a measurable function. Recall that the Baire sets of X (denoted here by $\mathrm{Ba}(X)$) are those sets contained in the smallest σ-algebra making all the continuous real valued functions measurable, or equivalently, those belonging to the smallest σ-algebra generated by $\mathcal{Z}(X)$, the collection of all zero sets of continuous real valued functions on X. In [6], Henson found that for a subset A of a completely regular Hausdorff space X, $\mathrm{st}^{-1}(A)$ belongs to the σ-algebra generated by the internal sets if and only if A is a Baire set in some compactification of X. In particular, if X is a complete metric space, only the separable Borel sets satisfy this condition. It seemed that in this regard, the usefulness of Loeb measures was rather limited, especially considering that Henson worked with the largest possible internal algebra, ${}^*\mathcal{P}(X)$. However, the situation greatly improves if instead of the smallest σ-algebra generated by an internal algebra, we utilize its completion under some Loeb measure. Anderson ([2]) investigated the measurability of the standard part map using the $L({}^*\mu)$-completion of an internal algebra, where μ is some arbitrary standard measure. Loeb showed (in [11]) that if X is compact Hausdorff and B is a Borel subset of X, $\mathrm{st}^{-1}(B)$ belongs to the $L(\nu)$-completion of ${}^*\,\mathrm{Ba}(X)$, for any finite, internal, countably additive Baire measure ν. He has also given some results for noncompact spaces ([11], [12]). In [14], Landers and Rogge defined universal Loeb measurability with respect to all finite, internal, finitely additive measures, producing new results in the theory.

All the set functions considered here will be finite in the standard sense. So, for instance, in the expression "finite, finitely additive internal measure", "finite" means that there exists an $n \in \mathbb{N}$ such that the measure of the space being considered is smaller than

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n, while "finitely additive" means *finitely additive, for otherwise the measure would not be internal. Let μ be an internal measure on an algebra; recall that an internal or external set A is $L(\mu)$-measurable if given any standard $\varepsilon > 0$, there are internal μ-measurable sets A_1 and A_2 with $A_1 \subset A \subset A_2$ and $\mu(A_2) - \mu(A_1) < \varepsilon$. The first theorem of this paper (Theorem 2.2) shows that in order to approximate sets of the form $\mathrm{st}^{-1}(A)$ from above and from below, relatively few internal sets are needed; in particular, if X is a completely regular Hausdorff space, and if B_1 and B_2 are internal subsets of ${}^*X$ satisfying $B_1 \subset \mathrm{st}^{-1}(A) \subset B_2$, then there exist sets $Z_i \in {}^*\mathcal{Z}(X)$, $i = 1, 2$, with $B_1 \subset Z_1 \subset \mathrm{st}^{-1}(A) \subset Z_2 \subset B_2$. This explains why whenever st is measurable, it is already measurable with respect to ${}^*\,\mathrm{Ba}(X)$; in fact, it does not make any difference which internal algebra one is working with, provided it contains the internal zero sets. With these facts in mind, we are going to define universal Loeb measurability in a way that differs from Landers and Rogge's. We shall restrict ourselves to completely regular Hausdorff spaces; in this context, universality will refer to all finite internal contents defined on the lattice ${}^*\mathcal{Z}(X)$; a content is a positive real valued set function which is monotone, subadditive and additive. The basic difference between a content λ and a finite (finitely additive) measure is that the first may fail to be modular, *i.e.*, $\lambda(A) + \lambda(B)$ need not be equal to $\lambda(A \cup B) + \lambda(A \cap B)$. We remark that this definition of content is different from the one Landers and Rogge use in [14]: in their terminology, a content is just a finitely additive measure.

Section 2 also contains a characterization of the universally Loeb measurable spaces, that is, the standard spaces X for which the set of near standard points $\mathrm{ns}({}^*X)$ is universally Loeb measurable. Let st_x be the standard part map for the space X and $\mathrm{st}_{\beta x}$ the standard part map for its Stone-Čech compactification βX. Theorem 2.5 states that $\mathrm{ns}({}^*X)$ is measurable with respect to all finite internal contents defined on ${}^*\mathcal{Z}(X)$, if and only if $\mathrm{st}_{\beta x}^{-1}(X)$ is measurable with respect to all finite internal contents defined on ${}^*\mathcal{Z}(\beta X)$. In Section 3, we show that starting with any finite internal content λ on ${}^*\mathcal{Z}(X)$ one can produce a Radon measure on X, provided $\mathrm{ns}({}^*X)$ is λ-measurable. As in the case of measures, the universal Loeb measurability of $\mathrm{ns}({}^*X)$ is equivalent to the universal Loeb measurability of the standard part map. From this and the characterization given in Section 2, it follows that all Borel subsets of a compact Hausdorff space are universally Loeb measurable, a fact that implies the previously known results. For example, a theorem of Čech states that a completely metrizable space (a space homeomorphic to a complete metric space) is a G_δ subset of its Stone-Čech compactification. Hence completely metrizable spaces are universally Loeb measurable. Actually, all Borel subsets of a universally Loeb measurable space are universally Loeb measurable (Theorem 4.1). As an application, the above construction of Radon measures is used to give a proof of the Riesz Representation Theorem.

We mentioned before that while Loeb examined the Loeb measurability of the standard map st with respect to any finite, internal, countably additive Baire measure, Landers and Rogge [14] consider all finite, internal, finitely additive Baire measures, not just the countably additive ones. One might expect that the class of spaces for which st is universally Loeb measurable in the sense of Landers and Rogge is strictly smaller than in the case studied by Loeb. However, this is not true. By Theorem 3.8, ns($^*X$) is measurable with respect to all finite, internal, finitely additive Baire measures in $^*X$ if and only if it is measurable with respect to all nonstandard extensions of standard, finite countably additive Baire measures in X. An easy consequence of Landers and Rogge's results is that this is exactly the class of (completely regular and Hausdorff) pre-Radon spaces. We also obtain a characterization analogous to that of Theorem 2.5: ns($^*X$) is measurable with respect to all finite, internal, finitely additive Baire measures in $^*X$ if and only if $\mathrm{st}_{\beta x}^{-1}(X)$ is measurable with respect to all nonstandard extensions of finite, countably additive Baire measures in βX. As a corollary, a standard characterization due to Knowles [8] of completely regular and Hausdorff pre-Radon spaces follows. Finally, Section 4 investigates the topological properties of universally Loeb measurable spaces.

We will assume that our nonstandard model is α-saturated, where α is larger than the cardinality of the topology of βX. For easy reference, we state the following well known facts about the Stone-Čech compactification βX of a completely regular Hausdorff space X. A set $Z \subset X$ of the form $Z = \{x \in X : f(x) = 0\}$ for some continuous real valued f, is called a *zero set*. The space X is embedded in βX as a dense subspace, and every bounded, real valued, continuous function f on X has a unique continuous extension $\bar{f}$ to βX. As a consequence, for every zero set Z of X there exists a zero set $\bar{Z}$ of βX with $\bar{Z} \cap X = Z$; for this reason, the set X is said to be *z-embedded* in βX. Furthermore, if Z_1 and Z_2 are disjoint zero sets in X, then there exist disjoint zero sets $\bar{Z}_1$ and $\bar{Z}_2$ in βX such that $Z_i = \bar{Z}_i \cap X$, for $i = 1, 2$. To see why this is true, let $f^{-1}\{0\} = Z_1$ and $g^{-1}\{0\} = Z_2$. The bounded continuous function $h = \frac{f^2}{f^2+g^2}$ satisfies $h^{-1}\{0\} = Z_1$ and $h^{-1}\{1\} = Z_2$. Now $\bar{Z}_1 = \bar{h}^{-1}\{0\}$ and $\bar{Z}_2 = \bar{h}^{-1}\{1\}$ are the desired sets.

A *paving* on a set X is just a collection of subsets of X. It is a $[\emptyset, X, \cap f, \cup f]$-paving if it contains the empty set and X, and is closed under finite intersections and unions. We remark that the collections of zero sets of X, $\mathcal{Z}(X)$, and of cozero sets (complements of zero sets), $\mathcal{U}(X)$, are $[\emptyset, X, \cap f, \cup f]$-pavings. In addition to this, $\mathcal{U}(X)$ forms a base for X whenever X is completely regular and Hausdorff.

As was indicated before, in order to distinguish the standard part map for X from the standard part map for βX, we denote the first by st_x, and the second by $\mathrm{st}_{\beta x}$, so $\mathrm{st}_x\colon \mathrm{ns}(^*X) \longrightarrow X$ and $\mathrm{st}_{\beta x}\colon {}^*\beta X \longrightarrow \beta X$. Concerning the relation between these two maps, we mention that st_x is just the restriction of $\mathrm{st}_{\beta x}$ to ns($^*X$). Let $m(x)$ be the monad of the point x. If the topological space X fails to be Hausdorff, then the standard part map is not well defined. In this case, for any $A \subset X$ we set $\mathrm{st}_x^{-1}(A) = \cup_{x\in A} m(x)$.

2. **Main results.**

DEFINITION 2.1. Given a paving V, cV, *the complement paving of* V, is the collection of all complements of sets in V.

THEOREM 2.2. *Let X be a topological space, A a subset of X, B_1 and B_2 internal subsets of $^*X$ satisfying $B_1 \subset \mathrm{st}_x^{-1}(A) \subset B_2$, and $\mathcal{B}$ a base closed under finite unions and intersections. Then:*

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i) There exist sets $O_i \in {}^*\mathcal{B}$, $i = 1, 2$, *with* $B_1 \subset O_1 \subset \mathrm{st}_x^{-1}(A) \subset O_2 \subset B_2$.

*ii) If X is a regular space, there exist *closed sets* C_i, $i = 1, 2$, *with* $B_1 \subset C_1 \subset \mathrm{st}_x^{-1}(A) \subset C_2 \subset B_2$.

iii) If X is compact, there exist sets $C_i \in {}^*c\mathcal{B}$, $i = 1, 2$, *with* $B_1 \subset C_1 \subset \mathrm{st}_x^{-1}(A) \subset C_2 \subset B_2$.

iv) If X is completely regular and Hausdorff, there exist sets $Z_i \in {}^*\mathcal{Z}(X)$, $i = 1, 2$, *with* $B_1 \subset Z_1 \subset \mathrm{st}_x^{-1}(A) \subset Z_2 \subset B_2$.

PROOF. i) Let $x \in A$. The monad of x, $m(x)$, equals the intersection of all ${}^*O$ for which $O \in \mathcal{B}$ and $x \in O$. Since $m(x) \cap \neg B_2 = \emptyset$, by α-saturation there exist basic open sets O_i, $i = 1, \ldots, n$, $x \in O_i$, $n \in \mathbb{N}$, with

$$\left[\bigcap_{i=1}^{n} {}^*O_i\right] \cap \neg B_2 = \emptyset.$$

Let O_x be the internal set $\bigcap_{i=1}^n {}^*O_i$, and let $\mathcal{C} = \{O_x : x \in A\}$. Finite unions of sets in $\mathcal{C}$ are internal and contained in B_2, so we can extend $\mathcal{C}$ to a hyperfinite collection $D_1, \ldots, D_h$, $h \in {}^*\mathbb{N}$, $D_i \in {}^*\mathcal{B}$, that satisfies

$$O_2 := \bigcup_{i=1}^{h} D_i \subset B_2.$$

Thus

$$O_2 \in {}^*\mathcal{B} \text{ and } \mathrm{st}_x^{-1}(A) \subset \bigcup_{x \in A} O_x \subset O_2 \subset B_2.$$

Next, we find a set $O_1 \in {}^*\mathcal{B}$ with $B_1 \subset O_1 \subset \mathrm{st}_x^{-1}(A)$. Using α-saturation, extend $\{{}^*O : O \in \mathcal{B}\}$ to a hyperfinite collection $J = \{O_1, \ldots, O_h : h \in {}^*\mathbb{N}\}$ of subsets of ${}^*\mathcal{B}$. Select $x \in B_1$, and let O_x be the intersection of all sets in J having x as an element. Then $x \in O_x \in {}^*\mathcal{B}$, and since J is hyperfinite, the collection $\{O_x : x \in B_1\}$ contains only hyperfinitely many different sets. Thus $B_1 \subset \bigcup_{x \in B_1} O_x \in {}^*\mathcal{B}$. To see why $\bigcup_{x \in B_1} O_x \subset \mathrm{st}_x^{-1}(A)$, fix $x \in B_1$, and note that $O_x \subset {}^*G$ for each standard basic open set G containing $\mathrm{st}_x\, x$.

ii) Let X be a regular space. We will show that if B_1 is an internal set with $B_1 \subset \mathrm{st}_x^{-1}(A)$, then the internal closure of B_1 is also contained in $\mathrm{st}_x^{-1}(A)$. Fix $y \notin \mathrm{st}_x^{-1}(A)$. For each $x \in A$ choose a standard open set O_x containing x with $y \notin {}^*O_x$. By regularity of X, given x in A, there is an open set G_x satisfying $x \in G_x \subset c\ell\, G_x \subset O_x$, where $c\ell\, G_x$ stands for the closure of G_x. Since the internal set B_1 is contained in $\bigcup_{x \in A} {}^*G_x$, by α-saturation there exists a finite collection $G_{x_1}, \ldots, G_{x_n}$ with $B_1 \subset \bigcup_{i=1}^n {}^*G_{x_i} \subset \bigcup_{i=1}^n {}^*c\ell\, G_{x_i}$. Hence, if we denote the internal closure of B_1 by C_1, we have $y \notin C_1$, and thus

$$B_1 \subset C_1 \subset \mathrm{st}_x^{-1}(A).$$

To complete the proof of ii), we must show that there exists a *closed set C_2 with

$$\mathrm{st}_x^{-1}(A) \subset C_2 \subset B_2$$

Fix $x \in A$. Since $\neg B_2 \cap m(x) = \emptyset$, by α-saturation there exists a standard open O_x containing x with $\neg B_2 \cap {}^*O_x = \emptyset$. By regularity of X, there is an open set G_x satisfying $x \in G_x \subset c\ell\, G_x \subset O_x$. For any finite subset $x_1, \ldots, x_n$ of A,

$$\neg B_2 \subset \bigcap_{i=1}^{n} \neg{}^*c\ell\, G_{x_i},$$

so by α-saturation there exists an internal open set $\neg C_2$ such that

$$\neg B_2 \subset \neg C_2 \subset \bigcap_{x \in A} \neg{}^*c\ell\, G_x.$$

Therefore

$$\mathrm{st}_x^{-1}(A) \subset \bigcup_{x\in A} {}^*G_x \subset \bigcup_{x \in A} {}^*c\ell\, G_x \subset C_2 \subset B_2.$$

iii) Let X be compact. Then ${}^*X = \mathrm{st}_x^{-1}(A) \cup \mathrm{st}_x^{-1}(X \setminus A)$, so $B_1 \subset \mathrm{st}_x^{-1}(A) \subset B_2$ implies that $\neg B_2 \subset \neg\, \mathrm{st}_x^{-1}(A) = \mathrm{st}_x^{-1}(X \setminus A) \subset \neg B_1$. By Part i) there are internal sets $O_1, O_2 \in {}^*\mathcal{B}$ with

$$\neg B_2 \subset O_2 \subset \mathrm{st}_x^{-1}(X \setminus A) \subset O_1 \subset \neg B_1.$$

Hence $C_1 := \neg O_1 \in {}^*c\mathcal{B}$, $C_2 := \neg O_2 \in {}^*c\mathcal{B}$ and $B_1 \subset C_1 \subset \mathrm{st}_x^{-1}(A) \subset C_2 \subset B_2$.

iv) Now assume X is completely regular and Hausdorff. This case can be reduced to the previous one by embedding X in its Stone-Čech compactification. Since $B_1 \subset \mathrm{st}_x^{-1}(A) \subset B_2$, $B_1 \subset \mathrm{st}_{\beta x}^{-1}(A) \subset B_2 \cup \neg{}^*X$, so by Part iii), with $\mathcal{B} = \mathcal{U}(\beta X)$, there exist *zero subsets Z_1, Z_2 of ${}^*\beta X$ which satisfy

$$B_1 \subset Z_1 \subset \mathrm{st}_{\beta x}^{-1}(A) \subset Z_2 \subset B_2 \cup \neg{}^*X.$$

It is easy to check that ${}^*X \cap \mathrm{st}_{\beta x}^{-1}(A) = \mathrm{st}_x^{-1}(A)$. Hence

$$B_1 \subset Z_1 \cap {}^*X \subset \mathrm{st}_x^{-1}(A) \subset Z_2 \cap {}^*X \subset B_2.$$

This is the desired result, since $Z_1 \cap {}^*X$ and $Z_2 \cap {}^*X$ are in ${}^*\mathcal{Z}(X)$. ■

REMARK. By Theorem 2.2, the condition that two internal measures coincide on the *zero sets is sufficient to ensure that they generate the same standard measure via the Loeb construction. The converse, however, is not true: if λ is Lebesgue measure, and X the unit interval, the λ can be represented by means of ${}^*\lambda$, or by a hyperfinite counting measure μ, but clearly these do not coincide on ${}^*\mathcal{Z}(X)$.

Theorem 2.2 suggests that instead of considering internal finitely additive measures on internal algebras, we take a more general approach. We say that a nonnegative, real valued (and hence finite), set function λ defined on a $[\emptyset, X, \cap f, \cup f]$-paving is a *content* if the following conditions are satisfied:

i) $\lambda(\emptyset) = 0$.
ii) Monotonicity: $A \subset B$ implies that $\lambda(A) \leq \lambda(B)$.
iii) Subadditivity: $\lambda(A \cup B) \leq \lambda(A) + \lambda(B)$.
iv) Additivity: If $A \cap B = \emptyset$, then $\lambda(A) + \lambda(B) = \lambda(A \cup B)$.

Note that i) follows from iv) and the finiteness of λ. A content λ is *modular* (respectively *submodular*, or *strongly subadditive*) if

$$\lambda(A \cup B) + \lambda(A \cap B) = \lambda(A) + \lambda(B) \quad \text{(respectively } \leq\text{)}.$$

We do not require that a content be modular, so it may fail to be extensible to a finitely additive measure. (Such contents exist and are easy to construct.) The notion of content we are using is thus more general than that of Landers and Rogge [14].

DEFINITION 2.3. Let X be a completely regular Hausdorff space, and assume our nonstandard model is α-saturated, with α strictly greater than the cardinality of the topology of βX. Let V be an internal $[\emptyset, X, \cap f, \cup f]$-paving on ${}^*X$, and suppose λ is a finite, internal content on V. A set $A \subset {}^*X$ is (V, λ)-measurable if for every standard $\epsilon > 0$ there exist $A_1, A_2 \in V$ with $A_1 \subset A \subset A_2$ and $\lambda(A_2) - \lambda(A_1) < \epsilon$.

If A is (V, λ)-measurable for all finite internal contents λ then it is said to be *V-universally Loeb measurable*, and we write $A \in L_u(V)$. If $A \in L_u\big({}^*\mathcal{Z}(X)\big)$ we simply say that A is *universally Loeb measurable*. (As indicated in the introduction, $\mathcal{Z}(X)$ is a $[\emptyset, X, \cap f, \cup f]$-paving.)

LEMMA 2.4. *Let μ be a content on $\mathcal{Z}(\beta X)$. Then $\lambda(Z) = \inf\{\mu(\bar{Z}) : \bar{Z} \in \mathcal{Z}(\beta X)$ and $\bar{Z} \cap X = Z\}$ is a content on $\mathcal{Z}(X)$. If μ is submodular, so is λ.*

PROOF. Conditions i) and ii) are obviously satisfied.

iii) Subadditivity: Fix $A, B \in \mathcal{Z}(X)$. Select $\epsilon > 0$ and sets $\bar{A}, \bar{B} \in \mathcal{Z}(\beta X)$ with $\bar{A} \cap X = A$, $\bar{B} \cap X = B$ and $\lambda(A) > \mu(\bar{A}) - \epsilon$, $\lambda(B) > \mu(\bar{B}) - \epsilon$. Then $(\bar{A} \cup \bar{B}) \cap X = A \cup B$ and

$$\begin{aligned}\lambda(A \cup B) &\leq \mu(\bar{A} \cup \bar{B}) \text{ by the definition of } \lambda\\ &\leq \mu(\bar{A}) + \mu(\bar{B}) \text{ by the subadditivity of } \mu\\ &< \lambda(A) + \lambda(B) + 2\epsilon.\end{aligned}$$

The rest is proven in a similar way. ∎

Suppose λ is a content on $\mathcal{Z}(X)$. Define $\bar{\lambda}$, the natural extension of λ to $\mathcal{Z}(\beta X)$, by setting $\bar{\lambda}(\bar{A}) = \lambda(A)$, where $\bar{A} \cap X = A$. Clearly $\bar{\lambda}$ is also a content, which is submodular if λ is. When a subset A of ${}^*\beta X$ is measurable with respect to all natural extensions of finite, internal contents on ${}^*\mathcal{Z}(X)$ we write $A \in L_e\big({}^*\mathcal{Z}(\beta X)\big)$.

THEOREM 2.5. *Let X be a completely regular Hausdorff space. The following are equivalent:*

i) $\mathrm{ns}({}^*X) \in L_u\big({}^*\mathcal{Z}(X)\big)$

ii) $\mathrm{st}_{\beta x}^{-1}(X) \in L_u\big({}^*\mathcal{Z}(\beta X)\big)$

iii) $\mathrm{st}_{\beta x}^{-1}(X) \in L_e\big({}^*\mathcal{Z}(\beta X)\big)$.

PROOF. ii) implies iii) is trivial.

iii) implies i). Let λ be a finite internal content on ${}^*\mathcal{Z}(X)$, $\bar{\lambda}$ its natural extension to ${}^*\mathcal{Z}(\beta X)$. Fix a standard $\epsilon > 0$, and select $Z_1, Z_2 \in {}^*\mathcal{Z}(\beta X)$ with

$$Z_1 \subset \mathrm{st}_{\beta x}^{-1}(X) \subset Z_2 \text{ and } \bar{\lambda}(Z_2) - \bar{\lambda}(Z_1) < \epsilon.$$

Then for $i = 1, 2$, $Z_i \cap {}^*X \in {}^*\mathcal{Z}(X)$,

$$Z_1 \cap {}^*X \subset [\mathrm{st}_{\beta x}^{-1}(X)] \cap {}^*X = \mathrm{ns}({}^*X) \subset Z_2 \cap {}^*X, \text{ and } \lambda(Z_2 \cap {}^*X) - \lambda(Z_1 \cap {}^*X) < \epsilon.$$

i) implies ii). Let μ be a finite internal content on ${}^*\mathcal{Z}(\beta X)$ and set $\lambda(Z)$ = internal $\inf\{\mu(\bar{Z}) : \bar{Z} \in {}^*\mathcal{Z}(\beta X) \text{ and } \bar{Z} \cap {}^*X = Z\}$. By the transfer of Lemma 2.4, λ is an internal content on ${}^*\mathcal{Z}(X)$. Let ϵ be a positive standard real number. Choose $Z_1, Z_2 \in {}^*\mathcal{Z}(X)$ with

$$Z_1 \subset \mathrm{ns}({}^*X) \subset Z_2 \text{ and } \lambda(Z_2) - \lambda(Z_1) < \epsilon.$$

By Theorem 2.2 we can select $\bar{Z}_1 \in {}^*\mathcal{Z}(\beta X)$ so that

$$Z_1 \subset \bar{Z}_1 \subset \mathrm{st}_{\beta x}^{-1}(X).$$

We also can choose $\bar{Z}_2 \in {}^*\mathcal{Z}(\beta X)$ satisfying $\bar{Z}_2 \cap {}^*X = Z_2$ and $\lambda(Z_2) + \epsilon > \mu(\bar{Z}_2)$, whence

$$\mu(\bar{Z}_2) < \lambda(Z_1) + 2\epsilon \leq \mu(\bar{Z}_1) + 2\epsilon.$$

It remains to be shown that $\mathrm{st}_{\beta x}^{-1}(X)$ is a subset of $\bar{Z}_2$. Fix $y \in \mathrm{st}_{\beta x}^{-1}(X)$. We shall prove that y is a point of closure of the *closed set $\bar{Z}_2$, whence $y \in \bar{Z}_2$. Let O be any internal neighborhood of y. By α-saturation there exists an internal cozero O_y satisfying $y \in O_y \subset O \cap m[\mathrm{st}_{\beta x}(y)]$. Now, ${}^*X$ is dense in ${}^*\beta X$, and thus there exists some $z \in {}^*X \cap O_y$. But then $z \in {}^*X \cap \mathrm{st}_{\beta x}^{-1}(X) = \mathrm{ns}({}^*X) \subset \bar{Z}_2$, so we are done. ∎

3. **Construction of standard measures and applications.** In the process of representing standard measures via Loeb measures and the standard part map, one starts with some internal, finitely additive measure μ defined on an internal algebra of sets. By Theorem 2.2, the same results can be achieved when X is completely regular and Hausdorff, if we restrict μ to ${}^*\mathcal{Z}(X)$. In this section we prove that any finite internal content λ on ${}^*\mathcal{Z}(X)$ generates a Radon measure defined on the Borel sets of X, provided ns(${}^*X$) is λ-measurable; this is so even if λ is not the restriction to ${}^*\mathcal{Z}(X)$ of an internal measure. The lack of closure under complementation of ${}^*\mathcal{Z}(X)$ adds some technical difficulties, but the proofs are basically the same as in the case of measure on algebras. As applications we give a proof of the Riesz Representation Theorem and a nonstandard characterization of pre-Radon spaces, from which the usual standard characterization immediately follows. Recall that given a paving V, cV denotes the collection of all complements of sets in V.

LEMMA 3.1. *Let λ be a content on a $[\emptyset, X, \cap f, \cup f]$-paving V. Then λ can be extended to a monotone set function μ on $V \cup cV$ by setting $\mu(Z) = \lambda(Z)$ for $Z \in V$ and $\mu(O) = \lambda(X) - \lambda(\neg O)$ for $O \in cV$.*

PROOF. Fix $A, B \in V \cup cV$, with $A \subset B$.
If $A, B \in V$, the result is obvious since λ is monotone.
If $A \in V$, $B \in cV$, then

$$\lambda(A) + \lambda(\neg B) = \lambda(A \cup \neg B) \text{ by the additivity of } \lambda$$
$$\leq \lambda(X) \text{ by the monotonicity of } \lambda, \text{ so}$$
$$\mu(A) = \lambda(A) \leq \lambda(X) - \lambda(\neg B) = \mu(B).$$

The other cases are similar. ∎

LEMMA 3.2. *Let λ and μ be given as in Lemma 3.1. If $Z \in V$, $O \in cV$, and $Z \subset O$, then $\mu(O \setminus Z) = \mu(O) - \mu(Z)$.*

PROOF.

$$\begin{aligned}\mu(O \setminus Z) &= \mu(X) - \mu(\neg O \cup Z) \text{ by definition}\\ &= \lambda(X) - \lambda(\neg O) - \lambda(Z) \text{ by the additivity of } \lambda \text{ on } V\\ &= \mu(O) - \mu(Z).\end{aligned}$$

■

From now on, we will assume that any internal content λ on ${}^*\mathcal{Z}(X)$ has already been extended to ${}^*\mathcal{Z}(X) \cup {}^*\mathcal{U}(X)$ in the way indicated by the transfer of Lemma 3.1; the extension will also be denoted by λ. Recall (Definition 2.3) that the set $\mathrm{st}_x^{-1}(A)$ is $({}^*\mathcal{Z}(X), \lambda)$-measurable if for all standard $\epsilon > 0$, there exist internal zero sets Z_i, $i = 1, 2$, such that $Z_1 \subset \mathrm{st}_x^{-1}(A) \subset Z_2$ and $\lambda(Z_2) - \lambda(Z_1) < \epsilon$. By Theorem 2.2 and the transfer of Lemma 3.1, $\mathrm{st}_x^{-1}(A)$ is $({}^*\mathcal{Z}(X), \lambda)$-measurable if and only if there are sets $O_1, O_2 \in {}^*\mathcal{U}(X)$ with

$$O_1 \subset \mathrm{st}_x^{-1}(A) \subset O_2 \text{ and } \lambda(O_2) - \lambda(O_1) < \epsilon.$$

An analogous statement holds if either O_1 or O_2 is replaced by a set Z in ${}^*\mathcal{Z}(X)$.

Next we show that the $({}^*\mathcal{Z}(X), \lambda)$-measurability of $\mathrm{ns}({}^*X)$ implies that the collection of subsets A of X for which $\mathrm{st}_x^{-1}(A)$ is $({}^*\mathcal{Z}(X), \lambda)$-measurable is a σ-algebra $\mathcal{A}$, containing the Borel sets of X. The set function $L(\lambda) \circ \mathrm{st}_x^{-1}$, defined on $\mathcal{A}$ by:

$$\begin{aligned}L(\lambda) \circ \mathrm{st}_x^{-1}(A) &:= \inf\{{}^\circ\lambda(Z) : Z \in {}^*\mathcal{Z}(X) \text{ and } Z \supset \mathrm{st}_x^{-1}(A)\}\\ &= \sup\{{}^\circ\lambda(Z) : Z \in {}^*\mathcal{Z}(X) \text{ and } Z \subset \mathrm{st}_x^{-1}(A)\},\end{aligned}$$

is a complete, countably additive measure. The proof of countable additivity is the same as in the usual case, when one starts with an internal measure (see, for instance, [9]). It will, therefore, be omitted.

THEOREM 3.3. *Let X be a completely regular Hausdorff space, and assume $\mathrm{ns}({}^*X)$ is $({}^*\mathcal{Z}(X), \lambda)$-measurable. Then the collection $\mathcal{C}$ of subsets C of X for which $\mathrm{st}_x^{-1}(C)$ is $({}^*\mathcal{Z}(X), \lambda)$-measurable is a complete σ-algebra.*

PROOF. Fix a positive standard ϵ.

i) $\mathcal{C}$ is closed under complementation: Let $\mathrm{st}_x^{-1}(C)$ be $({}^*\mathcal{Z}(X), \lambda)$-measurable. For $i = 1, 2$, select internal cozero sets O_i, D, and internal zero sets L_i, Z, with

$$\begin{gathered}O_1 \subset \mathrm{ns}({}^*X) \subset Z \subset O_2,\\ \lambda(O_2) - \lambda(O_1) < \epsilon, \text{ and}\\ L_1 \subset D \subset \mathrm{st}_x^{-1}(C) \subset L_2,\\ \lambda(L_2) - \lambda(L_1) < \epsilon.\end{gathered}$$

Then

$$\begin{aligned}\lambda(Z \setminus D) &\le \lambda(O_2 \setminus L_1) \text{ by Lemma 3.1}\\ &= \lambda(O_2) - \lambda(L_1) \text{ by Lemma 3.2}\\ &< \lambda(Z) + \epsilon - \lambda(L_1),\end{aligned}$$

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and

$$\begin{aligned}\lambda(O_1 \setminus L_2) &= \lambda(X) - \lambda(\neg O_1 \cup L_2)\\ &\geq \lambda(X) - \lambda(\neg O_1) - \lambda(L_2) \text{ by the subadditivity of } \lambda\\ &= \lambda(O_1) - \lambda(L_2).\end{aligned}$$

Now

$$(Z \setminus D) \supset \mathrm{st}_x^{-1}(\neg C) = \mathrm{ns}(^*X) \setminus \mathrm{st}_x^{-1}(C) \supset (O_1 \setminus L_2),$$

and also

$$\begin{aligned}\lambda(Z \setminus D) - \lambda(O_1 \setminus L_2) &< \lambda(Z) + \epsilon - \lambda(L_1) - [\lambda(O_1) - \lambda(L_2)]\\ &= \lambda(Z) - \lambda(O_1) + \lambda(L_2) - \lambda(L_1) + \epsilon\\ &< 3\epsilon.\end{aligned}$$

ii) $\mathcal{C}$ is closed under countable unions: For each standard positive integer n select $C_n \in \mathcal{C}$, and $Z_{ni} \in {}^*\mathcal{Z}(X)$, $O_{ni} \in {}^*\mathcal{U}(X)$, $i = 1, 2, 3$, with

$$Z_{n1} \subset O_{n1} \subset Z_{n2} \subset \mathrm{st}_x^{-1}(C_n) \subset O_{n2} \subset Z_{n3} \subset O_{n3} \text{ and } \lambda(O_{n3}) - \lambda(Z_{n1}) < \epsilon 2^{-n}.$$

Then

$$\bigcup_{n=1}^{k} Z_{n2} \subset \mathrm{st}_x^{-1}\Big(\bigcup_{n=1}^{k} C_n\Big) \subset \bigcup_{n=1}^{k} O_{n2}.$$

Also

$$\begin{aligned}\lambda\Big(\bigcup_{n=1}^{k} O_{n2}\Big) - \lambda\Big(\bigcup_{n=1}^{k} Z_{n2}\Big) &= \lambda\Big(\bigcup_{n=1}^{k} O_{n2} \setminus \bigcup_{n=1}^{k} Z_{n2}\Big) \text{ by Lemma 3.2}\\ &\leq \lambda\Big(\bigcup_{n=1}^{k} [O_{n2} \setminus Z_{n2}]\Big) \text{ by Lemma 3.1}\\ &\leq \lambda\Big(\bigcup_{n=1}^{k} [Z_{n3} \setminus O_{n1}]\Big) \text{ by Lemma 3.1}\\ &\leq \sum_{n=1}^{k} \lambda(Z_{n3} \setminus O_{n1}) \text{ by the subadditivity of } \lambda\\ &\leq \sum_{n=1}^{k} \lambda(O_{n3} \setminus Z_{n1}) \text{ by Lemma 3.1}\\ &= \sum_{n=1}^{k} [\lambda(O_{n3}) - \lambda(Z_{n1})] \text{ by Lemma 3.2}\\ &< \epsilon.\end{aligned}$$

Let $a = \lim_k {}^\circ\lambda(\bigcup_{n=1}^{k} Z_{n2})$. Then for all standard positive integers k

$$\lambda\Big(\bigcup_{n=1}^{k} O_{n2}\Big) < a + \epsilon.$$

By α-saturation we can extend the collection

$$\{O_{n2} : n \text{ is a standard positive integer}\}$$

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to a hyperfinite set

$$\{O_{n2} : 1 \leq n \leq h, h \in {}^*\mathbb{N}\} \text{ with } \lambda\Big(\bigcup_{n=1}^{h} O_{n2}\Big) < a + \epsilon.$$

Let $m \in \mathbb{N}$ be such that

$$\lambda\Big(\bigcup_{n=1}^{m} Z_{n2}\Big) > a - \epsilon.$$

Then we have

$$\bigcup_{n=1}^{m} Z_{n2} \subset \mathrm{st}_x^{-1}\Big(\bigcup_{n\in\mathbb{N}} C_n\Big) \subset \bigcup_{n=1}^{h} O_{n2}, \text{ and } \lambda\Big(\bigcup_{n=1}^{h} O_{n2}\Big) - \lambda\Big(\bigcup_{n=1}^{m} Z_{n2}\Big) < 2\epsilon.$$

Finally, if C is a subset of X and for every standard $\epsilon > 0$ there exists an internal zero set Z such that $\mathrm{st}_x^{-1}(C) \subset Z$ and $\lambda(Z) < \epsilon$, then C belongs to $\mathcal{C}$, whence $\mathcal{C}$ is complete. ■

Before we prove that $\mathcal{C}$ contains the Borel sets of X, some preliminary results are needed.

DEFINITION 3.4. The standard part map of X, st_X, is said to be $({}^*\mathcal{Z}(X), \lambda)$*-measurable* (respectively *universally Loeb measurable*) if $\mathrm{st}_x^{-1}(B)$ is $({}^*\mathcal{Z}(X), \lambda)$-measurable for every Borel subset B of X (respectively if $\mathrm{st}_x^{-1}(B) \in L_u\big({}^*\mathcal{Z}(X)\big)$ for every Borel subset B of X).

The proofs of the next two theorems are essentially identical to those appearing in Theorems 1 and 5 of [14], so they will be omitted. Theorem 3.5 was established by Landers and Rogge for finite, finitely additive measures. The result appearing in Theorem 3.6 was proven by Loeb when λ is a finite internal measure and X is compact [11], or locally compact [12], and by Landers and Rogge for all completely regular spaces (Theorem 5, [14]).

THEOREM 3.5. *Let V be an internal $[\emptyset, X, \cap f, \cup f]$-paving on a set X, and let $\mathcal{C} \subset V$ be a collection with cardinality less than α. Then $\cup\mathcal{C}$ and $\cap\mathcal{C}$ are V-universally Loeb measurable.*

THEOREM 3.6. *Let X be a completely regular Hausdorff space, and let λ be a finite internal content on ${}^*\mathcal{Z}(X)$. If* $\mathrm{ns}({}^*X)$ *is $({}^*\mathcal{Z}(X), \lambda)$-measurable then so is* st_x.

It now follows that if λ is a finite internal content on ${}^*\mathcal{Z}(X)$ and $\mathrm{ns}({}^*X)$ is $({}^*\mathcal{Z}(X), \lambda)$-measurable, then $L(\lambda) \circ \mathrm{st}_x^{-1}$ is a complete, countably additive measure defined on a σ-algebra containing the Borel sets of X. Furthermore, $L(\lambda) \circ \mathrm{st}_x^{-1}$ is Radon. Recall that a finite, countably additive Borel measure is *Radon* if it is inner regular with respect to the collection of compact sets. Let A be a Borel subset of X and B an internal zero set contained in $\mathrm{st}_x^{-1}(A)$. By Theorem 3.6.1 of [15], $\mathrm{st}_x(B)$ is compact, and hence Borel, so $\mathrm{st}_x(B)$ is $L(\lambda) \circ \mathrm{st}_x^{-1}$-measurable and $B \subset \mathrm{st}_x^{-1}\big(\mathrm{st}(B)\big) \subset \mathrm{st}_x^{-1}(A)$, from which it follows that $L(\lambda) \circ \mathrm{st}_x^{-1}$ is Radon.

REMARKS. 1) Theorem 3.6 implies that st_x is universally Loeb measurable if (and only if) $\mathrm{ns}({}^*X)$ is. The set $\mathrm{ns}({}^*\beta X) = {}^*\beta X$ is obviously universally Loeb measurable in ${}^*\beta X$, so by Theorems 2.5 and 3.6, if X is a Borel subset of βX then $\mathrm{ns}({}^*X)$ is universally

Loeb measurable. This fact entails the previously known results: ns($^*X$) is universally Loeb measurable if X is compact [14, 11], σ-compact [14], locally compact [14, 12] (for a locally compact space is open in any of its compactifications), or completely metrizable [14, 1] (since a completely metrizable space is a G_δ set in its Stone-Čech compactification). A completely regular Hausdorff space X is said to be *Čech complete* if it is G_δ in βX. Hence ns($^*X$) is universally Loeb measurable whenever X is Čech complete.

2) It was indicated in the introduction that if Z_1 and Z_2 are disjoint zero subsets of a completely regular Hausdorff space X, then there exists a continuous function h which is identically zero on Z_1 and identically one on Z_2. Hence it is possible to find disjoint cozero sets O_1 and O_2 containing Z_1 and Z_2 respectively (just take $O_1 = h^{-1}\{[0, \frac{1}{2})\}$, $O_2 = h^{-1}\{(\frac{1}{2}, 1]\}$). We can use this observation to derive the Riesz Representation Theorem:

THEOREM 3.7. *Let X be a compact Hausdorff space and I a positive linear functional on $C(X)$, the space of continuous real valued functions on X. Then there exists a Radon measure μ that represents I, i.e., for every $f \in C(X)$, $I(f) = \int_X f\,d\mu$.*

PROOF. Define λ on $^*\mathcal{Z}(X)$ by setting $\lambda(Z) = {}^*\inf\{^*I(f) : f \in {}^*C(X) \text{ and } f \geq \chi_Z\}$. The set function λ is clearly finite, monotone, subadditive, $\lambda(\emptyset) = 0$, and by transfer of the above observation, finitely additive, so it is a finite internal content. Thus $\mu := L(\lambda) \circ \mathrm{st}_x^{-1}$ is a Radon measure. Next we prove that for all f in $C(X)$, $I(f) = \int_X f\,d\mu$. Fix $f \in C(X)$ and a positive standard ϵ. Assume the range of f is contained in the interval $[a, b]$. Select $y_0 < a < y_1 < \cdots < y_n = b$ so that for $i = 1, 2, \ldots, n$, $y_i - y_{i-1} < \epsilon$. Let $E_i = \{x : y_i \geq f(x) > y_{i-1}\}$. Because each E_i is Borel, we can find internal zero sets $Z_i \subset \mathrm{st}_x^{-1}(E_i)$ with $\mu(E_i) < \lambda(Z_i) + \frac{\epsilon}{n}$. Choose (disjoint) internal cozero sets O_i such that $Z_i \subset O_i \subset \mathrm{st}_x^{-1}(E_i)$ (by Theorem 2.2 i) and internal continuous $^*[0, 1]$-valued functions h_i satisfying $h_i \equiv 1$ on Z_i, $h_i \equiv 0$ off O_i. The continuity of f implies that for $z \in \mathrm{st}_x^{-1}(E_i)$, $^*f(z) > y_{i-1}$. Since $^*f \geq \sum_1^n h_i {}^*f$, it follows that

$$\begin{aligned} I(f) = {}^*I(^*f) &\geq \sum_1^n {}^*I(h_i {}^*f) > \sum_1^n (y_i - \epsilon)^*I(h_i) \geq \sum_1^n (y_i - \epsilon)\,\lambda(Z_i) \\ &> \sum_1^n (y_i - \epsilon)\Big(\mu(E_i) - \frac{\epsilon}{n}\Big) \geq \sum_1^n y_i \mu(E_i) - b\epsilon - \epsilon\mu(X) + \epsilon^2 \\ &\geq \int_X f\,d\mu - b\epsilon - \epsilon\mu(X). \end{aligned}$$

Therefore

$$I(f) \geq \int_X f\,d\mu$$

and also

$$-I(f) = I(-f) \geq \int_X (-f)\,d\mu = -\int_X f\,d\mu,$$

completing the proof. ∎

Note that in the preceding proof, the theorem on partitions of unity was not needed; by Theorem 2.2 we were able to choose the internal functions h_i so that they had disjoint supports. For prior use of nonstandard analysis in proving the Riesz Representation Theorem, see [4], [12], [16], [20].

In the rest of this section we will investigate the question of when $\mathrm{ns}({}^*X)$ is measurable with respect to all finite, finitely additive internal measures defined on the internal algebra generated by ${}^*\mathcal{Z}(X)$. A surprising answer is given by Theorem 3.8: one only needs to check whether $\mathrm{ns}({}^*X)$ is measurable with respect to all nonstandard extensions of finite, countably additive Baire measures. Afterwards, we present a theorem, analogous to Theorem 2.5, which gives a characterization of the spaces X for which $\mathrm{ns}({}^*X)$ has the above property. A standard characterization of such spaces is also provided.

Some preliminary definitions follow. Let $\mathcal{G}$ be an upwards directed collection of open subsets of a topological space X, that is, if A and B are in $\mathcal{G}$, then there is a set C in $\mathcal{G}$ with $A \cup B \subset C$. A Borel measure μ in X is called *τ-additive* (or *τ-smooth*) if for each such $\mathcal{G}$, $\mu(\cup \mathcal{G}) = \sup\{\mu(G) : G \in \mathcal{G}\}$. A Hausdorff space is called *Radon* if each finite (countably additive) Borel measure in X is Radon, *pre-Radon* if each finite τ-additive Borel measure in X is Radon. A subspace X of a Hausdorff space Y is *Radon measurable in* Y if it is $\hat{\mu}$-measurable for every finite Radon measure μ in Y, where $\hat{\mu}$ is the outer measure generated by μ. We denote by $\mathrm{Ba}(X)$ (respectively $\mathrm{aBa}(X)$) the σ-algebra (respectively the algebra) generated by the zero sets of X, and designate the Borel sets of X by $\mathrm{Bo}(X)$. A finite, internal, finitely additive measure λ defined on ${}^*\,\mathrm{aBa}(X)$ is said to be **zero set regular* if for all A in ${}^*\,\mathrm{aBa}(X)$, $\lambda(A) = {}^*\sup\{\lambda(Z) : Z \subset A$ and $Z \in {}^*\mathcal{Z}(X)\}$. We write $A \in L_{\mathrm{um}}\big({}^*\,\mathrm{aBa}(X)\big)$ (respectively $A \in L_{\mathrm{urm}}\big({}^*\,\mathrm{aBa}(X)\big)$) if A is universally Loeb measurable with respect to all finite, internal, finitely additive measures on $\big({}^*X, {}^*\,\mathrm{aBa}(X)\big)$, (respectively if A is universally Loeb measurable with respect to all finite, internal, finitely additive measures that are *zero set regular on $\big({}^*X, {}^*\,\mathrm{aBa}(X)\big)$). If instead of finitely additive measures we are dealing with the countably additive ones, we use $L_{\mathrm{u}}\sigma\big({}^*\,\mathrm{Ba}(X)\big)$.

Let λ be a finite, internal, finitely additive measure on ${}^*\,\mathrm{aBa}(X)$. Landers and Rogge define for each $A \subset {}^*X$

$$\bar{L}(\lambda)(A) = \inf\{L(\lambda)(B) : A \subset B \in {}^*\,\mathrm{aBa}(X)\} \text{ and}$$
$$\underline{L}(\lambda)(A) = \sup\{L(\lambda)(B) : A \supset B \in {}^*\,\mathrm{aBa}(X)\}.$$

They also prove that if μ is a finite τ-smooth Borel measure in X, then $\bar{L}({}^*\mu) \circ \mathrm{st}_x^{-1} \equiv \mu$ on $\mathrm{Bo}(X)$ (Theorem 9. ii of [14], together with the fact that X is a completely regular Hausdorff space, so μ is regular).

THEOREM 3.8. *Let X be a completely regular Hausdorff space. The following are equivalent:*

1) $\mathrm{ns}({}^*X) \in L_{\mathrm{um}}\big({}^*\,\mathrm{aBa}(X)\big)$.
2) $\mathrm{ns}({}^*X) \in L_{\mathrm{um}}\big({}^*\mathcal{P}(X)\big)$.
3) $\mathrm{ns}({}^*X) \in L_{\mathrm{urm}}\big({}^*\,\mathrm{aBa}(X)\big)$.
4) $\mathrm{ns}({}^*X) \in L_{\mathrm{u}}\sigma\big({}^*\,\mathrm{Ba}(X)\big)$.
5) $\mathrm{ns}({}^*X)$ *is* $({}^*\,\mathrm{Ba}(X), {}^*\lambda)$*-measurable for all finite, countably additive measures* λ *on* $\mathrm{Ba}(X)$.

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6) X is pre-Radon.

PROOF. 1) $\Leftrightarrow$ 2) One direction is obvious. The other follows by Theorem 2.2. iv together with the transfer of a theorem of Horn and Tarski [7] stating that every finite measure on an algebra can be extended to a finite measure on the power set of X. 1) $\Rightarrow$ 3) and 4) $\Rightarrow$ 5) are obvious. 3) $\Rightarrow$ 4) is a consequence of the fact that a countably additive measure defined on the Baire sets is automatically zero set regular. 5) $\Rightarrow$ 6) Let the countably additive Borel measure λ be τ-smooth in X, and denote its restriction to $\mathrm{Ba}(X)$ also by λ. Then on $\mathrm{Bo}(X)$, $L({}^*\lambda)\circ\mathrm{st}_x^{-1} \equiv \bar{L}({}^*\lambda)\circ\mathrm{st}_x^{-1}$ is Radon and $\lambda \equiv \bar{L}({}^*\lambda)\circ\mathrm{st}_x^{-1}$, by the result of Landers and Rogge mentioned before, so X is pre-Radon. 6) $\Rightarrow$ 1) Let λ be a finite internal finitely additive measure on ${}^*\,\mathrm{aBa}(X)$. $\bar{L}(\lambda)\circ\mathrm{st}_x^{-1}$ is a τ-smooth measure on $\mathrm{Bo}(X)$ (Theorem 4. ii of [14]) so it is Radon. Therefore $\mathrm{ns}({}^*X)$ is λ-measurable. ∎

In particular, this theorem tells us that for every (completely regular and Hausdorff) Radon space X, the set of near standard points of X is μ-measurable with respect to all finite, internal, finitely additive Baire measures μ. Theorem 3.8 raises the following question: is universal Loeb measurability with respect to all finite internal contents equivalent to all the other notions appearing in Theorem 3.8? At the present moment, the answer is unknown to me. If there is a counter example, then by Theorem 3.10 and the first remark after Theorem 3.6, such a space must be a non-Borel, Radon measurable subspace in its Stone-Čech compactification.

Next we give a characterization of the spaces X for which $\mathrm{ns}({}^*X)$ is universally Loeb measurable with respect to all finite, internal, finitely additive measures on $\big({}^*X, {}^*\,\mathrm{aBa}(X)\big)$.

THEOREM 3.9. *Let X be a completely regular Hausdorff space. The following are equivalent:*

i) $\mathrm{ns}({}^*X) \in L_{\mathrm{um}}\big({}^*\,\mathrm{aBa}(X)\big)$

ii) $\mathrm{st}_{\beta x}^{-1}(X) \in L_{\mathrm{urm}}\big({}^*\,\mathrm{aBa}(\beta X)\big)$

iii) $\mathrm{st}_{\beta x}^{-1}(X)$ *is* $({}^*\,\mathrm{Ba}(\beta X), {}^*\lambda)$*-measurable for all finite, countably additive measures* λ *on* $\mathrm{Ba}(\beta X)$.

PROOF. ii) implies iii) is obvious. For i) implies ii) we need, given an arbitrary, finite, internal, finitely additive *zero set regular measure μ on ${}^*\,\mathrm{aBa}(\beta X)$, a corresponding λ on ${}^*\,\mathrm{aBa}(X)$ with the same property as in Lemma 2.4; that is, if $Z \in \mathcal{Z}(X)$, then $\lambda(Z)$ is the internal infimum of all $\mu(\bar{Z})$ where $\bar{Z} \in {}^*\mathcal{Z}(\beta X)$ and $\bar{Z} \cap {}^*X = Z$. By transfer of the Alexandroff Representation Theorem, we know that given such a μ there is an internal bounded linear functional $\bar{I}$ defined on the space of internal, *continuous, *real valued functions on ${}^*\beta X$ with $\bar{I}(\bar{f}) = \int_{{}^*\beta X} \bar{f}\, d\mu$. Now define I on the space of internal, *bounded, *continuous, *real valued functions on ${}^*X$ by $I(f) = \bar{I}(\bar{f})$, where $\bar{f}$ is the unique internal, *continuous extension of f to ${}^*\beta X$. We apply the transfer of the Alexandroff Representation Theorem once more to obtain the measure λ corresponding to I. If $O \in {}^*\,\mathcal{U}(X)$, then

$$\lambda(O) = {}^*\sup\{I(f) : 0 \leq f \leq \chi_O\}$$

(see, for instance, the proof of the Alexandroff Representation Theorem in [19]). In what follows, $\bar{O}$ *does not* denote the closure of the cozero set $O \subset {}^*X$, but a cozero subset of ${}^*\beta X$ whose intersection with ${}^*X$ is O. Given any $\bar{O} \in {}^*\mathcal{U}(\beta X)$ with $\bar{O} \cap {}^*X = O$, and any $\bar{f}$ such that $0 \leq \bar{f} \leq \chi_{\bar{O}}$, the function $f = \bar{f}|{}^*X$ satisfies $0 \leq f \leq \chi_O$. Clearly,

$$\begin{aligned}\lambda(O) &= {}^*\sup\{\bar{I}(\bar{f}) : 0 \leq \bar{f} \leq \chi_{\bar{O}}, \bar{O} \in {}^*\mathcal{U}(\beta X) \text{ and } \bar{O} \cap {}^*X = O\}\\ &= {}^*\sup\{\mu(\bar{O}) : \bar{O} \in {}^*\mathcal{U}(\beta X) \text{ and } \bar{O} \cap {}^*X = O\}.\end{aligned}$$

Hence for any $Z \in {}^*\mathcal{Z}(X)$,

$$\begin{aligned}\lambda(Z) = \lambda({}^*X) - \lambda(\neg Z) = \mu({}^*\beta X) - {}^*\sup\{\mu(\bar{O}) : \bar{O} \in {}^*\mathcal{U}(\beta X) \text{ and } \bar{O} \cap X = \neg Z\}\\ = {}^*\inf\{\mu(\bar{Z}) : \bar{Z} \cap X = Z \text{ and } \bar{Z} \in {}^*\mathcal{Z}(\beta X)\}.\end{aligned}$$

The rest of the proof that Part i) implies Part ii) is as in Theorem 2.5, [i) implies ii)].

iii) $\Rightarrow$ i) By Theorem 3.8 it is enough to prove that for every finite, countably additive measure λ on $\text{Ba}(X)$, the set $\text{ns}({}^*X)$ is $({}^*\text{Ba}(X), {}^*\lambda)$-measurable, so select any such λ. The natural extension $\bar{\lambda}$ of λ, defined by $\bar{\lambda}(A) = \lambda(A \cap X)$ for all $A \in \text{Ba}(\beta X)$, is a finite, countably additive Baire measure in βX. Now the rest of the proof is the same as in iii) $\Rightarrow$ i) of Theorem 2.5. ∎

THEOREM 3.10. *Let X be a completely regular Hausdorff space. Then* $\text{ns}({}^*X) \in L_{\text{um}}\big({}^*\text{aBa}(X)\big)$ *if and only if X is Radon measurable in βX.*

PROOF. By Theorem 3.9, $\text{ns}({}^*X) \in L_{\text{um}}\big({}^*\text{aBa}(X)\big)$ if and only if $\text{st}_{\beta x}^{-1}(X)$ is $({}^*\text{Ba}(\beta X), {}^*\lambda)$-measurable for all finite, countably additive measures λ on $\text{Ba}(\beta X)$. Assume $\text{ns}({}^*X) \in L_{\text{um}}\big({}^*\text{aBa}(X)\big)$, and let λ be a fixed Radon measure in βX. Restrict λ to the Baire sets of βX, and denote the restriction also by λ. Then $\text{st}_{\beta x}^{-1}(X)$ is ${}^*\lambda$-measurable, and so X is $\hat{\lambda}$-measurable, since $\lambda \equiv L({}^*\lambda) \circ \text{st}_{\beta x}^{-1}$ (Theorem 3.3 of [2]).

For the other direction, again by Theorem 3.9, it is enough to show that if λ is a finite, countably additive measure on $\text{Ba}(\beta X)$, then the set $\text{st}_{\beta x}^{-1}(X)$ is $({}^*\text{Ba}(\beta X), {}^*\lambda)$-measurable. But $L({}^*\lambda) \circ \text{st}_{\beta x}^{-1}$ is a Radon measure in βX, so by hypothesis X is $L({}^*\lambda) \circ \text{st}_{\beta x}^{-1}$-measurable, concluding the proof. ∎

From Theorems 3.8 and 3.10 we obtain the following standard result, due to Knowles ([8], Theorem 3.4, also Corollary 11.8 of [5], whose terminology we follow).

COROLLARY 3.11. *Let X be a completely regular Hausdorff space. Then X is pre-Radon if and only if it is Radon measurable in its Stone-Čech compactification βX.*

This corollary is important because, among other reasons, one can obtain from it the most general known criterion for deciding whether a completely regular Hausdorff space is Radon (see [5]).

4. **Topological considerations.** Recall that subspaces and products of completely regular Hausdorff spaces are completely regular and Hausdorff. Clearly, the universal Loeb measurability of ns($^*X$) is a topological property of the space X. When ns($^*X$) is universally Loeb measurable, or equivalently (by Theorem 3.6), when st_x is universally Loeb measurable, we also call the space X universally Loeb measurable. In this section we examine how these spaces behave under various topological operations.

"Nice" subspaces of universally Loeb measurable spaces inherit this property. We have already seen that the Borel subsets of βX are universally Loeb measurable. In general, we have:

THEOREM 4.1. *Every Borel subset of a universally Loeb measurable space is universally Loeb measurable.*

We omit the proof, since the argument is very similar to that of Theorem 2.5, iii $\Rightarrow$ i. One only needs to notice that if Z is an internal zero set of X, then $Z \cap {}^*B$ is an internal zero set of ${}^*B$, and therefore every finite internal content on ${}^*\mathcal{Z}(B)$ has a natural extension to ${}^*\mathcal{Z}(X)$.

Universal Loeb measurability is not in general preserved under continuous mappings, and we will see later that this is so even for quotient maps. The identity map between $[0, 1]$ with the discrete metric and $[0, 1]$ with the Sorgenfrey topology (the topology generated by intervals of the form $[a, b)$) is continuous. We denote the latter space by $[0, 1]$-s. The interval $[0, 1]$ with the discrete metric is both locally compact and a complete metric space, hence universally Loeb measurable, while ns($^*[0, 1]$-s) fails to be measurable with respect to the transfer of Lebesgue measure. (See [17], Example 6.4, or [18], Example 1.1.13.)

A completely regular Hausdorff space is *Lindelöf* if every open cover has a countable subcover, *zero dimensional* if it has a base of clopen sets, *paracompact* if every open cover has a locally finite open refinement, and *realcompact* if it is homeomorphic to a closed subset of $\mathbb{R}^\gamma$ for some cardinal γ (see [3]). If a space is Lindelöf then it is both paracompact and realcompact. Now $[0, 1]$-s provides us with an example of a Lindelöf zero dimensional space for which universal Loeb measurability of ns($^*X$) fails. Therefore universal Loeb measurability is not preserved under arbitrary products, since for some cardinal γ, $[0, 1]$-s is homeomorphic to a closed subset of $\mathbb{R}^\gamma$, and closed subsets of universally Loeb measurable spaces are universally Loeb measurable. A Hausdorff space X is called a *k-space* if it is the image of a locally compact Hausdorff space under a quotient map. Theorem 3.3.20 of [3] states that every first countable Hausdorff space is a k-space. Thus $[0, 1]$-s is a complete regular Hausdorff k-space, and it follows that universal Loeb measurability is not preserved by quotient maps. Landers and Rogge [14] give additional examples showing that separable metric spaces, complete uniform spaces and arbitrary products of universally Loeb measurable spaces need not be universally Loeb measurable. For finite or countable products we have the following.

THEOREM 4.2. *Universal Loeb measurability is preserved under finite and countable products.*

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PROOF. We give the proof only for the finite case. The countable case is similar. Let X and Y be universally Loeb measurable. Let λ be a finite internal content on ${}^*\mathcal{Z}(X \times Y)$, π_1 the projection onto X and π_2 the projection onto Y. Then $\lambda \circ \pi_1^{-1}$ and $\lambda \circ \pi_2^{-1}$ are finite internal contents on ${}^*\mathcal{Z}(X)$ and ${}^*\mathcal{Z}(Y)$ respectively. Note that $\mathrm{ns}({}^*X) \times \mathrm{ns}({}^*Y) = \mathrm{ns}({}^*X \times Y)$. If A and B are zero (respectively cozero) subsets of X and Y, then $A \times B$ is a zero (respectively cozero) subset of $X \times Y$. To see why this is true, select nonegative continuous functions $f: X \longrightarrow \mathbb{R}$ and $g: Y \longrightarrow \mathbb{R}$ such that $A = \{f = 0\}$, $B = \{g = 0\}$ (respectively $A = \{f > 0\}$, $B = \{g > 0\}$) and consider the function $f + g$ (respectively $f \cdot g$). Now choose $Z_{xi} \in {}^*\mathcal{Z}(X)$, $O_{xi} \in {}^*\mathcal{U}(X)$, $Z_{yi} \in {}^*\mathcal{Z}(Y)$, $O_{yi} \in {}^*\mathcal{U}(Y)$, $i = 1, 2, 3$, with

$$
\begin{gathered}
Z_{x1} \subset O_{x1} \subset Z_{x2} \subset \mathrm{ns}({}^*X) \subset O_{x2} \subset Z_{x3} \subset O_{x3}, \\
\lambda \circ \pi_1^{-1}(O_{x3}) - \lambda \circ \pi_1^{-1}(Z_{x1}) < \epsilon, \\
Z_{y1} \subset O_{y1} \subset Z_{y2} \subset \mathrm{ns}({}^*Y) \subset O_{y2} \subset Z_{y3} \subset O_{y3} \text{ and} \\
\lambda \circ \pi_2^{-1}(O_{y3}) - \lambda \circ \pi_2^{-1}(Z_{y1}) < \epsilon.
\end{gathered}
$$

Then

$$Z_{x2} \times Z_{y2} \subset \mathrm{ns}({}^*X) \times \mathrm{ns}({}^*Y) \subset O_{x2} \times O_{y2}.$$

By Lemmas 3.1 and 3.2,

$$
\begin{aligned}
&\lambda(O_{x2} \times O_{y2}) - \lambda(Z_{x2} \times Z_{y2}) \\
&\quad = \lambda(O_{x2} \times O_{y2} \setminus Z_{x2} \times Z_{y2}) \\
&\quad = \lambda([O_{x2} \times O_{y2} \setminus Z_{x2} \times {}^*Y] \cup [O_{x2} \times O_{y2} \setminus {}^*X \times Z_{y2}]) \\
&\quad \leq \lambda([Z_{x3} \times {}^*Y \setminus O_{x1} \times {}^*Y] \cup [{}^*X \times Z_{y3} \setminus {}^*X \times O_{y1}]) \\
&\quad \leq \lambda([Z_{x3} \times {}^*Y \setminus O_{x1} \times {}^*Y) + \lambda({}^*X \times Z_{y3} \setminus {}^*X \times O_{y1}) \\
&\quad = \lambda([Z_{x3} \setminus O_{x1}] \times {}^*Y) + \lambda({}^*X \times [Z_{y3} \setminus O_{y1}]) \\
&\quad \leq \lambda([O_{x3} \setminus Z_{x1}] \times {}^*Y) + \lambda({}^*X \times [O_{y3} \setminus Z_{y1}]) \\
&\quad \leq \lambda \circ \pi_1^{-1}(O_{x3} \setminus Z_{x1}) + \lambda \circ \pi_2^{-1}(O_{y3} \setminus Z_{y1}) \\
&\quad = \lambda \circ \pi_1^{-1}(O_{x3}) - \lambda \circ \pi_1^{-1}(Z_{x1}) + \lambda \circ \pi_2^{-1}(O_{y3}) - \lambda \circ \pi_2^{-1}(Z_{y1}) \\
&\quad < \epsilon + \epsilon.
\end{aligned}
$$

The result for finite products follows now by induction. ∎

Let Λ be a directed index set for the topological spaces X_γ. Suppose that for any indices $\varphi \leq \gamma \leq \eta \in \Lambda$ there are continuous mappings $\pi_\varphi^\gamma: X_\gamma \longrightarrow X_\varphi$ satisfying:

i) $\pi_\gamma^\gamma = \mathrm{id}_{X_\gamma}$

ii) $\pi_\varphi^\gamma \circ \pi_\gamma^\eta = \pi_\varphi^\eta$.

Then, we say that $S = \{X_\gamma, \pi_\varphi^\gamma, \Lambda\}$ is an *inverse system* of the spaces X_γ. If $\Lambda = \mathbb{N}$, S is called an *inverse sequence*. Let S be an inverse system. An element (x_γ) of the Cartesian product $\prod_{\gamma \in \Lambda} X_\gamma$ is called a *thread* of S if $\pi_\varphi^\gamma(x_\gamma) = x_\varphi$ for all $\varphi \leq \gamma \in \Lambda$. The subspace of $\prod_{\gamma \in \Lambda} X_\gamma$ consisting of all threads of S is called the *limit of the inverse system S*.

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PROPOSITION 4.3. *The limit of an inverse system S of Hausdorff spaces X_γ is a closed subset of the Cartesian product $\prod_{\gamma\in\Lambda} X_\gamma$.*

PROOF. See [3] page 98.

This Proposition, together with Theorems 4.1 and 4.2, implies that if $\prod_{\gamma\in\Lambda} X_\gamma$ is universally Loeb measurable, then so is the limit of the inverse system $S = \{X_\gamma, \pi^\gamma_\varphi, \Lambda\}$, and that the limit of an inverse sequence is universally Loeb measurable if each space X_n is.

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces. Give $\bigoplus_{\alpha\in\Lambda} X_\alpha := \bigcup_{\alpha\in\Lambda} X_\alpha \times \{\alpha\}$ the strongest topology making all the canonical injections $i_\alpha \colon X_\alpha \longrightarrow \bigoplus_{\alpha\in\Lambda} X_\alpha$ continuous. Then $\bigoplus_{\alpha\in\Lambda} X_\alpha$ is the *topological sum* of the spaces X_α. The easy proof of the following theorem is left to the reader.

THEOREM 4.4. *The topological sum of a family of universally Loeb measurable spaces is universally Loeb measurable.*

If we consider finite or countable products, and arbitrary topological sums of spaces X_α such that ns($^*X$) is measurable with respect to all finite, finitely additive internal measures on * aBa(X), then the analogues of Theorems 4.2 and 4.4 also hold. The proof for topological sums is the same, and for products it is actually simpler, since one then works with internal algebras, which are closed under set differences. By Theorem 3.8, it follows that the class of completely regular and Hausdorff pre-Radon spaces is closed under countable products and topological sums. On the other hand, all the counter examples provided use internal measures, and not just contents, so again by Theorem 3.8 they can be interpreted in terms of pre-Radon spaces. Thus, for instance, a completely regular Hausdorff space which is the image of a pre-Radon space under a quotient map need not be pre-Radon.

OPEN PROBLEM. Is there a completely regular Hausdorff space X such that ns($^*X$) is measurable with respect to all finite, internal, finitely additive measures defined on * aBa(X), but fails to be measurable with respect to some finite internal content defined on $^*\mathcal{Z}(X)$? The nonstandard model is assumed to be α-saturated, with α larger than the cardinality of βX.

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Department of Mathematics
The University of the West Indies
Kingston 7
Jamaica