BULL. AUSTRAL. MATH. SOC. 05C45
VOL. 29 (1984), 389-399. (05C38)

GRAPHS WITH EULERIAN CHAINS

Roger B. Eggleton and Donald K. Skilton

An *eulerian chain* in a graph is a continuous route which traces every edge exactly once. It may be finite or infinite, and may have 0, 1 or 2 end vertices. For each kind of eulerian chain, there is a characterization of the graphs which admit such a tracing. This paper derives a uniform characterization of graphs with an eulerian chain, regardless of the kind of chain. Relationships between the edge complements of various kinds of finite subgraphs are also investigated, and hence a sharpened version of the eulerian chain characterization is derived.

1. Introduction

An *eulerian chain* in a graph is a continuous route which traces every edge of the graph exactly once. Eulerian chains are of four possible types:

(1) *eulerian trail*, a finite eulerian chain with two end vertices;

(2) *eulerian circuit*, a finite eulerian chain with no end vertices;

(3) *eulerian one-way chain*, an infinite eulerian chain with exactly one end vertex;

Received 19 December 1983.

https://doi.org/10.1017/S0004972700021638 Published online by Cambridge University Press

(4) *eulerian two-way chain*, an infinite eulerian chain with no end vertex.

A characterization of the graphs which have an eulerian circuit is well-known; so too is a characterization of those graphs which have an eulerian trail. These results are due to Euler [3], hence the present terminology. Less widely known are characterizations of the graphs which have an eulerian one-way chain, and those which have an eulerian two-way chain. The problem of characterizing such graphs was solved by Erdös, Grünwald (= Gallai) and Vázsonyi [2]. For convenience all these results are stated in Section 3. The purpose of the present paper is to synthesize these results into a single characterization of the graphs which have an eulerian chain, without explicit separation into cases dependent on the nature of that chain. In giving such a characterization, we shall be able to place certain additional restrictions on the conditions given by [2].

The following terminology is convenient. An *even* graph is any graph with no vertex of odd degree (though vertices of infinite degree are not excluded). An *eulerian* graph is a connected finite even graph. A *quasi-eulerian* graph is a connected countably infinite even graph. (Some authors omit connectedness from their definitions of the last two terms, and allow quasi-eulerian graphs to be uncountably infinite.)

If H is a subgraph of a graph G, the *edge complement* of H in G is the graph $G\backslash EH$ induced by the edges of G which are not edges of H. The *cofinite rank* of G is the smallest cardinal α such that for every finite subgraph H of G, the number of infinite components in $G\backslash EH$ is at most α. We similarly define the *connected cofinite rank*, the *even cofinite rank* and the *eulerian cofinite rank* of G to correspond to the cases in which H is further restricted to be connected, even, or eulerian, respectively.

Clearly, any graph which has an eulerian chain must be connected and countable, so this restriction is built into the following synthesis of the conditions of [2] and [3].

THEOREM 1. *Let G be a connected countable graph with n odd vertices, cofinite rank r and even cofinite rank s. Then G has an eulerian chain if and only if $n + r \leq 2$ and $s \leq 1$.*

We shall in fact prove the following sharper form of this

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characterization.

THEOREM 2. *Let G be a connected countable graph with n odd vertices, connected cofinite rank c and eulerian cofinite rank e. Then G has an eulerian chain if and only if $n + c \leq 2$ and $e \leq 1$.*

In [*1*] we recently announced, without proof, an intermediate version of these formulations. That version is an immediate corollary of the results in the present paper.

In Section 2 we shall examine some relationships between the various kinds of cofinite rank. The results obtained are applied in Section 3 to deduce Theorems 1 and 2 from the characterizations of Euler and Erdős *et al.* The paper concludes in Section 4 by briefly examining the independence of the various kinds of cofinite rank.

2. Cofinite rank relationships

First consider the relationship between cofinite rank and connected cofinite rank.

THEOREM 3. *For any connected graph, the cofinite rank and connected cofinite rank are equal.*

Proof. The theorem is clearly true for finite graphs. Let G be a connected infinite graph with cofinite rank r and connected cofinite rank c. By definition, $c \leq r$. We now prove that equality holds. Let H be a disconnected finite subgraph of G, and suppose $G\backslash EH$ has n infinite components. Because G is connected, there is a path P in $G\backslash EH$ between two components H_1 and H_2 of H. Let $H' := H \cup P$. Then $H_1 \cup H_2 \cup P$ belongs to a single component of H', so H' has fewer components than H. Also $G\backslash EH'$ has at least n infinite components. Iteration of this procedure yields a connected finite subgraph of G whose edge complement has at least n infinite components, so $n \leq c$. Hence $r \leq c$, and so $r = c$, as required. □

Next we examine the relationship between even cofinite rank and eulerian cofinite rank.

We shall say that a graph G has property P if, *for each finite even subgraph H having exactly two components, the graph $G\backslash EH$ has a*

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single infinite component. An important class of quasi-eulerian graphs has this property.

LEMMA 1. *If G is a quasi-eulerian graph with eulerian cofinite rank* 1 *, then G has property* P .

Proof. Let G be a quasi-eulerian graph with eulerian cofinite rank 1 . Suppose G does not have property P : we shall show this leads to a contradiction.

By hypothesis, G has an even subgraph H , comprising two vertex disjoint eulerian subgraphs H_1 and H_2 , such that $G\backslash EH$ has at least two infinite components. Since H is finite and G is quasi-eulerian, every component of $G\backslash EH$ is even. If K is any component of $G\backslash EH$, its *vertices of attachment* are the vertices shared by K and H , and its *edges of attachment* are the edges of H which are incident with the vertices of attachment of K . A *connecting path* for H is a path in $G\backslash EH$ with end vertices in H_1 and H_2 , and no other vertices in common with H . Three properties of such paths will now be established.

(1) Every infinite component of $G\backslash EH$ contains a connecting path for H .

Let K and K' be any two infinite components of $G\backslash EH$, and suppose K does not contain a connecting path for H . Then all vertices of attachment of K must be in one component of H , say H_1 . Therefore all edges of attachment of K are in H_1 , so K is a component of $G\backslash EH_1$. Also, $G\backslash EH_1$ has a component which contains K' , and K' is disjoint from K , so $G\backslash EH_1$ has at least two infinite components. Because G has eulerian cofinite rank 1 , this contradicts the fact that H_1 is eulerian. Hence there is no infinite component K of $G\backslash EH$ which does not contain a connecting path for H .

(2) No two edge disjoint connecting paths for H have the same end vertices.

Suppose P and P' are edge disjoint connecting paths for H . Each infinite component of $G\backslash EH$ must contain at least one infinite component of $G\backslash E(H \cup P \cup P')$, so this graph has at least two infinite components.

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But G has eulerian cofinite rank 1 so $H \cup P \cup P'$ cannot be eulerian. But $H \cup P \cup P'$ is connected, and both components of H are eulerian, so P and P' cannot have the same end vertices.

(3) No finite component of $G \backslash EH$ contains a connecting path for H .

Let K be any finite component of $G \backslash EH$. Since G has eulerian cofinite rank 1 , and the infinite components of $G \backslash E(H \cup K)$ are the same as those of $G \backslash EH$, at least two in number, it follows that $H \cup K$ is not eulerian. But K and each component of H are edge disjoint eulerian graphs, so $H \cup K$ is not connected. Therefore K cannot contain a connecting path for H .

Let K_1 and K_2 be any two infinite components of $G \backslash EH$. Each is incident with H_1 and H_2 , by property (1). Since H_1 is eulerian, its edges can be labelled $a_1, a_2, \ldots, a_m$ so that $v_1 a_1 v_2 a_2 \ldots v_m a_m v_1$ is an eulerian circuit with vertices v_i . Moreover, we may suppose the labelling to be such that K_1 is incident with v_1 , and K_2 is incident with v_r , while no intervening vertex v_i $(1 < i < r)$ is incident with any infinite component of $G \backslash EH$. Again, H_2 is eulerian so its edges can be labelled $b_1, b_2, \ldots, b_n$ so that $w_1 b_1 w_2 b_2 \ldots w_n b_n w_1$ is an eulerian circuit with vertices w_i . We may also suppose the labelling to be such that w_1 is incident with K_1 , and w_s is incident with K_2 , while no intervening vertex w_i $(1 < i < s)$ is incident with K_1 or K_2 .

Since K_1 is connected, it contains a path P_1 with end vertices v_1 and w_1 . Similarly, there is a path P_2 in K_2 with end vertices v_r and w_s . Let Q_1 and T_1 be the trails in H_1 specified by $v_1 a_1 v_2 a_2 \ldots v_r$ and $v_r a_r \ldots v_m a_m v_1$, respectively. Thus Q_1 and T_1 are edge disjoint and $H_1 = Q_1 \cup T_1$. Similarly, let Q_2 and T_2 be the trails in H_2 specified by $w_1 b_1 w_2 b_2 \ldots w_s$ and $w_s b_s \ldots w_n b_n w_1$ respectively. Clearly, $C := P_1 \cup T_1 \cup P_2 \cup T_2$ is a circuit, so an

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eulerian subgraph of G . We shall now show that $G\backslash EC$ must have at least two infinite components, contrary to the fact that G has eulerian cofinite rank 1 . This shows that H does not exist, whence the lemma.

All vertices of attachment of K_1 and K_2 are in $T_1 \cup T_2$. Hence the only edges of attachment of K_1 which are not in $T_1 \cup T_2$ are a_1 and b_1 . Similarly, the only edges of attachment of K_2 which are not in $T_1 \cup T_2$ are a_{r-1} and b_{s-1} .

Since there are not two edge disjoint paths in K_1 between v_1 and w_1 , by property (2), it follows that v_1 and w_1 lie in different components of $K_1\backslash EP_1$, which we shall denote by K_1' and K_1'' respectively. With the possible exception of v_1 , no vertex in K_1' is odd, because K_1 is quasi-eulerian; v_1 may be odd or infinite. In either case it follows that K_1' is infinite. Similarly K_1'' is infinite.

Suppose there is a path P in $G\backslash EC$ between v_1 and w_1 . Since K_1' and K_1'' are disjoint, and the only edges of attachment of K_1 in $G\backslash E(T_1 \cup T_2)$ are a_1 and b_1 , so the edges of P incident with v_1 and w_1 must be a_1 and b_1 . This implies that P is edge disjoint from K_1 . Also, it follows that P contains a connecting path P' for with end vertices in Q_1 and Q_2 . By property (3), P' lies in an infinite component of $G\backslash EH$. This must be K_2 , since P is edge disjoint from K_1 , and no other infinite component of $G\backslash EH$ is incident with Q_1 . Hence the end vertices of P' are v_r and w_s , so cannot be edge disjoint from P_2 , by property (2), and so P' cannot exist in $G\backslash EC$. It follows that there is no path in $G\backslash EC$ between v_1 and w_1 . Hence $G\backslash EC$ has at least two infinite components, one containing K_1' and one containing K_1'' . This is the required contradiction which shows that H does not exist, and hence that G has property P . □

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If the graph G has property P , there is only one infinite component after the edges of any two disjoint eulerian subgraphs are deleted, so if H is any eulerian subgraph then $G \backslash EH$ has eulerian cofinite rank 1 . This equivalent of property P is useful in proving the next lemma.

LEMMA 2. *A quasi-eulerian graph which has eulerian cofinite rank* 1 *has even cofinite rank* 1 .

Proof. Let G be a quasi-eulerian graph with eulerian cofinite rank 1 . Then G has property P , by Lemma 1. Let H be a finite even subgraph of G having components $H_1, H_2, \ldots, H_n$. Since H_1 is eulerian it follows that $G \backslash EH_1$ is even, has a single infinite component, and has eulerian cofinite rank 1 . Let G_1 be the infinite component of $G \backslash EH_1$. Then G_1 is quasi-eulerian, has eulerian cofinite rank 1 , and so has property P . Also $G \backslash EG_1$ is finite and contains H_1 . Clearly $H_2, H_3, \ldots, H_n$ each lie in some component of $G \backslash EH_1$. Thus $G_1 \backslash EH_2$ is even, has a single infinite component, and has eulerian cofinite rank 1 . Let G_2 be the infinite component of $G_1 \backslash EH_2$. Then G_2 is quasi-eulerian, has eulerian cofinite rank 1 , and so has property P . Also $G \backslash EG_2$ is finite and contains $H_1 \cup H_2$. Continuing thus, the quasi-eulerian graph G_n which is iteratively defined by this procedure is such that $G \backslash EG_n$ is finite and contains H . It follows that one component of $G \backslash EH$ contains G_n , and any other components are finite. Thus $G \backslash EH$ has a single infinite component.

Since H was arbitrary, G has even cofinite rank 1 . □

THEOREM 4. *For any quasi-eulerian graph, the eulerian cofinite rank is* 1 *precisely when the even cofinite rank is* 1 .

Proof. Let G be a quasi-eulerian graph with even cofinite rank s and eulerian cofinite rank e , so $e \leq s$. If $s = 1$ then $e \leq 1$; but G is infinite, so $e = 1$. Conversely, if $e = 1$ then $s = 1$, by Lemma 2. □

3. Characterizing graphs with eulerian chains

The theorems of Euler and Erdős, Gallai and Vázsonyi characterizing graphs with various kinds of eulerian chain are our essential starting point. They may be stated as follows.

THEOREM 5 (Euler). *A connected finite graph has an eulerian circuit precisely if it is even, and has an eulerian trail precisely when it has two odd vertices.*

THEOREM 6 (Erdős, Gallai, Vázsonyi). *A connected countably infinite graph has an eulerian one-way chain precisely if it has cofinite rank* 1 *, has at most one odd vertex, and has at least one infinite vertex if it has no odd vertex.*

THEOREM 7 (Erdős, Gallai, Vázsonyi). *A quasi-eulerian graph has an eulerian two-way chain precisely if it has cofinite rank at most* 2 *and even cofinite rank* 1 .

We now verify that Theorem 1 follows from Theorems 5, 6 and 7. A graph with exactly one odd vertex must be infinite, so the parameters n, r, s in Theorem 1 must have $r \geq 1$ if $n = 1$. Also, cofinite rank cannot be less than even cofinite rank, and deletion of a single vertex shows that even cofinite rank is at least 1 for any infinite graph. Thus $s \geq 1$ exactly when $r \geq 1$. Only five triples of parameters meet these conditions and the inequalities of Theorem 1, namely

$$(n, r, s) = (0, 0, 0), (2, 0, 0), (1, 1, 1), (0, 1, 1) \text{ and } (0, 2, 1) .$$

The existence of an eulerian chain is guaranteed in the first two cases by Theorem 5, in the next case by Theorem 6, and in the last two cases by Theorem 7. Conversely, for each of the four kinds of eulerian chain, it is trivial to check that the inequalities of Theorem 1 are satisfied. Thus, Theorem 1 characterizes all graphs with an eulerian chain.

Theorem 2 now follows from Theorem 1, using the results established in Theorems 3 and 4.

4. Independence of the kinds of cofinite rank

In Theorems 3 and 4 we established relations between various kinds of cofinite rank. How far do such relations extend? This question will now be briefly discussed, with regard to connected graphs. (The results for disconnected graphs easily follow from this discussion.)

Theorem 3 shows that the cofinite and connected cofinite ranks are always equal for a connected graph. However, the even and eulerian cofinite ranks are not always equal. For example, Figure 1 shows a graph which has cofinite rank 2 but even cofinite rank which is infinite. Indeed, it is not difficult to construct a graph which has any countable number of odd vertices and any eulerian cofinite rank greater than 1 , but for which the even cofinite rank is infinite.

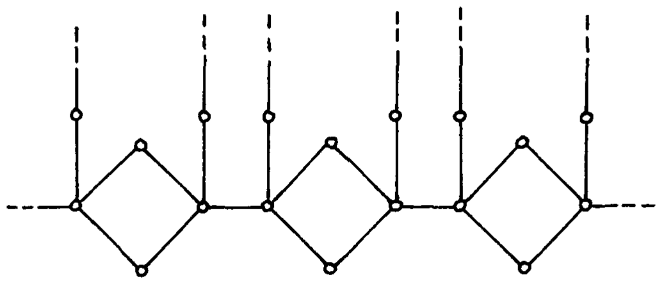

FIGURE 1. A graph with eulerian cofinite rank 2 but even cofinite rank which is infinite.

Theorem 4 shows that, for a quasi-eulerian graph, the eulerian cofinite rank is 1 precisely when the even cofinite rank is 1 . It can readily be shown that this also holds for any connected countable graph with a single odd vertex. On the other hand, Figure 2 shows two graphs which both have more than one odd vertex, and have eulerian cofinite rank 1 but have even cofinite rank greater than 1 . The graphs in Figure 2 generalize to give a connected, countably infinite graph which has eulerian cofinite rank 1 and any given even cofinite rank.

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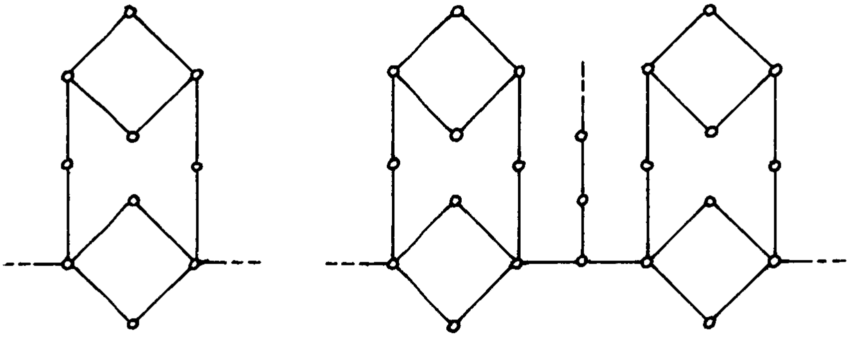

FIGURE 2. Two graphs with eulerian cofinite rank 1; G has even cofinite rank 2 and H has even cofinite rank 3.

In conclusion, we note that connected and eulerian cofinite ranks are independent. Figure 3 shows a graph which has eulerian cofinite rank 1 but connected confinite rank which is infinite.

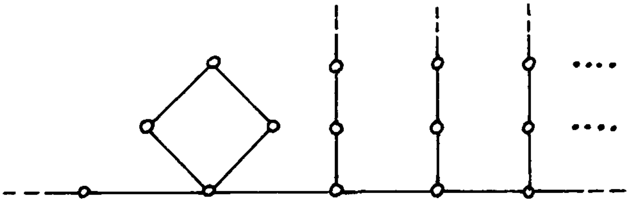

FIGURE 3. A graph with eulerian confinite rank 1 and connected cofinite rank which is infinite.

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Department of Mathematics,
The University of Newcastle,
Newcastle,
New South Wales 2308,
Australia.

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