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# EXACT ANALYSIS FOR A CLASS OF SIMPLE, CIRCUIT-SWITCHED NETWORKS WITH BLOCKING

YAAKOV KOGAN,* *Technion—IIT*

**Abstract**

We consider the same circuit switching problem as in Mitra [1]. The calculation of the blocking probabilities is reduced to finding the partition function for a closed exponential pseudo-network with $L$-1 customers. This pseudo-network differs from that in [1] in one respect only: service rates at nodes $1, 2, \cdots, p$ depend on the queue length. The asymptotic expansion developed in [1] follows from our exact expression for the partition function.

GENERATING AND PARTITION FUNCTIONS

## 1. Introduction

Consider the same circuit switching, with blocking, as in Mitra's network [1]. There are $K_j$ lines from center $j$ to a hub, center $(p+1)$, $1 \leqq j \leqq p$ and $K_{p+1}$ lines from the hub to the destination, center $(p+2)$. A call originating at center $j$, of class $j$, requires two lines, one line from center $n$ to the hub, another line from the hub to the destination. The holding times for circuits of class $j$ are independent random variables with an arbitrary distribution and mean $1/\mu_j$. At the termination of a call, both links are simultaneously released. The total offered traffic of call-requests at center $j$, $1 \leqq j \leqq p$, is Poisson with rate parameter $\lambda_j$. A call-request at center $j$ may be blocked either if all lines from center $j$ to center $(p+1)$ are in use, or if all lines from center $(p+1)$ to center $(p+2)$ are in use. Blocked calls are cleared. The problem is the calculation of equilibrium blocking probabilities at each of the originating centers $1, 2, \cdots, p$.

Let

$$\rho_j \triangleq \lambda_j/\mu_j, \ 1 \leqq j \leqq p,$$

and consider the case

$$L \triangleq \sum_{j=1}^{p} K_j - K_{p+1} \leqq 1.$$

Let $n_j$, $1 \leqq j \leqq p$, denote the number of calls of class $j$ in progress, and write $\boldsymbol{n} = (n_1, \cdots, n_p)$. Then the unique equilibrium distribution

$$\pi(\boldsymbol{n}) = \frac{1}{G} \prod_{j=1}^{p} \frac{\rho_j^{n_j}}{n_j!} \qquad \boldsymbol{n} \in \mathscr{S}$$

where

$$\mathscr{S} = \left\{ 0 \leqq n_j \leqq K_j, \ 1 \leqq j \leqq p, \text{ and} \sum_{j=1}^{p} n_j \leqq K_{p+1} \right\},$$

and $G$ is the normalizing constant. $G$ is the partition function

$$G(\boldsymbol{K}; L) = G(K_1, \cdots, K_p; L) \triangleq \sum_{\boldsymbol{n} \in \mathscr{S}} \prod_{j=1}^{p} \frac{\rho_j^{n_j}}{n_j!}.$$

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* Postal address: Faculty of Industrial Engineering and Management, Technion–Israel Institute of Technology, Haifa 32 000, Israel.

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The equilibrium probability that a call of class $j$ is not blocked is

$$\frac{G(K_1,\dots,K_j-1,\dots,K_p;L)}{G(K_1,\dots,K_j,\dots,K_p;L)}.$$

In [1] it is shown that

$$G(\boldsymbol{K};L)=\left[\prod_{j=1}^{p}\rho_j^{K_j}/K_j!\right]\left[\prod_{j=1}^{p}B^{-1}(K_j,\rho_j)-I(\boldsymbol{K};L)\right]$$

where

$$B(K_j,\rho_j)\triangleq[\rho_j^{K_j}/K_j!]\Big/\sum_{n=0}^{K_j}(\rho_j^n/n!)$$

and a rather cumbersome procedure is given for generating the coefficients $A_n(L)$ of the complete expansion of $I(\boldsymbol{K};L)$ in terms of the inverse powers of a parameter $N\gg 1$. The parameter $N$ is introduced in such a way that

$$\begin{aligned}\beta_j&=K_j/N,\\ \Gamma_j&=N/\rho_j,\end{aligned}\qquad j=1,2,\cdots,p,$$

are quantities of order 1.

We show that for a fixed $\boldsymbol{K}$, $I(\boldsymbol{K};L)$ is the partition function for a closed exponential pseudo-network with a single class of $L-1$ customers. This pseudo-network differs from that in [1] in one respect only: service rate $\nu_j$, at node $j$, $1\leqq j\leqq p$, is dependent on the queue length $l_j$ and is given by

$$\nu_j=\nu_j(l_j)=\frac{\rho_j}{K_j}\left(1-\frac{l_j-1}{K_j}\right)^{-1}=\Gamma_j^{-1}\left(\beta_j-\frac{l_j-1}{N}\right)^{-1}. \tag{1}$$

Thus, the large parameter $N$ does not interfere with the calculation of the partition function of the pseudo-network. Moreover, the expansion of $I(\boldsymbol{K};L)$ derived in [1] follows easily from the exact expression for this partition function, but our derivation is considerably shorter and simpler than Mitra's.

## 2. Main result

We start with the explicit expression for generating function

$$\mathcal{G}_{(z;x)}\triangleq\sum_{L=0}^{\infty}\sum_{K_1=0}^{\infty}\cdots\sum_{K_p=0}^{\infty}x^Lz_1^{K_1}\cdots z_p^{K_1}G(\boldsymbol{K};L)$$

$$=\frac{e^{\sum_j\rho_jz_j}}{1-x}\left[1\Big/\prod_{j=1}^{p}(1-z_j)-x\Big/\prod_{j=1}^{p}(1-xz_j)\right]$$

given in [1]. Denote by $\mathcal{J}(\boldsymbol{z};x)$ the generating function of the partition function

$$J(\boldsymbol{K};L)=I(\boldsymbol{K};L)\prod_{j=1}^{r}(\rho_j^{K_j}/K_j!)$$

where

$$I(\boldsymbol{K};L)=\sum_{0\leqq\boldsymbol{n}\leqq\boldsymbol{K}}1_{\boldsymbol{l}\boldsymbol{n}\leqq L-1}\prod_{j=1}^{p}\frac{K_j!}{(K_j-n_j)!}(1/\rho_j)^{n_j}.$$

Then one can easily see that

$$\mathcal{J}(\boldsymbol{z};x)=\frac{x}{1-x}\prod_{j=1}^{p}\frac{e^{\rho_jz_j}}{1-xz_j}.$$

By Cauchy's formula

$$J(\boldsymbol{K};L)=(2\pi i)^{-(p+1)}\oint_{C_x}\frac{x^{-L}}{1-x}\,dx\prod_{j=1}^{p}\oint_{C_j}\frac{z_j^{-K_j-1}e^{\rho_j z_j}}{1-xz_j}\,dz_j.$$

Denote

(2)
$$R(x)=\prod_{j=1}^{p}\oint_{C_j}\frac{z_j^{-K_j-1}e^{\rho_j z_j}}{1-xz_j}\,dz_j.$$

Then

$$J(\boldsymbol{K};L)=\oint_{C_x}\frac{x^{-L}R(x)}{1-x}\,dx=\frac{1}{(L-1)!}\frac{d^{L-1}[R(x)(1-x)^{-1}]}{dx^{L-1}}\bigg|_{x=0}.$$

We now show that

(3)
$$R(x)=\left(\prod_{j=1}^{p}\left(\rho_j^{K_j}/K_j!\right)\right)\prod_{j=1}^{p}f_j(x)$$

where

$$f_j(x)=\rho_j\int_0^\infty e^{-\rho_j t}(1+xt_j)^{K_j}\,dt_j.$$

Define the function

(4)
$$g(x;M)\triangleq\sum_{n_1+n_2=M}\frac{\rho^{n_1}}{n_1!}x^{n_2}.$$

It is easily seen that

(5)
$$\sum_{M=0}^{\infty}z^M(g(x;M)=e^{\rho z}/(1-xz)$$

and

(6)
$$\begin{aligned}g(x;M)&=\frac{\rho^M}{M!}\sum_{n=0}^{M}\int_0^\infty e^{-t}\binom{M}{n}\left(\frac{tx}{\rho}\right)^n dt\\&=\frac{\rho^M}{M!}\rho\int_0^\infty e^{-\rho t}(1+xt)^M\,dt.\end{aligned}$$

Now (3) is implied by (2) and (4)–(6). Thus we have

$$I(\boldsymbol{K};L)=\frac{1}{(L-1)!}\frac{d^{L-1}[F(x)(1-x)^{-1}]}{dx^{L-1}}\bigg|_{x=0}$$

where

$$F(x)=\prod_{j=1}^{p}f_j(x).$$

Leibnitz's rule for differentiating the product of several functions yields

(7)
$$\begin{aligned}I(\boldsymbol{K};L)&=\frac{1}{(L-1)!}\sum_{m=0}^{L-1}\binom{L-1}{m}F^{(m)}(0)(L-1-m)!\\&=\sum_{m=0}^{L-1}(F^{(m)}(0)/m!)\end{aligned}$$

where

(8)
$$F^{(m)}(0)=\sum_{l_1+\cdots l_p=m}\frac{m!}{l_1!\cdots l_p!}\prod_{j=1}^{p}f_j^{(l_j)}(0)$$

while

(9)
$$f_j^{(l)}(0)=\frac{(K_j)_l l!}{\rho_j^l}=l!\Gamma_j^l\prod_{i=0}^{l-1}\left(\beta_j-\frac{i}{N}\right).$$

(Here $(K_j)_l = K_j, \cdots, (K_j - l + 1)$). The final result is implied by (7)–(9) and given by

$$I(\boldsymbol{K}; L) = \sum_{m=0}^{L-1} \sum_{l_1+\cdots+l_p=m} \prod_{j=1}^{p} \Gamma_j^{l_j} \prod_{i=1}^{l_j} \left(\beta_j - \frac{i-1}{N}\right). \tag{10}$$

It is easily seen that (10) is the partition function for the closed exponential network with $p+1$ tandem queues and $L-1$ customers of a single class. The queueing discipline in each queue is FCFS. The service rate at queue $(p+1)$ is 1 while the service rate $\nu_j$ at queue $j$, $1 \leqq j \leqq p$, is dependent on the queue length $l_j$ and given by (1). Thus $I(\boldsymbol{K}; L)$ may be computed by well-known recursive formulas. Starting from (11), one can easily obtain the same final formulae for the expansion coefficients $A_n(L)$ as in [1].

## Reference

[1] Mitra, D. (1987) Asymptotic analysis and computational methods for a class of simple, circuit-switched networks with blocking. *Adv. Appl. Prob.* **19,** 219–239.