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THE HADAMARD THREE-CIRCLES THEOREMS FOR NONLINEAR EQUATIONS

R. VÝBORNÝ

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Abstract

The aim of this paper is is to establish Hadamard's type three-circles theorems for fully nonlinear elliptic and parabolic inequalities.

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1. Introduction

The famous Hadamard three-circles theorem of function theory has been generalized to solutions of elliptic and parabolic equations and these generalizations have various applications; see [4] and [2], where further references can be found. The following sharpened version of the boundary point lemma was established in [6] as a consequence of the Hadamard three-circles theorem for subharmonic functions.

If u is a continuous subharmonic function which attains its maximum in a ball $B = \{x; |x| < a\}$ at y with $|y| = a$ and if $M(r) = \sup\{u(x); |x| = r\}$ then $M'_{-}(a) > 0$ and

$$\limsup \frac{u(x) - u(y)}{|x - y|} \leq -M'_{-}(a) < 0, \tag{1}$$

where x approaches y along the normal at y.

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As soon as Hadamard's three-circles theorem is established for a class of functions, then a maximum principle like (1) follows.

In this paper we prove Hadamard type theorems for the difference of two solutions to parabolic equations of the form

$$u_t = f(t, x, u, Du, D^2u), \tag{2}$$

and for the corresponding elliptic equation, where Du and D^2u denote the gradient and the Hessian matrix of u, respectively. The boundary point lemma (1) can then be used in the usual way [4] to prove uniqueness of the second and third boundary value problem under very mild assumptions.

2. The theorems

The function f in (2) is said to be uniformly elliptic with respect to u with the constant κ (or just uniformly elliptic) if there exists a positive κ such that

$$f(t, x, u, Du, Q_1) - f(t, x, u, Du, Q_2) \geq \kappa \operatorname{Tr}(Q_1 - Q_2) \tag{1}$$

whenever $Q_1 - Q_2 \geq 0$ (that is, whenever $Q_1 - Q_2$ is positive semidefinitive).

This definition of ellipticity has the advantage that it does not require any smoothness of f.

A function M is said to be a strictly convex function of z on $[a, b]$ if for all $r_1 < r < r_2$, $[r_1, r_2] \subset [a, b]$,

$$M(r) < M(r_1)\frac{z(r_2) - z(r)}{z(r_2) - z(r_1)} + M(r_2)\frac{z(r) - z(r_1)}{z(r_2) - z(r_1)}.$$

In what follows $| \, |$ denotes the Euclidean norm of a vector or a matrix and

$$A = \{x ; 0 < R_1 < |x| < R_2\}.$$

THEOREM 1 (elliptic case). *Given L, κ, n there exists a function $z\colon [R_1, R_2] \to \mathbb{R}$ such that if*

(i) *u and v belong to $C^2(A) \cap C(\overline{A})$,*
(ii) *$u \geq v$ in A,*
(iii) *$f(x, u(x), Du(x), D^2u(x)) \geq f(x, v(x), Dv(x), D^2v(x))$ for $x \in A$,*
(iv) *$f(x, \cdot, p, Q)$ is decreasing for $x \in A$, $p \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n^2}$,*
(v) *$M(r) = \operatorname{Max}\{u(x) - v(x); |x| = r\}$, M is strictly increasing in $[R_1, R_2]$,*

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(vi) $|f(x, y, p, Q) - f(x, y, \overline{p}, \overline{Q})| \leq L(|p - \overline{p}| + |Q - \overline{Q}|)$ *for* $x \in A$, $y \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n^2}$, $\overline{p} \in \mathbb{R}^n$, $\overline{Q} \in \mathbb{R}^{n^2}$,

(vii) f *is uniformly elliptic with respect to* u *with the constant* κ,

then M *is a strictly convex function of* z.

REMARK 1. If inequality (iii) is satisfied in B rather than in A and if the strong maximum applies then M is strictly increasing. Strong maximum principles for nonlinear elliptic and parabolic equations and inequalities were established in [1], [3], [5] and [7]. Generally, (v) is an extra assumption; however this assumption is also needed in the linear case. A similar theorem holds if M is strictly decreasing. If (v) does not hold, one must consider intervals where M decreases or increases separately.

REMARK 2. If f is independent of u then (ii) is superfluous.

REMARK 3. If one considers a linear differential operator

$$Lu = \sum_{i,j=1}^{n} a_{ij} D_{ij} u + b_i D_i u + cu,$$

instead of f, the assumption (iv) corresponds to $c \leq 0$.

REMARK 4. If f does not contain the mixed second order derivatives of u then the assumption $u \in C^2(A)$ can be weakened to the mere existence of the pure second order derivatives.

For the proof we shall use the following.

LEMMA. *Let* M *and* z *be strictly increasing on* $[a, b]$. *Then* M *is a strictly convex function of* z *if and only if the following condition is satisfied*:

> *for every* $\gamma > 0$ *and every interval* $[\alpha, \beta] \subset [a, b]$ *the function* $r \to M(r) - \gamma z(r)$ *attains its maximum either at* α *or at* β.

We omit the fairly straightforward proof of the lemma.

PROOF OF THEOREM 1. We choose z such that

$$z''(t) + \frac{L}{\kappa}\left(1 + \frac{2n}{t}\right) z'(t) = 0,$$

and $z'(t)$ positive for positive t. Let us now assume, contrary to what we want to prove, that $M - \gamma z$ attains its maximum over $[\alpha, \beta] \subset [R_1, R_2]$ in (α, β) for some positive γ. Then the function $w : x \to u(x) - v(x) - \gamma z(|x|)$ attains its maximum over $\{x; \alpha \leq |x| \leq \beta\}$ at an interior point $\overline{x}$. At $\overline{x}$

we have $Dw = 0$, that is, $Du - Dv = \gamma Dz$, and $D^2 w \leq 0$. Further

$$\begin{aligned}
0 &\leq f(\overline{x}, u(\overline{x}), Du(\overline{x}), D^2u(\overline{x})) - f(\overline{x}, v(\overline{x}), Dv(\overline{x}), D^2v(\overline{x})) \\
&\leq f\left(\overline{x}, u(\overline{x}), Du(\overline{x}), D^2w(\overline{x}) + \gamma z''(|\overline{x}|)\frac{\overline{x}_i\overline{x}_j}{|\overline{x}|^2}\right. \\
&\qquad\qquad \left. + D^2v(\overline{x}) + \gamma z'(|\overline{x}|)\left(\frac{\delta_{ij}}{|\overline{x}|} - \frac{\overline{x}_i\overline{x}_j}{|\overline{x}|^3}\right)\right) \\
&\quad - f(x, v(\overline{x}), Dv(\overline{x}), D^2v(\overline{x})) \\
&\leq f\left(\overline{x}, u(\overline{x}), Du(\overline{x}), D^2w(\overline{x}) + \gamma z''(|\overline{x}|)\frac{\overline{x}_i\overline{x}_j}{|\overline{x}|^2}\right. \\
&\qquad\qquad \left. + D^2v(\overline{x}) + \gamma z'(|\overline{x}|)\left(\frac{\delta_{ij}}{|\overline{x}|} - \frac{\overline{x}_i\overline{x}_j}{|\overline{x}|^3}\right)\right) \\
&\quad - f\left(\overline{x}, u(\overline{x}), Du(\overline{x}), D^2v(\overline{x}) + \gamma z'(|\overline{x}|)\left(\frac{\delta_{ij}}{|\overline{x}|} - \frac{\overline{x}_i\overline{x}_j}{|\overline{x}|^3}\right)\right) \\
&\quad + f\left(\overline{x}, u(\overline{x}), Du(\overline{x}), D^2v(\overline{x}) + \gamma z'(\overline{x})\left(\frac{\delta_{ij}}{|\overline{x}|} - \frac{\overline{x}_i\overline{x}_j}{|\overline{x}|^3}\right)\right) \\
&\quad - f(\overline{x}, v(\overline{x}), Dv(\overline{x}), D^2v(\overline{x})) \\
&\leq \kappa \sum_{i=1}^{n}\left(D_{ii}w(\overline{x}) + \gamma z''(|\overline{x}|)\frac{\overline{x}_i\overline{x}_j}{|\overline{x}|^2}\right) \\
&\quad + L\left\{|Du(\overline{x}) - Dv(\overline{x})| + \gamma z'(|\overline{x}|)\left(\frac{\sqrt{n}}{|\overline{x}|} + \frac{n}{|\overline{x}|}\right)\right\} \\
&< \gamma\left\{\kappa z''(|\overline{x}|) + Lz'(|\overline{x}|) + \frac{2nL}{|\overline{x}|}z'(|\overline{x}|)\right\} = 0.
\end{aligned}$$

This contradiction completes the proof.

A similar theorem holds for parabolic inequalities; however the function M must be modified. For u, v defined on $\overline{S}$ with $S = A \times (0, T]$ let

$$\begin{aligned}
&M_1 = \operatorname{Max}\{u(x, 0) - v(x, 0), x \in \overline{A}\}, \\
&M_2(r) = \operatorname{Max}\{u(x, t) - v(x, t); |x| = r, 0 \leq t \leq T\}, \quad \text{and} \\
&M(r) = \operatorname{Max}(M_1, M_2(r)).
\end{aligned} \tag{4}$$

THEOREM 2 (parabolic case). *Given L, κ, n there exists a function z such that if*

(i) *u and v have continuous second order partial derivatives with respect to the variables x_i, continuous derivative with respect to t in S and are continuous in $\overline{S}$,*

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(ii) $u \geq v$ *in* S,

(iii) $$f(t, x, u(t, x), Du(t, x), D^2u(t, x)) - u_t \geq f(t, x, v(t, x), Dv(t, x), D^2v(t, x)) - v_t \quad \textit{for } (t, x) \in S,$$

(iv) $f(t, x, \cdot, p, Q)$ *is decreasing for* $(t, x) \in S$, $p \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n^2}$,

(v) M *defined in* (4) *is strictly increasing,*

(vi) $|f(t, x, y, p, Q) - f(t, x, y, \overline{p}, \overline{Q})| \leq L(|p - \overline{p}| + |Q - \overline{Q}|)$ *for* $(t, x) \in S$, $y \in \mathbb{R}$, $p \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n^2}$, $\overline{p} \in \mathbb{R}^n$, $\overline{Q} \in \mathbb{R}^{n^2}$,

(vii) f *is uniformly elliptic with respect to* u *with the constant* κ,

then M *is a strictly convex function of* z.

We sketch the proof insofar as it differs from the proof of Theorem 1. We proceed again indirectly and assume that $M - \gamma z$ attains its maximum over $[\alpha, \beta]$ at some interior point $(\overline{t}, \overline{x})$ and that this maximum is larger than the values at the end-points. First we rule out the possibility $\overline{t} = 0$ indirectly. Clearly $u(\overline{t}, \overline{x}) - v(\overline{t}, \overline{x}) = u(0, \overline{x}) - v(0, \overline{x}) \leq M_1$ and also $u(0, \overline{x}) - v(0, \overline{x}) = M(\overline{r}) \geq M_1$, and consequently $u(0, \overline{x}) - v(0, \overline{x}) = M_1$. Now we have

$$\begin{aligned} M(\overline{r}) - \gamma z(\overline{r}) &= u(0, \overline{r}) - v(0, \overline{r}) - \gamma z(\overline{r}) \\ &= M_1 - \gamma z(\overline{r}) < M_1 - \gamma z(\alpha) \leq M(\alpha) - \gamma z(\alpha), \end{aligned}$$

a contradiction. Hence $\overline{t} > 0$. We now follow the pattern of the proof of Theorem 1.

$$\begin{aligned} 0 \leq\ & f(\overline{t}, \overline{x}, u(\overline{t}, \overline{x}), Du(\overline{t}, \overline{x}), D^2u(\overline{t}, \overline{x})) - u_t(\overline{t}, \overline{x}) \\ & - f(\overline{t}, \overline{x}, v(\overline{t}, \overline{x}), Dv(\overline{t}, \overline{x}), D^2v(\overline{t}, \overline{x})) + v_t(\overline{t}, \overline{x}). \end{aligned} \tag{5}$$

The term $v_t(\overline{t}, \overline{x}) - u_t(\overline{t}, \overline{x}) \leq 0$ (because $u - v - \gamma z$ has a maximum at $(\overline{t}, \overline{x})$) and consequently $v_t - u_t$ can be omitted in (5) and the rest is similar to the proof of Theorem 1.

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15 Rialanna
Kenmore
Queensland 4069
Australia