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ON SOME CONGRUENCES INVOLVING CENTRAL BINOMIAL COEFFICIENTS

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Abstract

We prove the following conjecture of Z.-W. Sun [‘On congruences related to central binomial coefficients’, *J. Number Theory* **13**(11) (2011), 2219–2238]. Let p be an odd prime. Then

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k2^k}\equiv -\frac{1}{2}H_{(p-1)/2}+\frac{7}{16}p^2B_{p-3} \pmod{p^3},$$

where H_n is the nth harmonic number and B_n is the nth Bernoulli number. In addition, we evaluate $\sum_{k=0}^{p-1}(ak+b)\binom{2k}{k}/2^k$ modulo p^3 for any p-adic integers a,b.

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1. Introduction

In 2006, Adamchuk [1] proposed the following congruence involving central binomial coefficients: for any prime $p\equiv 1 \pmod 3$,

$$\sum_{k=1}^{2(p-1)/3}\binom{2k}{k}\equiv 0 \pmod{p^2}.$$

This conjecture was confirmed by the author [6]. Many researchers studied congruences for sums of binomial coefficients (see, for instance, [7, 10, 14, 17]). Pan and Sun [10] used a combinatorial identity to deduce that if p is a prime, then

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right) \pmod{p}\quad \text{for } d=0,1,\dots,p,$$

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where $(\frac{\cdot}{\cdot})$ is the Jacobi symbol. They also showed that for any odd prime p,

$$\sum_{k=0}^{p-1}(3k+1)\binom{2k}{k} \equiv -\left(\frac{p}{3}\right) \pmod{p}.$$

In 2018, Apagodu [2] conjectured that for any odd prime p,

$$\sum_{k=0}^{p-1}(5k+1)\binom{4k}{2k} \equiv -\left(\frac{p}{3}\right) \pmod{p}.$$

Mao and Cao [7] confirmed this conjecture and also showed that for any odd prime p,

$$\sum_{k=0}^{p-1}(15k+5)\binom{4k}{2k} \equiv -\left(\frac{p}{5}\right) \pmod{p}.$$

The Bernoulli numbers $\{B_n\}$ and the Bernoulli polynomials $\{B_n(x)\}$ are given by

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (0 < |x| < 2\pi), \quad B_n(x) = \sum_{k=0}^{n}\binom{n}{k} B_k x^{n-k} \quad (n \in \mathbb{N}).$$

Mattarei and Tauraso [8] deduced that for any prime $p > 3$,

$$\sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) - \frac{1}{3}p^2 B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.$$

Sun [13] obtained many congruences involving central binomial coefficients and proposed many conjectures. Our first goal is to prove a conjecture of Sun ([13, Conjecture 5.2] or [16, Conjecture 6]).

THEOREM 1.1. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k2^k} \equiv -\frac{1}{2}H_{(p-1)/2} + \frac{7}{16}p^2 B_{p-3} \pmod{p^3}.$$

Sun [14] proved that for any odd prime p,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3}, \tag{1.1}$$

where the Euler numbers $\{E_n\}$ are given by

$$E_0 = 1 \quad \text{and} \quad E_{2n} = -\sum_{k=1}^{n}\binom{2n}{2k} E_{2n-2k} \quad (n \geq 1).$$

Our second goal is to generalise (1.1) as follows.

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THEOREM 1.2. *For any odd prime p and p-adic integers a, b,*

$$\sum_{k=0}^{p-1}(ak+b)\frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2}(b-a) + ap(2-2^{p-1}) - (b-a)p^2E_{p-3} \pmod{p^3}.$$

REMARK 1.3. If we set $a = 0, b = 1$, we obtain (1.1), and with $a = b = 1$, we obtain

$$\sum_{k=0}^{p-1}(k+1)\frac{\binom{2k}{k}}{2^k} \equiv p(2-2^{p-1}) \pmod{p^3}. \tag{1.2}$$

Finally, we prove the following result.

THEOREM 1.4. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}(k+1)^2\frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} + p(2^{p-1}-2) + p^2 - p^2E_{p-3} \pmod{p^3}.$$

REMARK 1.5. Combining (1.1), (1.2) and Theorem 1.4,

$$\sum_{k=0}^{p-1}(k^2+3k+1)\frac{\binom{2k}{k}}{2^k} \equiv p^2 \pmod{p^3}.$$

Combining Theorem 1.2 with $a = 2, b = 1$ and Theorem 1.4,

$$\sum_{k=0}^{p-1}k^2\frac{\binom{2k}{k}}{2^k} \equiv 2(-1)^{(p-1)/2} + 3p(2^{p-1}-2) + p^2 - 2p^2E_{p-3} \pmod{p^3}.$$

Consequently, we can evaluate $\sum_{k=0}^{p-1}(ak^2+bk+c)\binom{2k}{k}/2^k$ modulo p^3.

We prove Theorem 1.1 in Section 2. Sections 3 and 4 are devoted to proving Theorems 1.2 and 1.4. Our proofs make use of some congruences involving harmonic numbers and combinatorial identities which can be found and proved by the package `Sigma` [11] via the software `Mathematica` and some known congruences.

2. Proof of Theorem 1.1

For $n, m \in \{1, 2, 3, \ldots\}$, define the harmonic numbers of order m by

$$H_n^{(m)} := \sum_{1\le k\le n}\frac{1}{k^m}, \quad H_0^{(m)} := 0.$$

When $m = 1$, these numbers are the classical harmonic numbers.

To prove Theorem 1.1, we need some lemmas and identities. For each positive integer n,

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$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2k}{k}\binom{n}{2k}2^{n-2k}=\binom{2n}{n}, \tag{2.1}$$

$$\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}(H_k^2-H_k^{(2)})=2\sum_{k=1}^{n}\frac{H_{k-1}}{k^2}, \tag{2.2}$$

$$\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}=-H_n, \tag{2.3}$$

$$\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}H_k=-H_n^{(2)}. \tag{2.4}$$

REMARK 2.1. Equation (2.1) follows from [3, (3.99)]; (2.2) can be proved by induction on n; (2.3) and (2.4) follow from [3, (1.45)] and an identity of Hernández [4] (or [15, (3.4)]), respectively.

LEMMA 2.2 [12, Theorems 5.1 and 5.2]. *Let $p>3$ be a prime. Then*

$$H_{p-1}\equiv-\tfrac{1}{3}p^2B_{p-3}\pmod{p^3},\qquad H^{(3)}_{(p-1)/2}\equiv-2B_{p-3}\pmod{p},$$
$$H_{(p-1)/2}\equiv-2q_p(2)+pq_p(2)^2\pmod{p^2},\qquad H^{(2)}_{(p-1)/2}\equiv\tfrac{7}{3}pB_{p-3}\pmod{p^2}.$$

LEMMA 2.3 [15, Lemma 4.2]. *Let $p=2n+1$ be an odd prime and $k\in\{0,\ldots,n\}$. Then*

$$\frac{\binom{n}{k}(-4)^k}{\binom{2k}{k}}\equiv1-p\sum_{j=1}^{k}\frac{1}{2j-1}+\frac{p^2}{2}\Big(\sum_{j=1}^{k}\frac{1}{2j-1}\Big)^2-\frac{p^2}{2}\sum_{j=1}^{k}\frac{1}{(2j-1)^2}\pmod{p^3}.$$

LEMMA 2.4. *Let $p=2n+1$ be an odd prime and $k\in\{0,\ldots,n\}$. Then*

$$\frac{\binom{2k}{k}\binom{p-1}{2k}}{4^k}\equiv\binom{n}{k}(-1)^k\Big(1-\frac{p}{2}H_k+\frac{p^2}{8}(H_k^2-H_k^{(2)})\Big)\pmod{p^3}.$$

PROOF. It is easy to check that

$$\binom{p-1}{2k}=\prod_{j=1}^{2k}\Big(\frac{p-j}{j}\Big)=\prod_{j=1}^{2k}\Big(1-\frac{p}{j}\Big)$$
$$\equiv1-pH_{2k}+\frac{p^2}{2}(H_{2k}^2-H_{2k}^{(2)})\pmod{p^3}.$$

By Lemma 2.3, modulo p^3,

$$\frac{\binom{2k}{k}}{4^k}\equiv\frac{\binom{n}{k}(-1)^k}{1-p(H_{2k}-\frac{1}{2}H_k)+\frac{1}{2}p^2(H_{2k}-\frac{1}{2}H_k)^2-\frac{1}{2}p^2(H_{2k}^{(2)}-\frac{1}{4}H_k^{(2)})}$$
$$\equiv\binom{n}{k}(-1)^k\Big(1+p\Big(H_{2k}-\frac{1}{2}H_k\Big)+\frac{p^2}{2}\Big(H_{2k}-\frac{1}{2}H_k\Big)^2+\frac{p^2}{2}\Big(H_{2k}^{(2)}-\frac{1}{4}H_k^{(2)}\Big)\Big).$$

From this, we immediately obtain the desired result. □

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PROOF OF THEOREM 1.1. The cases $p = 3, 5$ can be checked directly. We will assume $p > 5$ from now on. By (2.1),

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k2^k}=\sum_{k=1}^{p-1}\frac{1}{k2^k}\sum_{j=0}^{\lfloor k/2\rfloor}\binom{2j}{j}\binom{k}{2j}2^{k-2j}=H_{p-1}+\sum_{j=1}^{(p-1)/2}\frac{\binom{2j}{j}}{4^j}\sum_{k=2j}^{p-1}\frac{\binom{k}{2j}}{k}.$$

By Sigma [11], we find the following identity which can be proved by induction on n:

$$\sum_{k=2j}^{n-1}\frac{1}{k}\binom{k}{2j}=\frac{1}{2j}\binom{n-1}{2j}.$$

This, with Lemma 2.4, yields

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k2^k}-H_{p-1}&=\frac{1}{2}\sum_{j=1}^{(p-1)/2}\frac{\binom{2j}{j}\binom{p-1}{2j}}{j4^j}\\&\equiv\frac{1}{2}\sum_{j=1}^{(p-1)/2}\frac{(-1)^j}{j}\binom{\frac{1}{2}(p-1)}{j}\Big(1-\frac{p}{2}H_j+\frac{p^2}{8}(H_j^2-H_j^{(2)})\Big)\pmod{p^3}.\end{aligned}$$

Substituting $n = (p-1)/2$ into (2.2)–(2.4),

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k2^k}-H_{p-1}\equiv\frac{1}{2}\Big(-H_{(p-1)/2}+\frac{p}{2}H^{(2)}_{(p-1)/2}+\frac{p^2}{4}\sum_{k=1}^{(p-1)/2}\frac{H_{k-1}}{k^2}\Big)\pmod{p^3}.$$

In view of [5, Lemma 3.2] and Lemma 2.2,

$$\sum_{k=1}^{(p-1)/2}\frac{H_{k-1}}{k^2}=\sum_{k=1}^{(p-1)/2}\frac{H_k}{k^2}-H^{(3)}_{(p-1)/2}\equiv\frac{3}{2}B_{p-3}\pmod{p}.$$

This, with Lemma 2.2, yields the desired result. □

3. Proof of Theorem 1.2

LEMMA 3.1. *For any prime $p > 3$,*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{(p-1)/2}{k}(-1)^k}{(2k+1)(2k+2)}\equiv p-1+(-1)^{(p-1)/2}(1-q_p(2)-p+pq_p(2)^2)\pmod{p^2},$$

where $q_p(2) = (2^{p-1}-1)/p$ stands for the Fermat quotient.

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PROOF. By `Sigma`, we find the following identity which can be proved by induction on n:

$$\sum_{k=0}^{n-1}\frac{\binom{n}{k}(-1)^k}{(2k+1)(2k+2)}=-\frac{1}{2n+2}-\frac{(-1)^n}{(2n+1)(2n+2)}+\frac{4^n}{(2n+1)\binom{2n}{n}}.$$

Setting $n=(p-1)/2$,

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{(p-1)/2}{k}(-1)^k}{(2k+1)(2k+2)}=-\frac{1}{p+1}-\frac{(-1)^{(p-1)/2}}{p(p+1)}+\frac{2^{p-1}}{p\binom{p-1}{(p-1)/2}}.$$

The well-known Morley's congruence [9] gives

$$\binom{p-1}{(p-1)/2}\equiv(-1)^{(p-1)/2}4^{p-1}\pmod{p^3}\quad\text{for } p>3. \tag{3.1}$$

This, with $2^{p-1}=1+pq_p(2)$, yields

$$\begin{aligned}\sum_{k=0}^{(p-3)/2}\frac{\binom{(p-1)/2}{k}(-1)^k}{(2k+1)(2k+2)}&\equiv-\frac{1}{p+1}+\frac{(-1)^{(p-1)/2}}{p}\Big(\frac{1}{2^{p-1}}-\frac{1}{p+1}\Big)\\&\equiv p-1+(-1)^{(p-1)/2}\frac{1-q_p(2)}{2^{p-1}(p+1)}\\&\equiv p-1+(-1)^{(p-1)/2}(1-q_p(2)-p+pq_p^2(2))\pmod{p^2}.\end{aligned}$$

This proves Lemma 3.1. □

LEMMA 3.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{(p-1)/2}{k}H_k(-1)^k}{(2k+1)(2k+2)}\equiv-2q_p(2)+2E_{p-3}+2(-1)^{(p-1)/2}(q_p(2)^2-q_p(2))\pmod{p},$$
$$\sum_{k=0}^{(p-3)/2}\frac{\binom{(p-1)/2}{k}kH_k(-1)^k}{(2k+1)(2k+2)}\equiv2q_p(2)-E_{p-3}+(-1)^{(p-1)/2}(2q_p(2)-q_p(2)^2)\pmod{p}.$$

PROOF. By `Sigma`, we find the following identity which can be proved by induction on n,

$$\sum_{k=0}^{n-1}\frac{\binom{n}{k}H_k(-1)^k}{(2k+1)(2k+2)}=\frac{H_n}{2n+2}-\frac{(-1)^nH_n}{(2n+1)(2n+2)}-\frac{4^n}{(2n+1)\binom{2n}{n}}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k4^k}.$$

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Substituting $n=(p-1)/2$ into the above identity and by (3.1),

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{(p-1)/2}{k}H_k(-1)^k}{(2k+1)(2k+2)}\equiv\frac{H_{(p-1)/2}}{p+1}-\frac{(-1)^{(p-1)/2}H_{(p-1)/2}}{p(p+1)}-\frac{(-1)^{(p-1)/2}}{p2^{p-1}}\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k4^k}\pmod p.$$

Tauraso [18] and Sun [14, (1.5)] respectively proved

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k4^k}\equiv -H_{(p-1)/2}\pmod{p^3},\tag{3.2}$$

$$\sum_{k=\frac{p+1}{2}}^{p-1}\frac{\binom{2k}{k}}{k4^k}\equiv(-1)^{(p-1)/2}2pE_{p-3}\pmod{p^2}.\tag{3.3}$$

These, with Lemma 2.2, yield

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{(p-1)/2}{k}H_k(-1)^k}{(2k+1)(2k+2)}\equiv -2q_p(2)+2E_{p-3}+2(-1)^{(p-1)/2}(q_p(2)^2-q_p(2))\pmod p.$$

Similarly, by `Sigma`, we find the following identity which can be proved by induction on n:

$$\sum_{k=0}^{n-1}\frac{\binom{n}{k}kH_k(-1)^k}{(2k+1)(2k+2)}=-\frac{H_n}{2n+2}-\frac{n(-1)^nH_n}{(2n+1)(2n+2)}+\frac{4^n}{2(2n+1)\binom{2n}{n}}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k4^k}.$$

Setting $n=(p-1)/2$, and invoking (3.1), (3.2), (3.3) and Lemma 2.2,

$$\begin{aligned}\sum_{k=0}^{(p-3)/2}\frac{\binom{(p-1)/2}{k}kH_k(-1)^k}{(2k+1)(2k+2)}&\equiv-\frac{H_{(p-1)/2}}{p+1}-\frac{(p-1)(-1)^{(p-1)/2}H_{(p-1)/2}}{2p(p+1)}\\&\quad+\frac{(-1)^{(p-1)/2}}{2p2^{p-1}}\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k4^k}\\&\equiv 2q_p(2)-E_{p-3}+(-1)^{\frac{p-1}{2}}(2q_p(2)-q_p(2)^2)\pmod p.\end{aligned}$$

This completes the proof of Lemma 3.2. □

Proof of Theorem 1.2. We can check the case $p=3$ directly. From now on, we assume that $p>3$. By `Sigma`, we find the following identity which can be proved by induction on n:

$$\sum_{k=2j}^{n-1}(ak+b)\binom{k}{2j}=\frac{an(2j+1)+2bj+2b-a}{2j+2}\binom{n}{2j+1}.$$

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Substituting $n = p$ into this identity and using (2.1),

$$\begin{aligned}\sum_{k=0}^{p-1}(ak+b)\frac{\binom{2k}{k}}{2^k} &= \sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{p}{2j+1}}{4^j}\frac{ap(2j+1)+2bj+2b-a}{2j+2}\\ &= p\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{p-1}{2j}}{4^j}\frac{ap(2j+1)+2bj+2b-a}{(2j+1)(2j+2)}\\ &= \frac{\binom{p-1}{(p-1)/2}}{2^{p-1}}(ap+b-a)+p\sum_{j=0}^{(p-3)/2}\frac{\binom{2j}{j}\binom{p-1}{2j}}{4^j}\frac{(ap+b)(2j+1)+b-a}{(2j+1)(2j+2)}.\end{aligned}$$

This, with Lemma 2.4, yields, modulo p^3,

$$\begin{aligned}&\sum_{k=0}^{p-1}(ak+b)\frac{\binom{2k}{k}}{2^k}-\frac{\binom{p-1}{(p-1)/2}}{2^{p-1}}(ap+b-a)\\ &\equiv p\sum_{j=0}^{(p-3)/2}\binom{(p-1)/2}{j}(-1)^j\frac{(1-\frac{1}{2}pH_j)((ap+b)(2j+1)+b-a)}{(2j+1)(2j+2)}\\ &\equiv p\sum_{j=0}^{(p-3)/2}\binom{(p-1)/2}{j}(-1)^j\frac{(ap+b)(2j+1)+b-a-\frac{1}{2}p(2b-a)H_j-bpjH_j}{(2j+1)(2j+2)}.\end{aligned}$$

It is easy to check that

$$\sum_{k=0}^{n-1}\frac{\binom{n}{k}(-1)^k}{2k+2}=\frac{1}{2n+2}\sum_{k=0}^{n-1}\binom{n+1}{k+1}(-1)^k=-\frac{1}{2n+2}\sum_{k=1}^{n}\binom{n+1}{k}(-1)^k=\frac{1-(-1)^n}{2n+2}.$$

Setting $n = (p-1)/2$,

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{(p-1)/2}{k}(-1)^k}{2k+2}=\frac{1-(-1)^{(p-1)/2}}{p+1}\equiv(1-(-1)^{(p-1)/2})(1-p)\pmod{p^2}.$$

This, with Lemmas 3.1 and 3.2, yields, modulo p^3,

$$\begin{aligned}&\sum_{k=0}^{p-1}(ak+b)\frac{\binom{2k}{k}}{2^k}-\frac{\binom{p-1}{(p-1)/2}}{2^{p-1}}(ap+b-a)\\ &\qquad\equiv pa(1-(-1)^{(p-1)/2})-p(b-a)(-1)^{(p-1)/2}q_p(2)\\ &\qquad\quad -ap^2q_p(2)(1+(-1)^{(p-1)/2})-p^2(b-a)E_{p-3}.\end{aligned}$$

Simplifying this congruence using (3.1) and $2^{p-1} = 1 + pq_p(2)$ gives Theorem 1.2. □

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4. Proof of Theorem 1.4

LEMMA 4.1. *Let $p > 3$ be a prime. Then*

$$p(p+1)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{p-1}{2k}}{(k+1)4^k}\equiv 2p-2p^2q_p(2) \pmod{p^3}.$$

PROOF. By `Sigma`, we can find and prove the identity

$$\sum_{k=0}^{n}\frac{\binom{n}{k}(-1)^kH_k}{k+1}=-\frac{H_n}{n+1},$$

and it is easy to see that

$$\sum_{k=0}^{n}\frac{\binom{n}{k}(-1)^k}{k+1}=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k=-\frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^k=\frac{1}{n+1}.$$

Substituting $n=(p-1)/2$ into these identities and using Lemmas 2.4 and 2.2,

$$\begin{aligned}p(p+1)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{p-1}{2k}}{(k+1)4^k}&\equiv p(p+1)\sum_{k=0}^{(p-1)/2}\frac{\binom{(p-1)/2}{k}(-1)^k(1-\frac{1}{2}pH_k)}{k+1}\\&=p(p+1)\frac{2}{p+1}+\frac{p^2(p+1)}{2}\frac{2H_{(p-1)/2}}{p+1}=2p+p^2H_{(p-1)/2} \pmod{p^3}.\end{aligned}$$

This proves Lemma 4.1. □

LEMMA 4.2. *For any prime $p > 3$,*

$$\sum_{k=0}^{(p-1)/2}\frac{p\binom{2k}{k}\binom{p-1}{2k}}{(2k+3)4^k}\equiv(-1)^{(p-1)/2}\Big(\frac{1}{2}-\frac{3p}{4}+\frac{7p^2}{8}\Big)+\frac{p^2}{2}-\frac{p^2}{2}E_{p-3} \pmod{p^3}.$$

PROOF. By `Sigma`, we can find and prove the following identity:

$$\sum_{k=0}^{n}\frac{\binom{n}{k}(-1)^k}{2k+3}=\frac{4^n}{(2n+1)(2n+3)\binom{2n}{n}}.$$

Substituting $n=(p-1)/2$ into this identity and using Lemma 2.4,

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{p\binom{2k}{k}\binom{p-1}{2k}}{(2k+3)4^k}&\equiv\frac{\binom{p-3}{(p-3)/2}\binom{p-1}{p-3}}{2^{p-3}}+p\sum_{k=0}^{(p-1)/2}\frac{\binom{(p-1)/2}{k}(-1)^k(1-\frac{1}{2}pH_k)}{2k+3}\\&\quad-\binom{(p-1)/2}{(p-3)/2}(-1)^{(p-3)/2}\Big(1-\frac{p}{2}H_{(p-3)/2}\Big)=S_1-\frac{p}{2}S_2 \pmod{p^3},\end{aligned}$$

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where

$$S_1 = \frac{\binom{p-3}{(p-3)/2}\binom{p-1}{p-3}}{2^{p-3}} + \frac{1}{p+2}\frac{2^{p-1}}{\binom{p-1}{(p-1)/2}} - \binom{(p-1)/2}{(p-3)/2}(-1)^{(p-3)/2}\Big(1 - \frac{p}{2}H_{(p-3)/2}\Big),$$

$$S_2 = p\sum_{k=0}^{(p-1)/2}\frac{\binom{(p-1)/2}{k}(-1)^k H_k}{2k+3}.$$

In view of (3.1) and Lemma 2.2,

$$\begin{aligned} S_1 &= \frac{1}{2}(p-1)^2\frac{\binom{p-1}{(p-1)/2}}{2^{p-1}} + \frac{(-1)^{(p-1)/2}}{(p+2)2^{p-1}} + \frac{1}{2}(p-1)(-1)^{(p-1)/2}\Big(1 - \frac{p}{2}H_{(p-1)/2} + \frac{p}{p-1}\Big) \\ &\equiv \frac{1-2p+p^2}{2}(1+pq_p(2))(-1)^{(p-1)/2} + \frac{(-1)^{(p-1)/2}}{2}\Big(1 - \frac{p}{2} + \frac{p^2}{4}\Big) \\ &\quad \cdot (1 - pq_p(2) + p^2q_p(2)^2) - \frac{1}{2}(-1)^{(p-1)/2}\Big(1 - 2p + pq_p(2) - p^2q_p(2) - \frac{p^2}{2}q_p(2)^2\Big) \\ &\equiv \frac{(-1)^{(p-1)/2}}{2}\Big(1 - \frac{p}{2} + \frac{5p^2}{4} - pq_p(2) - \frac{p^2}{2}q_p(2) + \frac{3p^2}{2}q_p(2)^2\Big) \pmod{p^3}. \end{aligned}$$

By `Sigma`, we can find and prove the identity

$$\sum_{k=0}^{n}\frac{\binom{n}{k}(-1)^k H_k}{2k+3} = -\frac{2}{2n+3} + \frac{4^n}{(2n+1)(2n+3)\binom{2n}{n}}\Big(2 - \sum_{k=1}^{n}\frac{\binom{2k}{k}}{k4^k}\Big).$$

Setting $n = (p-1)/2$ in this identity and using (3.1), (3.2), (3.3) and Lemma 2.2,

$$\begin{aligned} S_2 &= -\frac{2p}{p+2} + \frac{1}{p+2}\frac{2^{p-1}}{\binom{p-1}{(p-1)/2}}\Big(2 - \sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k4^k}\Big) \\ &\equiv -p + \frac{1}{2}(-1)^{(p-1)/2}\Big(1 - \frac{p}{2} - pq_p(2)\Big)(2 - 2q_p(2) + pq_p(2)^2 + (-1)^{(p-1)/2}2pE_{p-3}) \\ &\equiv (-1)^{(p-1)/2}\Big(1 - q_p(2) - \frac{p}{2} - \frac{p}{2}q_p(2) + \frac{3p}{2}q_p(2)^2\Big) - p + pE_{p-3} \pmod{p^2}. \end{aligned}$$

Hence,

$$\sum_{k=0}^{(p-1)/2}\frac{p\binom{2k}{k}\binom{p-1}{2k}}{(2k+3)4^k} \equiv (-1)^{(p-1)/2}\Big(\frac{1}{2} - \frac{3p}{4} + \frac{7p^2}{8}\Big) + \frac{p^2}{2} - \frac{p^2}{2}E_{p-3} \pmod{p^3}.$$

This completes the proof of Lemma 4.2. □

PROOF OF THEOREM 1.4. It is easy to check by (2.1) that

$$\sum_{k=0}^{p-1}\frac{(k+1)^2\binom{2k}{k}}{2^k} = \sum_{k=0}^{p-1}(k+1)^2\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{2j}\frac{\binom{2j}{j}}{4^j} = \sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}}{4^j}\sum_{k=2j}^{p-1}(k+1)^2\binom{k}{2j}.$$

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By Sigma, we find the following identity which can be proved by induction on n:

$$\sum_{k=2j}^{n-1}(k+1)^2\binom{k}{2j} = \frac{n(n+1)(2nj+2n+2j+1)}{2(2j+1)(2j+3)}\binom{n-1}{2j}.$$

Substituting $n = p$ into this identity and using Lemmas 4.1 and 4.2,

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{(k+1)^2\binom{2k}{k}}{2^k} &= \sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}}{4^j}\frac{p(p+1)(2pj+2p+2j+1)}{2(2j+1)(2j+3)}\binom{p-1}{2j}\\ &= \sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{p-1}{2j}}{4^j}\frac{p^2(p+1)}{2j+3} + \sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{p-1}{2j}}{4^j}\Big(\frac{2p(p+1)}{2j+3} - \frac{p(p+1)}{2(j+1)}\Big)\\ &\equiv p^2\binom{(p-1)/2}{(p-3)/2}(-1)^{(p-3)/2} + 3p^2\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{p-1}{2j}}{(2j+3)4^j}\\ &\quad + 2p\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{p-1}{2j}}{(2j+3)4^j} - \frac{p(p+1)}{2}\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{p-1}{2j}}{(j+1)4^j}\\ &\equiv (-1)^{(p-1)/2} - p + p^2 - p^2E_{p-3} + p^2q_p(2) \pmod{p^3},\end{aligned}$$

which gives the result in Theorem 1.4. □

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