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A THEOREM ON POWER-OPEN LCA GROUPS AND ITS CONSEQUENCES

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An LCA group G is called largely open if every large subgroup (that is, subgroup of finite index) of G is open. G is power-open if the continuous endomorphism, $g \to ng$, $g \in G$, takes open sets of G onto open sets of G for every integer $n > 1$. The following are proved equivalent:

(1) G is power-open;

(2) $b\hat{G}/\hat{G}$ is torsion-free, where $b\hat{G}$ is the Bohr compactification of $\hat{G}$;

(3) every open subgroup of G is largely open;

(4) for every discontinuous ultimately measurable character ψ (in the sense of Edwin Hewitt and Kenneth A. Ross [*Math. Ann.* **160** (1965), 171-194]), $\psi(G)$ has outer measure one.

We apply these results to determine the structure of σ-compact, divisible LCA groups and divisible self-dual LCA groups. We show that the set of those characters of a nondiscrete, totally disconnected, power-open group G, which are not ultimately measurable, separates points of G. We also show that power-open LCA groups are precisely those LCA groups which admit no continuous extension of Haar measure (as defined by H. Leroy Peterson [*Trans. Amer. Math. Soc.* **228** (1977), 359-370]).

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1. Introduction and notation

Let H be a subgroup of an LCA group G. We say that H is a relatively large subgroup of G if H is a large subgroup of an open subgroup of G. H is called a strongly pure subgroup if G/H is torsion-free. $\hat{G}$, $B(G)$, G_0 and $T(G)$ denote respectively the group of continuous characters, the group of compact elements, the component of the identity and the maximal torsion subgroup of G. $E(G)$ is the minimal divisible extension of G topologized in the usual manner so that G is an open subgroup. Via the well-known continuous natural embedding, we regard $\hat{G}$ as a subgroup of $b\hat{G}$. The symbol $\cong$ denotes topological isomorphism. J_p, F_p and T are the group of p-adic integers, the group of p-adic numbers and the circle-group respectively, all with their usual topologies. Q and $Z(p^{\infty})$ denote the rational group and the quasi-cyclic group. In [8], an invariant countably additive extension of Haar measure λ in an LCA group G is defined with respect to a nonopen relatively large subgroup K, called the continuous extension of λ with respect to K. This led to the question: which LCA groups contain nonopen relatively large subgroups? Our answer: precisely those which are not power-open (see also [8], Theorem 1.2). As we had already met with some success in [5] in connecting power-open groups with a property of Haar measure, we tried further and found that these groups can be related to the concept of ultimately measurable characters as developed in [3]. (Briefly, a character ψ of G is ultimately measurable if there exists an invariant extension λ^* of Haar measure λ such that every character in the group generated by ψ and $\hat{G}$ is λ^*-measurable.) We also found that the dual group of a power-open group sits in a characteristic fashion (as a strongly pure subgroup) in its Bohr compactification. We have woven these ideas from diverse sources into our one and only theorem of this note.

2. Results

Before presenting our results, we need two lemmas.

LEMMA 1. *Let G be an abelian group and let n be a positive integer. Then nG is the intersection of all those subgroup of G whose indices in G divide n.*

Proof. Endow n with the discrete topology. Then $\hat{G}(n)$, the

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subgroup of $\hat{G}$ generated by elements whose orders divide n, is a closed subgroup of $\hat{G}$. Since nG is the annihilator of $\hat{G}(n)$ (see [2], 24.22), it follows by duality that nG is the intersection of all those subgroups of G whose indices in G divide n.

REMARK 1. It has been shown in ([5], 1.1) that the following are equivalent for a compact abelian group G:

(a) G is power-open;

(b) nG is open for every integer $n > 1$;

(c) $G/G_0 \cong \prod_p A_p$ where A_p is a topological direct product of at most finitely many copies of J_p and finitely many cyclic p-groups for each prime p.

We quote this result for subsequent use and also to bring into bold relief its importance in understanding the structure of power-open groups.

LEMMA 2. *A compact abelian group G is largely open if and only if nG is open for every $n > 1$.*

Proof. If nG is open for all $n > 1$, Lemma 1 shows that every large subgroup of G is open. Conversely suppose nG is not open for some n. Then G/nG is a compact, infinite, n-bounded group. By ([4], 5.1), G/nG contains a proper, dense, subgroup and by ([6], Theorem 2), it contains a proper, dense, maximal subgroup. Using the open, continuous, natural homomorphism of G onto G/nG, we conclude that G contains a proper, dense, maximal subgroup and so is not largely open. This completes the proof.

We are now ready to present our theorem. We would like to point out that the equivalence of *(a)* and *(b)* below was stated and proved in ([5], Theorem 1.1), and is included here for completeness and for use.

THEOREM. *The following are equivalent for an* LCA *group G:*

(a) G is power-open;

(b) $B(G)$ contains a compact, power-open subgroup open in $B(G)$;

(c) $\hat{G}$ is a strongly pure subgroup of $b\hat{G}$;

(d) for every discontinuous ultimately measurable character ψ

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of G , $\psi(G)$ *has outer measure one in* T ;

(e) every open subgroup of G *is largely open.*

Proof. *(a)* implies *(c)*. Let ψ be a character of G , not necessarily continuous, such that $n\psi \in \hat{G}$. Let V be a neighbourhood of the identity in T . Then $\psi^{-1}(V) \cap nG = n(n\psi)^{-1}(V)$. As G is power-open and $n\psi$ is continuous, this shows that $\psi^{-1}(V) \cap nG$ is open so that $\psi|nG$ is continuous. Since nG is open, ψ must be a continuous character of G so that $\psi \in \hat{G}$. This proves *(c)*.

(c) implies *(d)*. If ψ is a discontinuous ultimately measurable character of G , then $n\psi \in \hat{G}$ for some n or $\psi(G)$ has outer measure one ([3], 3.9). Since *(c)* precludes the first possibility, the second alternative must hold.

(d) implies *(e)*. Suppose K is an open subgroup of G with a large nonopen subgroup, which we can assume to be dense in K . Then K has a discontinuous character ϕ of finite order, which is ultimately measurable by ([3], 3.12). Let H be the (necessarily countable) minimal divisible extension of $\phi(K)$ in T . Since H is divisible, ϕ extends to a homomorphism ψ of G into H . Then ψ is an ultimately measurable discontinuous character of G ([3], 3.4), and $\psi(G)$ has measure zero. This contradicts *(d)*. Hence K is largely open.

(e) implies *(b)*. Let $G \cong R^m \times M$, where M has a compact, open subgroup H . Suppose nH is not open in H for some n . Then by Lemma 2, H contains a large nonopen subgroup H_1 and $R^m \times H_1$ is a large non-open subgroup of $R^m \times H$. This contradicts *(e)*. Hence nH is open in H for every $n > 1$ and so H is power-open by our Remark 1. As $B(G)$ is an open subgroup of M , it follows that $B(G)$ contains a compact, power-open subgroup open in $B(G)$. This completes the proof.

This theorem not only displays the various faces of power-open groups, it is also quite fruitful. For the rest of this note, we shall be harvesting its corollaries.

The statement *(e)* leads to the following corollary about continuous extensions of Haar measure (compare [8], Theorem 1.2 and Section 2).

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COROLLARY 1. *An* LCA *group* G *admits a continuous extension of Haar measure with respect to some nonopen relatively large subgroup if and only if* G *is not power-open.*

Proof. By *(e)*, G has a nonopen relatively large subgroup if and only if G is not power-open. The result now follows from ([8], Section 2).

The statement *(d)* not only reduces by one-half the necessary condition for a character to be ultimately measurable (see [3], 3.9), it also yields the following corollary.

COROLLARY 2. *Every nondiscrete, power-open* LCA *group* G *possesses characters which are not ultimately measurable. If* G *is also totally disconnected,* G *has sufficiently many characters of this kind to separate points of* G .

Proof. Since G is nondiscrete and power-open, $T(G)$ can not be open (see [5], Remark 1.1). Hence G has a proper dense subgroup H not contained in any maximal subgroup of G (see [6], Theorem 2). It follows that G/H is divisible ([1], Exercise 1, p. 99). As every nontrivial divisible abelian group has a quasicyclic quotient group, we can assume G/H is isomorphic to $z(p^{\infty})$ for some prime p . We get a character ψ of G with kernel H and $\psi(G)$ isomorphic to $z(p^{\infty})$. Then ψ is discontinuous and the measure of $\psi(G)$ is zero. So by *(d)* of our theorem, ψ is not ultimately measurable.

Next, assume G is totally disconnected and let x be an arbitrary nonzero element of G . There is an open subgroup H of G such that $x \notin H$. As H is largely open and $T(H)$ is not open, we get a discontinuous character ψ (as in the last paragraph) of H such that $\psi(H)$ is a copy of $z(p^{\infty})$ and the kernel of ψ is a proper dense subgroup of H . As $z(p^{\infty})$ is divisible, ψ extends to a character, necessarily discontinuous, of G onto $z(p^{\infty})$ and we call it ψ again. If $\psi(x)$ is nontrivial, we are through. Suppose $\psi(x)$ is trivial. Let ϕ be the natural homomorphism of G onto the discrete group G/H . We send the cyclic subgroup of G/H generated by $x + H$ into a suitable quasicyclic subgroup of T , say $z(p^{\infty})$, in such a way that the image of $x + H$ is non-trivial. Let χ be an extension of this homomorphism into $z(p^{\infty})$. Then $\chi \circ \phi$ is a continuous character of G . As ψ is a discontinuous

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character of G, $\psi + \chi \circ \phi$ must also be discontinuous. Also $(\psi + \chi \circ \phi)(x) = (\chi \circ \phi)(x) = \chi(x+H)$ is nontrivial. Since $(\psi + \chi \circ \phi)(G)$ is countable $\left(\text{being contained in } z(p^{\infty}) + z(q^{\infty})\right)$ and G is power-open, the character $\psi + \chi \circ \phi$ is not ultimately measurable. This completes the proof.

It turns out that the statement *(c)* is not a mere algebraic curiosity. Its efficiency as an instrument of proof is borne out by the three corollaries that follow.

COROLLARY 3. *The following are equivalent for an* LCA *group* G :

(1) *both* G *and* $\hat{G}$ *are divisible;*

(2) *both* G *and* $\hat{G}$ *are torsion-free and power-open.*

Proof. If G and $\hat{G}$ are both divisible, then G, $\hat{G}$ and $b\hat{G}$ are all torsion-free. Then $\hat{G}$, as a divisible subgroup of a torsion-free group, is strongly pure in $b\hat{G}$. Hence G is torsion-free and power-open, and so also is $\hat{G}$, by symmetry. Conversely, G_d is torsion-free so that $b\hat{G}$ is divisible. As G is power-open, $\hat{G}$ is strongly pure in $b\hat{G}$ so that G is divisible. Similarly, G is divisible and the proof is complete.

In the next corollary below, we shall use local direct products of LCA groups for which we refer to ([2], 6.16 and 23.33).

COROLLARY 4. *Let* H *denote the topological direct product* $\prod_p J_p^{n_p}$, *where* n_p *is an arbitrary cardinal for every prime* p, *and let* $G = E(H)$. *Then* $\hat{G}$ *is divisible if and only if each* n_p *is finite. Moreover, if this condition is satisfied, then* G *is self-dual and topologically isomorphic to the local direct product of the groups* $F_p^{n_p}$ *relative to the compact, open subgroups* $J_p^{n_p}$, *as* p *varies over the set of primes.*

Proof. Firstly, if $\hat{G}$ is divisible, Corollary 3 implies that G must be power-open, which in turn implies that H must be power-open, so that by Remark 1, n_p is finite for every p. Conversely, G is

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torsion-free and power-open. Hence $b\hat{G}$ is divisible and as $\hat{G}$ is pure in $b\hat{G}$, $\hat{G}$ is also divisible. Finally, suppose each n_p is finite. Let G_1 be the local direct product. Then G_1 is self-dual ([2], 25.34). Now G_1 is torsion-free, and H is an open, compact, power-open subgroup. Hence G_1 is power-open (see statement *(b)* of our theorem). As G_1 is self-dual, Corollary 3 ensures that G_1 is divisible. Also G_1/H is clearly a torsion group. Hence G_1 is also a minimal divisible extension of H ([1], Exercise 3, p. 108). By ([1], Theorem 24.4), we have an (algebraic) isomorphism f of G onto G_1, keeping H pointwise fixed. Since H is open in both G_1 and G, f is also bicontinuous. Hence $G \cong G_1$ and so G is self-dual. This completes the proof.

In the corollary below, m denotes an arbitrary cardinal, $\hat{Q}^m$ is the topological direct product of m copies of $\hat{Q}$, the character group of Q taken discrete, and Q^m is the discrete direct sum of m copies of Q.

COROLLARY 5. *A divisible* LCA *group G is self-dual if and only if* $G \cong R^n \times \hat{Q}^m \times Q^m \times E(N)$, *where* $N \cong \prod_p J_p^{n_p}$, n_p *being a non-negative integer for every prime* p.

Proof. First let G be divisible and self-dual, then G is torsion-free. By self-duality, Structure Theorem 25.33 of [2] and our Corollary 4, the result follows immediately. Conversely, dualizing the given direct product and using Corollary 4 again, it follows that G is self-dual.

σ-compact, divisible LCA groups have been extensively used by various authors mainly because they are power-open (see, for example, [9]). In our last corollary, we deal with the structure of power-open, divisible LCA groups.

COROLLARY 6. *An* LCA *group G is divisible and power-open if and only if $G \cong R^n \times E(K) \times L$, where K is a compact, power-open group and L is a discrete divisible group. In particular, G is σ-compact and divisible if and only if G is of the form stated above with L a countable group.*

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Proof. First suppose $G \cong R^n \times E(K) \times L$ as stated above. As K is an open subgroup of $E(K)$, it is clear that K is an open subgroup of $B(G)$ also. Hence G is divisible and power-open. Conversely, suppose G is divisible and power-open. Now $G \cong R^n \times M$, where M has a compact, open subgroup. By Lemma 1 of [5] and our hypothesis, $B(G)$ is an open, divisible, power-open subgroup of M. Let K be a compact, open, power-open subgroup of $B(G)$. Then $M \cong B(G) \times (M/B(G)) \cong E(K) \times L_1 \times (M/B(G))$, where the two latter groups are discrete, divisible and respectively torsion and torsion-free. It is now clear that $G \cong R^n \times E(K) \times L$ as stated. Finally, we note that if K is a compact, power-open group, then $E(K)/K$ is countable. As an LCA group G is σ-compact if and only if every open subgroup is of countable index, the last assertion is obvious, and the proof is complete.

REMARK 2. In view of Corollary 6, the following fact is of interest: an LCA group is power-open if and only if it is an open subgroup of a power-open, divisible LCA group. For a proof, we refer to Corollaries 1.3 and 1.4 of [5].

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