Canad. Math. Bull.Vol. **34** (3), 1991 pp. 338–342

POORLY APPROXIMATED $\mathbb{Z}_2$-COCYCLES FOR TRANSFORMATIONS WITH RATIONAL DISCRETE SPECTRUM

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ABSTRACT. Let T be an ergodic automorphism with rational discrete spectrum and ϕ a $\mathbb{Z}_2$-cocyle for T. We show that the resulting two-point extension of T is cohomologous to a Morse cocycle if ϕ is approximated with speed $o(1/n)$.

On the other hand, we show by example that this is in general false when the speed of approximation is $O(1/n)$.

In this paper we answer a question raised in [1]. In order to formulate the problem and its solution, we begin with some preliminary definitions and results.

Throughout this paper we let T denote an ergodic measure-preserving automorphism of a non-atomic Lebesgue probability space $(\mathfrak{X}, \mathfrak{B}, \mu)$. If $\phi: (\mathfrak{X}, \mathfrak{B}, \mu) \longrightarrow \mathbb{Z}_2$ is a measurable function, we obtain a measure-preserving automorphism T_ϕ of $\mathfrak{X} \times \mathbb{Z}_2$ (endowed with the obvious product measure $\bar{\mu}$) given by $T_\phi(x, i) = \big(Tx, i + \phi(x)\big)$. (We write $\mathbb{Z}_2$ additively as $\{0, 1\}$). We refer to T_ϕ as a $\mathbb{Z}_2$-extension of T, and to ϕ (perhaps inappropriately) as a $\mathbb{Z}_2$-cocycle for T. We say two $\mathbb{Z}_2$-extensions T_ϕ and T_ψ of T are *T-relatively isomorphic* if there is an isomorphism between T_ϕ and T_ψ of the form $(x, i) \longrightarrow \big(x, i + \rho(x)\big)$, where $\rho: \mathfrak{X} \longrightarrow \mathbb{Z}_2$ is measurable.

The following lemma is elementary and well-known.

LEMMA 1. *Let T on $(\mathfrak{X}, \mathfrak{B}, \mu)$, ϕ and ψ be as above. The following are equivalent.*

1.) T_ϕ and T_ψ are T-relatively isomorphic.

2.) There is a measurable function $\rho: \mathfrak{X} \longrightarrow \mathbb{Z}_2$ satisfying $\rho(x) + \rho(Tx) = \phi(x) + \psi(x)$ almost everywhere.

3.) The extension $T_{\phi+\psi}$ is not ergodic.

PROOF. The equivalence of 1.) and 2.) is immediate from the definitions. To show that 3.) implies 2.), let A be a proper invariant set for $T_{\phi+\psi}$. Note that this implies that $\bar{\mu}(A) = 1/2$ and for μ-almost every $x \in \mathfrak{X}$, A contains exactly one of the points $(x, 0)$ and $(x, 1)$. Letting $B = \{x \in \mathfrak{X} \mid (x, 0) \in A\}$, one sets $\rho = \chi_B$ and verifies that ρ satisfies 2.). To show that 2.) implies 3.), one can let $A = \{\big(x, \rho(x)\big) \mid x \in \mathfrak{X}\}$, and verify that A is a proper invariant set for $T_{\phi+\psi}$. ∎

If the conditions of the lemma are satisfied, we say ϕ and ψ are *cohomologous*.

The automorphism T is said to have rational discrete spectrum if the eigenfunctions of T span a dense subspace of $L^2(\mathfrak{X}, \mathfrak{B}, \mu)$ and the eigenvalues are all roots of 1. An

Received by the editors September 20, 1989; revised: September 14, 1990.
AMS subject classification: 28D05.

equivalent, and for our purposes more useful formulation is the following. A stack τ of height $h(\tau) = n$ for an automorphism T is a pairwise disjoint sequence of measurable sets $\{T^iB\}_{i=0}^{n-1}$ such that $\mu(\bigcup_i T^iB) = 1$. The set B is called the *base* of τ and the sets T^iB, $i = 0, 1, \ldots, n-1$ are called the *levels* of τ. An ergodic automorphism T has rational discrete spectrum if and only if there exists a sequence τ_j of stacks for T whose levels generate $\mathfrak{B}$. This last condition means that for all $A \in \mathfrak{B}$ and $\epsilon > 0$ there exists a set U satisfying $\mu(U \triangle A) < \epsilon$, where U is a union of levels of some τ_j. It is not hard to see that given such a sequence of stacks for T, we may assume, by passing to a subsequence and permuting the levels of the stacks, that the stacks in fact satisfy

a.) $h(\tau_j) \mid h(\tau_{j+1})$

b.) $B_{j+1} \subset B_j$ (where B_j denotes the base of τ_j) and

c.) for all $A \in \mathfrak{B}$ and $\epsilon > 0$ there exists a $J \in \mathbb{N}$ such that for all $j > J$ there exists a set U satisfying $\mu(U \triangle A) < \epsilon$, where U is a union of levels of τ_j.

Condition c.) in fact follows from a.) and the assumption that the original sequence of stacks generates. The effect of conditions a.) and b.) is simply to arrange that each τ_{j+1} is obtained from τ_j by cutting τ_j into $h(\tau_{j+1})/h(\tau_j)$ columns of equal measure and concatenating them.

For brevity, we will refer to a sequence satisfying a.), b.) and c.) as a generating sequence of stacks for T. Note that every subsequence of a generating sequence of stacks for T is again such a sequence.

Now and for the remainder of the paper we suppose that T is an ergodic automorphism of $(\mathfrak{X}, \mathfrak{B}, \mu)$ with rational discrete spectrum. A cocycle ϕ for T is called a Morse cocycle if for some generating sequence of stacks τ_j, ϕ is constant on the sets $T^i(B_j)$, $i = 0, 1, 2, \ldots h(\tau_j) - 2$. The term *Morse cocycle* is used because the measure-preserving automorphisms arising from generalized Morse sequences can be given as $\mathbb{Z}_2$-extensions of automorphisms of rational discrete spectrum by such a cocycle. (See [2]).

Let $\phi: (\mathfrak{X}, \mathfrak{B}, \mu) \longrightarrow \mathbb{Z}_2$ be a cocycle for T and let $f: \mathbb{N} \longrightarrow \mathbb{R}$. We say that ϕ is *approximated with speed* $o\big(f(n)\big)$ (respectively $O\big(f(n)\big)$ if for some generating sequence of stacks τ_j, there is a subsequence τ_{j_k}, such that for each k, there is a set U_k which is a union of levels of τ_{j_k} and satisfies

$$\mu\big(U_k \triangle \phi^{-1}(1)\big) = o\Big(f\big(h(\tau_{j_k})\big)\Big)$$

(respectively $O\Big(f\big(h(\tau_{j_k})\big)\Big)$).

THEOREM 1. *Let ϕ be a cocycle for T that is approximated with speed $o(1/n)$. Then there is a Morse cocycle ψ for T that is cohomologous to ϕ.*

REMARK. This is a strengthening of Theorem 1 of [1], where the hypothesis is that ϕ is approximated with speed $O(1/n^{1+\epsilon})$, for some $\epsilon > 0$. The proof we give here is a variant of their argument.

PROOF. Fix a generating sequence of stacks τ_j for T. We have by hypothesis a subsequence τ_{j_k} and sets U_k, each a union of levels of τ_{j_k} such that $\mu\big(U_k \triangle \phi^{-1}(1)\big) =$

$o\big(1/h(\tau_{j_k})\big)$. By passing to a subsequence and relabeling it as τ_k, we may assume that we have sets U_k, each a union of levels of τ_k such that $\mu\big(U_k \triangle \phi^{-1}(1)\big) \leq \epsilon_k\big(1/h(\tau_{j_k})\big)$, where $\sum_k \epsilon_k \leq 1/2$. On this sequence of stacks we will inductively define a Morse cocycle ψ such that $T_{\phi+\psi}$ is not ergodic. It will be convenient to let n_k denote $h(\tau_k)$ and q_k denote $\frac{n_{k+1}}{n_k}$. Note that on each τ_k there is a unique choice of U_k satisfying the given conditions. Indeed, for each $i = 0, 1, \ldots, n_k - 1$, either $\phi^{-1}(1)$ or $\phi^{-1}(0)$ occupies a fraction of at least $1 - \epsilon_k$ of T^iB_k. Let $\phi_k = \chi_{U_k}$. (Here we regard the values of ϕ_k as being elements of $\mathbb{Z}_2$).

Define $\psi(x) = 0$ for all $x \in \bigcup_{i=0}^{n_1-2} T^iB_1$. Having defined ψ on $\bigcup_{i=0}^{n_k-2} T^iB_k$, we extend the definition to $\bigcup_{i=0}^{n_{k+1}-2} T^iB_{k+1}$. The idea is to define ψ on the last levels of the columns obtained from τ_k in such a way as to make the sum of the values of $(\phi + \psi)(x)$ across such columns equal 0, for most x. In particular, for $x \in B_{k+1}$ and $i = 1, 2, \ldots, q_k - 1$, we set

$$\psi\big(T^{in_k-1}(x)\big) = \phi_{k+1}\big(T^{in_k-1}(x)\big) + \sum_{m=0}^{n_k-1}(\psi + \phi_{k+1})\big(T^{(i-1)n_k+m}(x)\big).$$

Now let $C_k = \bigcup_{i=0}^{n_k-1} T^i_{\phi+\psi}(B_k \times \{0\})$. We will show that these sets converge to a proper invariant set for $T_{\phi+\psi}$.

It is clear that for each k, $\bar{\mu}(C_k) = 1/2$ and $\bar{\mu}\big(C_k \triangle T_{\phi+\psi}(C_k)\big) \leq (2n_k)^{-1}$.

In the following estimates, we use the notation $A \underset{\eta}{\sim} A'$ to signify the condition $\bar{\mu}(A \triangle A') < \eta$. For each k we have $\mu(\{x \mid \phi_k(x) \neq \phi(x)\}) < \epsilon_k(1/n)$, so that

$$C_k = \bigcup_{i=0}^{n_k-1} T^i_{\phi+\psi}(B_k \times \{0\}) \underset{2\epsilon_k}{\sim} \bigcup_{i=0}^{n_k-1} T^i_{\phi_k+\psi}(B_k \times \{0\}).$$

Now $B_k \times \{0\} = \bigcup_{i=0}^{q_k-1} T^{in_k}(B_{k+1}) \times \{0\}$ so that $\bigcup_{i=0}^{n_k-1} T^i_{\phi_k+\psi}(B_k \times \{0\}) = \bigcup_{m=0}^{q_k-1}\bigcup_{i=0}^{n_k-1} T^i_{\phi_k+\psi}\big(T^{mn_k}(B_{k+1}) \times \{0\}\big)$. But if $T^{mn_k+i}B_{k+1}$ is a level of τ_{k+1} on which ϕ_{k+1} differs from ϕ_k (remember that both ϕ_{k+1} and ϕ_k are constant on levels of τ_{k+1}), then ϕ differs from ϕ_k on a subset of $T^{mn_k+i}B_{k+1}$ of measure at least $(1 - \epsilon_{k+1})/n_{k+1}$. Each such level contributes $(1 - \epsilon_{k+1})/n_{k+1}$ to $\mu\big(U_k \triangle \phi^{-1}(1)\big)$, and therefore the levels $T^{mn_k}B_{k+1} \subset B_k$ of $\tau_{k+1}(m = 0, 1, \ldots, q_k - 1)$ such that for some $i = 0, 1, \ldots, n_k - 1$, ϕ_{k+1} and ϕ_k differ on $T^{mn_k+i}B_{k+1}$ form a set of measure less than $\left(\frac{\epsilon_k}{1-\epsilon_{k+1}}\right) \cdot n_k^{-1}$. Thus, for all but a fraction $\left(\frac{\epsilon_k}{1-\epsilon_{k+1}}\right)$ of the $m \in [0, q_k - 1]$, $\bigcup_{i=0}^{n_k-1} T^i_{\phi_k+\psi}\big(T^{mn_k}(B_{k+1}) \times \{0\}\big) = \bigcup_{i=0}^{n_k-1} T^i_{\phi_{k+1}+\psi}\big(T^{mn_k}(B_{k+1}) \times \{0\}\big)$, and therefore,

$$\bigcup_{m=0}^{q_k-1}\bigcup_{i=0}^{n_k-1} T^i_{\phi_k+\psi}\big(T^{mn_k}(B_{k+1}) \times \{0\}\big) \underset{\epsilon_k/(1-\epsilon_{k+1})}{\sim}$$
$$\bigcup_{m=0}^{q_k-1}\bigcup_{i=0}^{n_k-1} T^i_{\phi_{k+1}+\psi}\big(T^{mn_k}(B_{k+1}) \times \{0\}\big) = \bigcup_{i=0}^{n_{k+1}-1} T^i_{\phi_{k+1}+\psi}(B_{k+1} \times \{0\}) \underset{2\epsilon_{k+1}}{\sim}$$
$$\bigcup_{i=0}^{n_{k+1}-1} T^i_{\phi+\psi}(B_{k+1} \times \{0\}) = C_{k+1}.$$

https://doi.org/10.4153/CMB-1991-054-7 Published online by Cambridge University Press

This establishes that the sets C_k form a Cauchy sequence in the metric of symmetric difference, and hence converge to a necessarily $T_{\phi+\psi}$-invariant set of measure $1/2$. ∎

The question raised in [1] is whether the conclusion of Theorem 1 holds when ϕ is approximated with speed $O(1/n)$. We show here by example that it does not.

Let T be the ergodic automorphism of $(\mathfrak{X}, \mathfrak{B}, \mu)$ with rational discrete spectrum, whose eigenvalues are all the roots of unity of order a power of 2. Thus, T admits a generating sequence of stacks τ_k of heights $h(\tau_k) = 2^k$, $k \in \mathbb{N}$. We will construct a cocycle ϕ for T such that ϕ is approximated with speed $O(1/n)$, but such that for all Morse cocycles ψ for T, $T_{\phi+\psi}$ is ergodic. In other words, for all Morse cocycles ψ for T, ϕ and ψ are not cohomologous.

The desired cocycle ϕ is defined as follows. As before, we let B_k denote the base of τ_k, and we recall that, by assumption, $B_k \supset B_{k+1}$. For each k, let

$$\phi(x) = \left\{ \begin{matrix} 0, & \text{if } x \in T^{2^{k-1}-1}(B_{k+1}) \\ 1, & \text{if } x \in T^{3\cdot 2^{k-1}-1}(B_{k+1}) \end{matrix} \right\}.$$

It is clear that ϕ is approximated with speed $O(1/n)$ (and not with speed $o(1/n)$).

Now let ψ be a Morse cocycle for T. Then there is a sequence $k_i \longrightarrow \infty$ such that ψ is constant on all the sets $T^j B_{k_i}$, $i \in \mathbb{N}$, $j = 0, 1, \ldots, 2^{k_i} - 2$. The sequence of stacks τ_{k_i} is again a generating sequence of stacks for T. Furthermore, since the levels of the stacks τ_{k_i} generate $\mathfrak{B}$, the sets of the form $L \times \{c\}$, $c \in \mathbb{Z}_2$, where L is a level of a stack τ_{k_i}, generate the product σ-algebra in $X \times \mathbb{Z}_2$. Let $A \subset X \times \mathbb{Z}_2$ be a set of measure $\bar{\mu}(A) \in (0, 1)$ and $\epsilon > 0$. We will show that $\bar{\mu}\left(\bigcup_{m \in \mathbb{Z}} T^m_{\phi+\psi}(A)\right) > 1 - 4\epsilon$. Since A and ϵ are arbitrary, it will follow that $T^m_{\phi+\psi}$ is ergodic. Choose i so that for some level $L = T^p B_{k_i}$ of τ_{k_i}, and some $c \in \mathbb{Z}_2$, $\bar{\mu}\big(A \cap (L \times \{c\})\big) > (1 - \epsilon)\bar{\mu}(L \times \{c\})$.

LEMMA 2. *(Notation as above). For all $x \in B_{k_{i+1}+1}$ and all $y \in T^{2^{k_{i+1}}}(B_{k_{i+1}+1})$, the sequences $\{(\phi+\psi)(T^m x)\}_{m=0}^{2^{k_{i+1}}-1}$ and $\{(\phi+\psi)(T^m y)\}_{m=0}^{2^{k_{i+1}}-1}$ are identical, except for the terms corresponding to $m = 2^{k_{i+1}} - 1$, at which they differ.*

PROOF. Since x and y both lie in $B_{k_{i+1}}$, this follows from the fact that ψ is constant on all but the last level of $\tau_{k_{i+1}}$, and ϕ is constant on all but the central level $T^{2^{k_{i+1}-1}}(B_{k_{i+1}})$, on which it takes the two values 0 and 1, as described above. ∎

The set L is a union of levels $T^{p+j\cdot 2^{k_i}}(B_{k_{i+1}+1})$, $j = 0, 1, 2, \ldots, 2^{k_{i+1}-k_i+1} - 1$ of the stack $\tau_{k_{i+1}}$, so that $L \times \{c\}$ is a union of the sets $T^{p+j\cdot 2^{k_i}}(B_{k_{i+1}+1}) \times \{c\}$. If L_j is one of these levels, where $0 \le j \cdot 2^{k_i} \le 2^{k_{i+1}} - 1$ (in other words, L_j is in the first half of $\tau_{k_{i+1}+1}$), let L'_j denote $T^{2^{k_{i+1}}} L_j$. We say $L_j \times \{c\}$ is good if both $L_j \times \{c\}$ and $L'_j \times \{c\}$ are contained in $\bigcup_{m \in \mathbb{Z}} T^m_{\phi+\psi}(A)$ except for a fraction less than 4ϵ of their measure. It follows that there must exist good sets $L_{j_1} \times \{c\}$ and $L_{j_2} \times \{c\}$ such that L_{j_1} occurs in the first quarter of $\tau_{k_{i+1}+1}$ $(0 \le j_1 \cdot 2^{k_i} \le 2^{k_{i+1}-1} - 1)$ and L_{j_2} occurs in the second quarter of $\tau_{k_{i+1}+1}$ $(2^{k_{i+1}-1} \le j_2 \cdot 2^{k_i} \le 2^{k_{i+1}} - 1)$. Let $r = (j_1 - j_2)2^{k_i}$ so that $T^r(L_{j_1}) = L_{j_2}$. Because of Lemma 2, we see that either $T^r_{\phi+\psi}(L_{j_1} \times \{c\}) = L_{j_2} \times \{c+1\}$, or $T^r_{\phi+\psi}(L'_{j_1} \times \{c\}) = L'_{j_2} \times \{c+1\}$. Suppose, without loss of generality, that the first is the case. Then $A \cup T^r_{\phi+\psi} A$ contains all but a fraction 4ϵ of $(L_{j_2} \times \{c\}) \cup (L_{j_2} \times \{c+1\})$, and therefore $\bigcup_{m=-2^{k_{i+1}+1}}^{m=2^{k_{i+1}+1}} T_{\phi+\psi}^{m}(A)$ has $\bar{\mu}$ measure greater than $1 - 4\epsilon$, and we are done.

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