

ARTICLE

A special case of Vu’s conjecture: colouring nearly disjoint graphs of bounded maximum degree

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Abstract

A collection of graphs is *nearly disjoint* if every pair of them intersects in at most one vertex. We prove that if G_1, \dots, G_m are nearly disjoint graphs of maximum degree at most D , then the following holds. For every fixed C , if each vertex $v \in \bigcup_{i=1}^m V(G_i)$ is contained in at most C of the graphs G_1, \dots, G_m , then the (list) chromatic number of $\bigcup_{i=1}^m G_i$ is at most $D + o(D)$. This result confirms a special case of a conjecture of Vu and generalizes Kahn’s bound on the list chromatic index of linear uniform hypergraphs of bounded maximum degree. In fact, this result holds for the correspondence (or DP) chromatic number and thus implies a recent result of Molloy and Postle, and we derive this result from a more general list colouring result in the setting of ‘colour degrees’ that also implies a result of Reed and Sudakov.

Keywords: Graph colouring; colour degrees; edge-colouring; hypergraphs; probabilistic method; nibble

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1. Introduction

A *proper colouring* of a graph G is an assignment of colours to the vertices of G such that no two vertices assigned the same colour are adjacent, and the *chromatic number* of G , denoted $\chi(G)$, is the minimal number of colours needed to properly colour G . A *list assignment* for a graph G is a map L whose domain is the vertex set $V(G)$ of G where $L(v)$ is a finite set called the ‘list of available colours for v ’, and an *L -colouring* of G is a proper colouring ϕ of G such that $\phi(v) \in L(v)$ for every $v \in V(G)$. If G has an L -colouring, then G is *L -colourable*. The *list chromatic number* of G , denoted $\chi_\ell(G)$, is the minimum $k \in \mathbb{N}$ such that G is L -colourable for every list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$.

We say graphs G_1, \dots, G_m are *nearly disjoint* if $|V(G_i) \cap V(G_j)| \leq 1$ for all distinct $i, j \in [m]$. In this paper, we prove an asymptotically optimal bound on the (list) chromatic number of a union of nearly disjoint graphs of bounded maximum degree (see Theorem 1.1), which confirms a special case of a conjecture of Vu (see Conjecture 1.2) and generalizes several well-known results on hypergraph edge-colouring. We derive our bound from a more general result (see Theorem 1.3) concerning L -colourings when L is a list assignment for a union of nearly disjoint graphs such that L has bounded maximum *colour degree*. This result also implies a result of Reed and Sudakov [25]. We also prove a bound on the chromatic number of a union of nearly disjoint graphs of bounded chromatic number (see Theorem 1.4).

1.1 Line graphs of linear hypergraphs and nearly disjoint graph unions

A hypergraph \mathcal{H} is a pair $\mathcal{H} = (V, E)$ where V is a set whose elements are called *vertices* and $E \subseteq 2^V$ is a set of subsets of V whose elements are called *edges*. A *proper edge-colouring* of a hypergraph \mathcal{H} is an assignment of colours to the edges of \mathcal{H} such that no two edges of the same colour share a vertex, and the *chromatic index* of \mathcal{H} , denoted $\chi'(\mathcal{H})$, is the minimum number of colours used by a proper edge-colouring of \mathcal{H} . The *line graph* of a hypergraph \mathcal{H} is the graph G where $V(G)$ is the edge set of \mathcal{H} , and $e, f \in V(G)$ are adjacent in G if $e \cap f \neq \emptyset$. Note that the chromatic index of a hypergraph is the chromatic number of its line graph. The *list chromatic index* of a hypergraph \mathcal{H} , denoted $\chi'_\ell(\mathcal{H})$, is the list chromatic number of its line graph.

A hypergraph \mathcal{H} is *linear* if every two distinct edges of \mathcal{H} intersect in at most one vertex and *k-bounded* if every edge of \mathcal{H} has size at most k . A celebrated result of Pippenger and Spencer [22] implies that k -bounded linear hypergraphs of maximum degree at most D have chromatic index at most $D + o(D)$ for fixed k as $D \rightarrow \infty$, and a similarly influential result of Kahn [14] generalizes the Pippenger–Spencer theorem to list colouring. (Both of these results apply more generally to hypergraphs of small codegree, and not only to linear hypergraphs.) An intermediate result of Kahn [13, Theorem 3] was also crucial to proving the Erdős–Faber–Lovász conjecture asymptotically. This conjecture states that a nearly disjoint union of n complete graphs, each on at most n vertices, has chromatic number at most n , and it was recently confirmed for all large n by Kang, Methuku, and the authors [17]. Note that the line graph of a k -bounded linear hypergraph is a union of nearly disjoint complete graphs, such that at most k of them contain any given vertex. Thus, the following result strengthens both the Pippenger–Spencer theorem and Kahn’s list edge-colouring theorem for the case of linear hypergraphs.

Theorem 1.1. *For every $C, \varepsilon > 0$, the following holds for all sufficiently large D . If G_1, \dots, G_m are nearly disjoint graphs of maximum degree at most D , such that each vertex $v \in \bigcup_{i=1}^m V(G_i)$ is contained in at most C of them, then $\chi_\ell(\bigcup_{i=1}^m G_i) \leq (1 + \varepsilon)D$.*

Theorem 1.1 also confirms a special case of the following conjecture of Vu [27], and (as our discussion shows) it recovers several of its significant consequences.

Conjecture 1.2 (Vu [27]). *For every $\zeta, \varepsilon > 0$, the following holds for all sufficiently large D . If G is a graph of maximum degree at most D and every two distinct vertices have at most ζD common neighbours in G , then $\chi_\ell(G) \leq (\zeta + \varepsilon)D$.*

Indeed, Conjecture 1.2, if true, implies Theorem 1.1 with $1/C$ and $C(D + C^2)$ playing the roles of ζ and D , respectively. If G_1, \dots, G_m are nearly disjoint graphs of maximum degree at most D , such that each vertex $v \in \bigcup_{i=1}^m V(G_i)$ is contained in at most C of them, then every two distinct vertices in $G := \bigcup_{i=1}^m G_i$ have at most $D + C^2$ common neighbours, and G has maximum degree at most CD . Vu [27] initially observed that a similar argument shows that Conjecture 1.2 implies Kahn’s [14] bound on the list chromatic index of linear hypergraphs of bounded maximum degree, and this was a major motivation for the conjecture: ‘The bound in [Conjecture 1.2], if valid, would be an amazing result. For instance, it would immediately imply a deep theorem of Kahn on the list chromatic index of [linear] hypergraphs.’ [27, p. 109]. The only other nontrivial result towards Vu’s conjecture so far was recently obtained by Hurley et al. [12], who proved a bound on the chromatic number of graphs as in Conjecture 1.2. Their result confirms the weaker version of the conjecture that considers the chromatic number rather than the list chromatic number, in the case when $\zeta > 1 - o(\varepsilon^{2/3})$. We remark that even finding an independent set of the required size is still an open problem (which was already raised by Vu [27]).

1.2 colour degrees

Our main result in this paper is actually a generalization of Theorem 1.1 that also implies a result of Reed and Sudakov [25]. For a graph G with list assignment L , we define the *colour degree* of each $v \in V(G)$ and $c \in L(v)$ to be $d_{G,L}(v, c) := |\{u \in N_G(v) : c \in L(u)\}|$ and the *maximum colour*

degree of G and L to be $\Delta(G, L) := \max_{v \in V(G)} \max_{c \in L(v)} d_{G,L}(v, c)$. Note that $\Delta(G, L) \leq \Delta(G)$. The following is our main result.

Theorem 1.3. *For every $C, \varepsilon > 0$, the following holds for all sufficiently large D . Let G_1, \dots, G_m be graphs that*

- (i) *are nearly disjoint and*
- (ii) *satisfy $|\{i \in [m] : v \in V(G_i)\}| \leq C$ for every $v \in \bigcup_{i=1}^m V(G_i)$.*

If L is a list assignment for $G := \bigcup_{i=1}^m G_i$ satisfying

- (iii) $\Delta(G_i, L|_{V(G_i)}) \leq D$ *for every $i \in [m]$ and*
- (iv) $|L(v)| \geq (1 + \varepsilon)D$ *for every $v \in V(G)$,*

then G is L -colourable.

Note that Theorem 1.1 immediately follows from Theorem 1.3.

Theorem 1.3 actually holds more generally for *correspondence colouring*, also known as *DP-colouring* (see Theorem 5.2), and this result also implies the recent result of Molloy and Postle [19], that k -bounded linear hypergraphs of maximum degree at most D have *correspondence chromatic index* at most $D + o(D)$. A proof of Theorem 5.2 can be obtained with only minor modifications to our argument used to prove Theorem 1.3. Thus, for the sake of presentation, we choose to first provide a complete proof of Theorem 1.3 for list colouring. Then, in Section 5, we describe the modifications necessary to prove Theorem 5.2, its correspondence colouring generalization.

The case $m = 1$ (or equivalently, $C = 1$) in Theorem 1.3 is a result of Reed and Sudakov [25]. For every $C \in \mathbb{N}$, let $f_C(D)$ denote the smallest integer for which the following holds: If G_1, \dots, G_m are graphs satisfying (i) and (ii), and if L is a list assignment for $G := \bigcup_{i=1}^m G_i$ satisfying (iii) such that $|L(v)| \geq D + f_C(D)$ for every $v \in V(G)$, then G is L -colourable. Clearly, $f_C(D) \geq 1$. Theorem 1.3 implies that $f_C(D) = o(D)$, but it would be interesting to prove better asymptotics. A bound of $f_C(D) \leq D^{1-1/C}$ for $C \geq 2$ would match the best known bound on the list chromatic index of linear hypergraphs due to Molloy and Reed [20]. Reed [24] conjectured that $f_1(D) = 1$ for every $D \in \mathbb{N}$. Bohman and Holzman [4] disproved this conjecture, and Reed and Sudakov [25] asked whether $f_1(D) = O(1)$. Another interesting direction would be to generalize Theorem 1.3 so as to also imply the generalization of the Reed–Sudakov result in which the maximum colour degree condition is replaced with an average one, proved by Glock and Sudakov [10] and Kang and Kelly [18]. For other related results and open problems involving list colouring and colour degrees, see for example [1–3, 6]. More generally, for a recent survey on colouring results and open problems obtained via nibble methods, see [16].

1.3 colouring nearly disjoint graphs of bounded chromatic number

Erdős proposed several variations of the Erdős–Faber–Lovász conjecture. For example, relaxing the condition of nearly disjointness, Erdős [8, Problem 9] asked for the largest possible chromatic number of a union of n complete graphs, each on at most n vertices, that pairwise intersect in at most t vertices. Building on the methods of [17], this problem was recently solved by Kang, Methuku, and the authors [15]. Here we discuss a problem related to a question of Erdős [9, p. 26] on bounds on the chromatic number of a union of nearly disjoint graphs G_i if we know their chromatic number (rather than their maximum degree as in Section 1.1).

Accordingly, for a family \mathcal{G} of graphs and $m \in \mathbb{N}$, let $f(m, \mathcal{G})$ be the largest possible chromatic number of the union of at most m nearly disjoint graphs in \mathcal{G} . Recall that the Erdős–Faber–Lovász conjecture states that a nearly disjoint union of n complete graphs, each on at most n vertices, has chromatic number at most n . Thus, the Erdős–Faber–Lovász conjecture can be expressed as

follows: For all $n \in \mathbb{N}$, we have $f(n, \{K_1, \dots, K_n\}) \leq n$, where K_t denotes the complete graph on t vertices. However, it is straightforward to show that $f(n, \{K_1, \dots, K_n\}) \leq \max\{n, f(n, \{K_{n-1}\})\}$, and it is well-known that $f(n, \{K_{n-1}\}) \geq n$ for $n \geq 3$. Hence, the Erdős–Faber–Lovász conjecture can be reduced to the seemingly weaker statement that $f(n, \{K_{n-1}\}) \leq n$ for every $n \in \mathbb{N}$.

Erdős [9, p. 26] proposed the following variation of the Erdős–Faber–Lovász conjecture: ‘Let G_1, \dots, G_m be m graphs each of chromatic number n . Assume that no two G ’s have an edge in common. What is the smallest m for which $\bigcup_{i=1}^m G_i$ has chromatic number greater than n ? Perhaps one can further demand that any two G ’s have at most one vertex in common.’ It turns out the answers to these two questions are ‘two’ and ‘three’ respectively. (For the former, observe that a complete graph on $n + 1$ vertices can be expressed as the union of a complete graph on n vertices and a star with no edges in common; for the latter, see Theorem 1.4(ii) below.) The latter question can also be expressed as follows: What is the smallest m for which $f(m, \mathcal{G}_n^X) > n$, where \mathcal{G}_n^X is the set of graphs of chromatic number at most n ? Although this question is straightforward to answer, we believe that further analysis of the function $f(m, \mathcal{G}_n^X)$ itself is warranted (which was probably the original intention of the above question of Erdős). In this direction, we prove the following.

Theorem 1.4.

- (i) The following holds for all $m, n \in \mathbb{N}$ with $m + n$ sufficiently large: If G_1, \dots, G_m are nearly disjoint graphs, each of chromatic number at most n , then $\chi(\bigcup_{i=1}^m G_i) \leq m + n - 2$.
- (ii) For every $n \geq 2$, there exist nearly disjoint graphs G_1, G_2, G_3 , each of chromatic number at most n , such that $\chi(G_1 \cup G_2 \cup G_3) \geq n + 1$.

Theorem 1.4(i) implies that $f(m, \mathcal{G}_n^X) \leq m + n - 2$ when $m + n$ is sufficiently large, and Theorem 1.4(ii) implies that this bound is tight for $m = 3$. However, this bound can likely be improved for larger m . In particular, it is tempting to conjecture that $f(n, \mathcal{G}_n^X) \leq n$, which by the discussion above, would imply the Erdős–Faber–Lovász conjecture if true; however, this was disproved by Postle [23], as follows.

Theorem 1.5 (Postle [23]). For every $m \in \mathbb{N}$ divisible by three and $n \geq m - 1$, there exist nearly disjoint graphs G_1, \dots, G_m , each of chromatic number at most n , such that $\chi(\bigcup_{i=1}^m G_i) \geq n + m/6$.

Because of the connection to the Erdős–Faber–Lovász conjecture, we find it most interesting to study the function $f(n, \mathcal{G}_{n-1}^X)$. Theorems 1.4(i) and 1.5 imply that for large n we have

$$\frac{7n - 8}{6} \leq f(n, \mathcal{G}_{n-1}^X) \leq 2n - 3. \quad (1)$$

We think it would be interesting to determine $\lim_{n \rightarrow \infty} f(n, \mathcal{G}_{n-1}^X) / n$, assuming the limit exists. By (1), it would be in the range $[7/6, 2]$.

Considering Theorem 1.1, it is natural to also ask for $f(n, \mathcal{G}_{n-1}^\Delta)$ (or $f(n, \mathcal{G}_{n-2}^\Delta)$), where \mathcal{G}_n^Δ is the set of graphs of maximum degree at most n . However, it is straightforward to show that $f(n, \mathcal{G}_{n-1}^\Delta) = f(n, \{K_n\}) = f(n, \{K_{n-1}\}) = f(n, \mathcal{G}_{n-2}^\Delta)$ for every $n \geq 3$.

1.4 Outline of the paper

Sections 2–4 are devoted to the proof of Theorem 1.3. We prove Theorem 1.3 using a semi-random colouring procedure (also referred to as the ‘nibble method’). Each step of this procedure is obtained by an application of Lemma 2.1, which is proved in Section 4. Section 3 provides some probabilistic tools needed in the proof of Lemma 2.1. In Section 2, we prove Theorem 1.3 (assuming Lemma 2.1). As mentioned, Theorem 1.3 holds more generally for correspondence colouring. In Section 5, we formally state this generalization and describe how to modify the proof of Theorem 1.3 to prove it. In Section 6, we prove Theorems 1.4 and 1.5.

2. Proof of Theorem 1.3

In this section we prove Theorem 1.3, assuming the following lemma. Given $k \in \mathbb{Z}$, we write $\log^k D := (\log D)^k$, where the logarithm is base e . Given $\Lambda, D, C, p \in \mathbb{R}$, we also define

$$\Lambda_{\mathbb{E}}(\Lambda, D, C, p) := \left(1 - \frac{p}{\Lambda}\right)^{DC} \Lambda \quad \text{and}$$

$$D_{\mathbb{E}}(\Lambda, D, C, p) := \left((1-p) \left(1 - \frac{p}{\Lambda}\right)^{D(C-1)} + p \left(1 - \left(1 - \frac{p}{\Lambda}\right)^{DC}\right) \right) D.$$

Lemma 2.1. *For every $C, \varepsilon > 0$, there exists $D_{5.3}$ such that the following holds for every $D \geq D_{(5.3)}$. Let G_1, \dots, G_m be graphs, and let L be a list assignment for $G := \bigcup_{i=1}^m G_i$ satisfying (i)–(iv) of Theorem 1.3. If $|L(v)| = \lceil \Lambda \rceil$ for every $v \in V(G)$, where $(1 + \varepsilon)D \leq \Lambda \leq 10CD$, and if $\log^{-1} D \geq p \geq \log^{-2} D$, then there exist $X \subseteq V(G)$, an $L|_X$ -colouring ϕ of $G[X]$, and a list assignment L' for $G - X$ satisfying $L'(v) \subseteq L(v) \setminus \{\phi(u) : u \in N_G(v) \cap X\}$ for every $v \in V(G) \setminus X$, such that*

(2.1.1) $|L'(v)| = \lceil \Lambda_{\mathbb{E}}(\Lambda, D, C, p) - \Lambda^{4/5} \rceil$ for every $v \in V(G) \setminus X$ and

(2.1.2) $\Delta(G_i - X, L'|_{V(G_i - X)}) \leq D_{\mathbb{E}}(\Lambda, D, C, p) + D^{4/5}$ for every $i \in [m]$.

We prove Lemma 2.1 in Section 4 by analysing a random colouring procedure, but we already explain some of the ideas involved in the proof now. In our random colouring procedure, we randomly assign each vertex $v \in V(G)$ a colour $\psi(v) \in L(v)$ uniformly at random from its list, and we ‘activate’ vertices independently at random with probability p . (We really only need to assign colours to activated vertices, but for technical reasons it is convenient to define the procedure this way.) For each vertex, we remove any colour from its list assigned to an activated neighbour, and we let X be the set of activated vertices whose colour was not assigned to any of its activated neighbours. We obtain L' by further truncating each vertex’s list to have the desired size, and we show that with nonzero probability, X, L' , and $\phi := \psi|_X$ satisfy (2.1.1) and (2.1.2).

To that end, assuming without loss of generality that $d_{G_i, L}(v, c) = D$ for every $i \in [m]$ and $v \in V(G_i)$ with $c \in L(v)$ (see Proposition 4.1), it is straightforward to show that a vertex ‘keeps’ a given colour in its list with probability $(1 - p/\Lambda)^{DC}$ (see Proposition 4.2). Thus, after applying this procedure, $\Lambda_{\mathbb{E}}(\Lambda, D, C, p)$ is the expected number of remaining available colours for each vertex, and $D_{\mathbb{E}}(\Lambda, D, C, p)$ is an upper bound on the expected colour degree of each pair of vertex and colour from its list (see (4) and Lemma 4.3). We show that with very high probability (at least $1 - \exp(-D^{1/4})$ – see Lemma 4.5), the number of remaining colours available for a given vertex, and the colour degree in $G - X$ of a given pair of vertex and colour, are not significantly larger than $\Lambda_{\mathbb{E}}(\Lambda, D, C, p)$ and $D_{\mathbb{E}}(\Lambda, D, C, p)$, respectively. We complete the proof by applying the Lovász Local Lemma.

To prove Theorem 1.3, we iteratively apply Lemma 2.1 $O(C\varepsilon^{-1} \log D)$ times, each time reapplying the lemma with $p = \log^{-1} D$ and with $G_1 - X, \dots, G_m - X$ and L' playing the roles of G_1, \dots, G_m and L , respectively, before completing the colouring with Lemma 2.3 below. The requirement $L'(v) \subseteq L(v) \setminus \{\phi(u) : u \in N_G(v) \cap X\}$ ensures that adjacent vertices are not assigned the same colour in different iterations, so each iteration extends a proper partial colouring of G .

This approach – known as the ‘nibble method’ or the ‘semi-random method’ – has led to numerous important developments in graph colouring (see [16, 21]). Our random colouring procedure was in fact already used by Reed and Sudakov [25] to prove the special case of Theorem 1.3 when $C = 1$; however, our analysis of the procedure is different even in this special case, drawing ideas from a recent proof of Kang and Kelly [18], and we need new ideas to analyse the procedure for $C > 1$. Recall that Theorem 1.3 also implies Kahn’s [14] bound on the chromatic index of linear uniform hypergraphs and that Theorem 5.2 implies Molloy and Postle’s [19] generalization of Kahn’s result to correspondence colouring. These edge-colouring results are also proved with a semi-random approach, but the random colouring procedures used in these proofs are

slightly different from ours. Since vertices may lose colours unnecessarily in our procedure – it would suffice to only remove colours from a vertex’s list that were assigned to a neighbour in X , but this version of the procedure would be more challenging to analyse – Reed and Sudakov [25] called the procedure ‘wasteful’ (see also [21, Chapter 12]). Nevertheless, this wastefulness is negligible because our activation probability p is small. However, Kahn [14] did not consider activation probabilities and consequently could not afford to use a wasteful variant of his procedure. Molloy and Postle’s [19] proof does consider activation probabilities but is still different from ours, even in the special case of edge-colouring; in particular, vertices (of the line graph) may be assigned multiple colours in their colouring procedure.

Let us now explain how we use Lemma 2.1 to prove Theorem 1.3. Crucially, (2.1.1) and (2.1.2) together imply that after each iteration of Lemma 2.1, the ratio of the number of remaining available colours for each vertex to the maximum remaining colour degree in each G_i , while initially only $1 + \varepsilon$, improves by a factor of at least $1 + \varepsilon p/4$. Moreover, the number of available colours does not decrease too much in each iteration. The following proposition makes this calculation precise.

Proposition 2.2. *For every $C \geq 1$ and $0 < \varepsilon < 1$ there exists $D_{2,2}$ such that the following holds for every $D \geq D_{2,2}$. If $\log^{-1} D \geq p \geq \log^{-2} D$ and $10CD \geq \Lambda \geq (1 + \varepsilon)D$, then*

$$\frac{\Lambda_{\mathbb{E}}(\Lambda, D, C, p) - \Lambda^{4/5}}{D_{\mathbb{E}}(\Lambda, D, C, p) + D^{4/5}} \geq (1 + \varepsilon p/4) \frac{\Lambda}{D} \tag{2}$$

and

$$D_{\mathbb{E}}(\Lambda, D, C, p) \geq (1 - pC)D. \tag{3}$$

Proof. First we prove (3). Clearly,

$$(1 - p) \left(1 - \frac{p}{\Lambda}\right)^{D(C-1)} D \geq (1 - p)(1 - p(C - 1))D \geq (1 - pC)D,$$

and (3) follows immediately, since $p \left(1 - (1 - p/\Lambda)^{DC}\right) > 0$.

Now we prove (2). By (3), since $\Lambda \leq 10CD$ and $\log^{-1} D \geq p \geq \log^{-2} D$,

$$\frac{\Lambda^{4/5}}{D_{\mathbb{E}}(\Lambda, D, C, p) + D^{4/5}} \leq \frac{10CD^{4/5}}{(1 - pC)D} \leq \frac{\varepsilon p}{100} \cdot \frac{\Lambda}{D}.$$

Let

$$W := (1 - p) \left(1 - \frac{p}{\Lambda}\right)^{-D} + p \left(\left(1 - \frac{p}{\Lambda}\right)^{-DC} - 1 \right) + \left(1 - \frac{p}{\Lambda}\right)^{-DC} D^{-1/5},$$

and note that

$$\frac{\Lambda_{\mathbb{E}}(\Lambda, D, C, p)}{D_{\mathbb{E}}(\Lambda, D, C, p) + D^{4/5}} = \frac{1}{W} \cdot \frac{\Lambda}{D}.$$

Since $1 + x \leq (1 + x/n)^n$ for $x \leq |n|$ and $\Lambda \geq (1 + \varepsilon)D$, we have

$$\begin{aligned} W &\leq \frac{1 - p}{1 - p/(1 + \varepsilon)} + p \left(\frac{1}{1 - pC/(1 + \varepsilon)} - 1 \right) + \frac{D^{-1/5}}{1 - pC/(1 + \varepsilon)} \leq 1 - p + \frac{p}{1 + \varepsilon} + O(p^2) \\ &\leq 1 - \frac{p\varepsilon}{3}. \end{aligned}$$

Since $(1 - p\varepsilon/3)^{-1} \geq 1 + p\varepsilon/3$, by combining the inequalities above, we have that the left side of (2) is at least

$$(1 + p\varepsilon/3 - \varepsilon p/100) \frac{\Lambda}{D} \geq (1 + \varepsilon p/4) \frac{\Lambda}{D},$$

as desired. □

We apply Lemma 2.1 iteratively until the ratio of the number of remaining available colours to the maximum remaining colour degree reaches $8CD$: (2) will imply that this ratio improves by a factor of at least $1 + \varepsilon/(4 \log D)$, so that we only need at most $33C\varepsilon^{-1} \log D$ iterations, and (3) will ensure that carrying out this many iterations is indeed possible. After this process, we can finish with the following result of Reed [24] (which is proved via a simple application of the Lovász Local Lemma).

Lemma 2.3 (Reed [24]). *Let G be a graph with list assignment L . If $|L(v)| \geq 8D$ for every $v \in V(G)$ and $\Delta(G, L) \leq D$, then G is L -colourable.*

Now we prove Theorem 1.3.

Proof of Theorem 1.3. Without loss of generality, we may assume that $\varepsilon < 1$. Let $p := \log^{-1} D$, $D_0 := D$ and $\Lambda_0 := (1 + \varepsilon)D_0$, and for each integer $0 \leq i \leq 33C/(\varepsilon p)$, let

$$\begin{aligned} \Lambda_{i+1} &:= \Lambda_{\mathbb{E}}(\Lambda_i, D_i, C, p) - \Lambda_i^{4/5} && \text{and} \\ D_{i+1} &:= D_{\mathbb{E}}(\Lambda_i, D_i, C, p) + D_i^{4/5}. \end{aligned}$$

Applying Proposition 2.2 inductively, for every integer $0 \leq i \leq 33C/(\varepsilon p)$, by (3) we have

$$D_i \geq (1 - pC)^{33C/(\varepsilon p)} D_0 \geq e^{-33C^2/\varepsilon} (1 - 33C^3 p/\varepsilon) D_0 \geq e^{-34C^2/\varepsilon} D_0,$$

where the second inequality uses that $(1 + x/n)^n \geq e^x(1 - x^2/n)$ for $|x| \leq n$. By (2),

$$\frac{\Lambda_i}{D_i} \geq \left(1 + \frac{\varepsilon p}{4}\right) \frac{\Lambda_{i-1}}{D_{i-1}}.$$

Moreover, we have

$$\frac{\lceil \Lambda_{\lfloor 33C/(\varepsilon p) \rfloor} \rceil}{D_{\lfloor 33C/(\varepsilon p) \rfloor}} \geq \left(1 + \frac{\varepsilon p}{4}\right)^{\lfloor 33C/(\varepsilon p) \rfloor} \frac{\Lambda_0}{D_0} \geq 8C.$$

In particular, there exists an integer $0 < i^* \leq 33C/(\varepsilon p)$ such that $\lceil \Lambda_{i^*} \rceil / D_{i^*} \geq 8C$ and $\lceil \Lambda_{i^*-1} \rceil / D_{i^*-1} < 8C$. We may assume D is sufficiently large so that $D_{i^*} \geq D_{2,1}$.

By (iv), we may assume without loss of generality that $|L(v)| = \lceil \Lambda_0 \rceil$ for every $v \in V(G)$, since we can truncate each list until equality holds. Now let $G_{j,0} := G_j$ for each $j \in [m]$, let $H_0 := G$, and let $L_0 := L$. Due to the above calculations, inductively by Lemma 2.1, for each integer $0 \leq i < i^*$, there is a set $X_i \subseteq V(H_i)$ and an $L_i|_{X_i}$ -colouring ϕ_i of $H_i|_{X_i}$ and a list assignment L_{i+1} for $H_{i+1} := H_i - X_i$ satisfying $L_{i+1}(v) \subseteq L_i(v) \setminus \{\phi_i(u) : u \in N_{H_i}(v) \cap X_i\}$ such that

- $\Delta(G_{j,i+1}, L_{i+1}|_{V(G_{j,i+1})}) \leq D_{i+1}$ for each $j \in [m]$ and
- $|L_{i+1}(v)| = \lceil \Lambda_{i+1} \rceil \geq (1 + \varepsilon)D_{i+1}$,

where $G_{j,i+1} := G_{j,i} - X_i$.

Let $X := \bigcup_{i=0}^{i^*-1} X_i$, let $\phi(v) := \phi_i(v)$ if $v \in X_i$ for some integer $0 \leq i < i^*$, let $G' := H_{i^*}$, and let $L' := L_{i^*}$. By construction, ϕ is an $L|_X$ -colouring of $G[X]$, and by the choice of i^* , we have

- $|L'(v)| \geq 8CD_{i^*}$ and
- $\Delta(G', L') \leq C \max\{\Delta(G_{j,i^*}, L_{i^*}|_{V(G_{j,i^*})}) : j \in [m]\} \leq CD_{i^*}$.

Therefore by Lemma 2.3, G' has an L' -colouring ϕ' , and we can combine ϕ and ϕ' to obtain an L -colouring of G , as desired. □

3. Probabilistic tools

In this section we provide some probabilistic tools used in the proof of Lemma 2.1 in Section 4. The first such tool is the Lovász Local Lemma.

Lemma 3.1 (Lovász Local Lemma [7]). *Let $p \in [0, 1)$ and \mathcal{A} a finite set of events such that for every $A \in \mathcal{A}$,*

- $\mathbb{P}[A] \leq p$, and
- A is mutually independent of a set of all but at most d other events in \mathcal{A} .

If $4pd \leq 1$, then the probability that none of the events in \mathcal{A} occur is strictly positive.

Next we need a concentration inequality of Bruhn and Joos [5], derived from Talagrand’s inequality [26]. To that end, we introduce the following definition.

Definition 3.2. Let $((\Omega_i, \Sigma_i, \mathbb{P}_i))$ be probability spaces, and let $(\Omega, \Sigma, \mathbb{P})$ be their product space. We say a random variable $\mathbf{X} : \Omega \rightarrow \mathbb{R}$ has upward (s, δ) -certificates with respect to a set of exceptional outcomes $\Omega^* \subseteq \Omega$ if for every $\omega \in \Omega \setminus \Omega^*$ and every $t > 0$, there exists an index set I of size at most s so that $\mathbf{X}(\omega') \geq \mathbf{X}(\omega) - t$ for every $\omega' \in \Omega \setminus \Omega^*$ for which the restrictions $\omega|_I$ and $\omega'|_I$ differ in at most t/δ coordinates.

Theorem 3.3 (Bruhn and Joos [5]). *Let $((\Omega_i, \Sigma_i, \mathbb{P}_i))$ be probability spaces, let $(\Omega, \Sigma, \mathbb{P})$ be their product space, and let $\Omega^* \in \Sigma$ be a set of exceptional outcomes. Let $\mathbf{X} : \Omega \rightarrow \mathbb{R}$ be a non-negative random variable, let $M := \max\{\sup \mathbf{X}, 1\}$, and let $\delta \geq 1$. If $\mathbb{P}[\Omega^*] \leq M^{-2}$ and \mathbf{X} has upward (s, δ) -certificates, then for $t > 50\delta\sqrt{s}$,*

$$\mathbb{P}[|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq t] \leq 4 \exp\left(-\frac{t^2}{16\delta^2 s}\right) + 4\mathbb{P}[\Omega^*].$$

4. Proof of Lemma 2.1

This section is devoted to the proof of Lemma 2.1. It will be convenient to assume that equality holds in (ii) and (iii) of Theorem 1.3 and that moreover $d_{G_i, L}(v, c) = D$ for every $v \in V(G)$ and $c \in L(v)$, so we first prove the following proposition which enables us to consider an embedding of G for which these properties hold.

Proposition 4.1. *Let $C, \Lambda, D \in \mathbb{N}$, let G_1, \dots, G_m be nearly disjoint graphs, and let L be a list assignment for $G := \bigcup_{i=1}^m G_i$. If $|\{i \in [m] : v \in V(G_i)\}| \leq C$ and $|L(v)| = \Lambda$ for every $v \in V(G)$ and $\Delta(G_i, L|_{V(G_i)}) \leq D$ for every $i \in [m]$, then there exist nearly disjoint graphs $G'_1, \dots, G'_{m'}$ for some $m' \geq m$ and a list assignment L' for $G' := \bigcup_{i=1}^{m'} G'_i$ such that*

- (4.1.1) $G_i \subseteq G'_i$ for every $i \in [m]$,
- (4.1.2) $L'(v) = L(v)$ for every $v \in V(G)$,
- (4.1.3) $|\{i \in [m'] : v \in V(G'_i)\}| = C$ for every $v \in V(G')$,
- (4.1.4) $|L'(v)| = \Lambda$ for every $v \in V(G')$, and
- (4.1.5) $d_{G'_i, L'}(v, c) = D$ for every $i \in [m']$ and $v \in V(G'_i)$ with $c \in L'(v)$.

The proof of Proposition 4.1 can be found in the appendix. For the remainder of this section, we let $C, \varepsilon > 0$, and we let D be sufficiently large, we let p satisfy $\log^{-1} D \geq p \geq \log^{-2} D$, and we let $G := \bigcup_{i=1}^m G_i$ with list assignment L satisfying the hypotheses of Lemma 2.1. For convenience, we assume Λ and D are integers where it does not affect the argument. By Proposition 4.1, we may assume without loss of generality that $|\{i \in [m] : v \in V(G_i)\}| = C$ for every $v \in V(G)$ and that $d_{G_i, L}(v, c) = D$ for every $i \in [m]$ and $v \in V(G)$ with $c \in L(v)$.

For every pair (A, ψ) satisfying $A \subseteq V(G)$ and $\psi(v) \in L(v)$ for each $v \in V(G)$,

- let $L_{A, \psi}(v) := L(v) \setminus \{\psi(u) : u \in N_G(v) \cap A\}$ for every $v \in V(G)$, and
- let $X_{A, \psi} := \{v \in A : \psi(v) \in L_{A, \psi}(v)\}$.

Note that $X_{A, \psi} = A \setminus \bigcup_{v \in A} \{u \in N_G(v) : \psi(u) = \psi(v)\}$.

We will consider the probability space $(\Omega, \Sigma, \mathbb{P})$ of such pairs (A, ψ) where

- each vertex in G is in A independently and uniformly at random with probability p and
- $\psi(v) \in L(v)$ is chosen independently and uniformly at random for each $v \in V(G)$.

Note that $(\Omega, \Sigma, \mathbb{P})$ is the product space of $(\Omega_v^{\text{act}}, \Sigma_v^{\text{act}}, \mathbb{P}_v^{\text{act}})$ and $(\Omega_v^{\text{col}}, \Sigma_v^{\text{col}}, \mathbb{P}_v^{\text{col}})$ taken over all $v \in V(G)$, where $(\Omega_v^{\text{act}}, \Sigma_v^{\text{act}}, \mathbb{P}_v^{\text{act}})$ models whether $v \in A$ and $(\Omega_v^{\text{col}}, \Sigma_v^{\text{col}}, \mathbb{P}_v^{\text{col}})$ models the choice of $\psi(v)$. We will use this fact to apply Theorem 3.3.

We will use the Lovász Local Lemma to show that with nonzero probability, $X = X_{A, \psi}$ and $L' = L_{A, \psi}|_{V(G) \setminus X}$ (with lists possibly truncated) satisfy the lemma with $\phi = \psi|_X$. To that end, we define the following random variables for each vertex $v \in V(G)$ and $c \in L(v)$ and $i \in [m]$:

- $\Lambda_v(A, \psi) := |L_{A, \psi}(v)|$,
- $\mathbf{D}_{v, c, i}(A, \psi) := |\{u \in N_{G_i}(v) : c \in L_{A, \psi}(u)\} \setminus X_{A, \psi}|$,
- $\mathbf{Y}_{v, c, i}(A, \psi) := |\{u \in N_{G_i}(v) \cap A : c \in L(u)\} \setminus X_{A, \psi}|$,
- $\mathbf{R}_{v, c, i}(A, \psi) := |\{u \in N_{G_i}(v) \setminus A : c \in L(u)\} \setminus \bigcup_{u \in A \setminus V(G_i) : \psi(u)=c} N_G(u)|$.

Note that

$$\begin{aligned} \mathbf{D}_{v, c, i}(A, \psi) &= \mathbf{Y}_{v, c, i}(A, \psi) - |\{u \in (N_{G_i}(v) \cap A) \setminus X_{A, \psi} : c \in L(u)\} \setminus \bigcup_{u \in A : \psi(u)=c} N_G(u)| \\ &\quad + |\{u \in N_{G_i}(v) \setminus A : c \in L(u)\} \setminus \bigcup_{u \in A : \psi(u)=c} N_G(u)| \leq \mathbf{Y}_{v, c, i}(A, \psi) + \mathbf{R}_{v, c, i}(A, \psi). \end{aligned} \tag{4}$$

To apply the Lovász Local Lemma, we first show that for each $v \in V(G)$, we have that Λ_v is at least the quantity in (2.1.1) with high probability, and for each $c \in L(v)$ and $i \in [m]$ such that $v \in V(G_i)$, we have that $\mathbf{D}_{v, c, i}$ is at most the quantity in (2.1.2) with high probability. To that end, we first compute the expectation of these random variables and then apply Theorem 3.3. However, it is not possible to apply Theorem 3.3 to $\mathbf{D}_{v, c, i}$ directly to obtain any meaningful bound. (Consider the case $C = m = 1$ when G_1 is complete. Either no vertex in A is assigned c , in which case $c \in L_{A, \psi}(u)$ for every vertex u , or some vertex in A is assigned c , in which case $c \notin L_{A, \psi}(u)$ for all $u \in V(G)$.) Nevertheless, we can apply Theorem 3.3 to both $\mathbf{Y}_{v, c, i}$ and $\mathbf{R}_{v, c, i}$ and moreover show that $\mathbf{Y}_{v, c, i} + \mathbf{R}_{v, c, i}$ is at most the quantity in (2.1.2) with high probability. Hence, (4) implies the required bound for $\mathbf{D}_{v, c, i}$ as well.

In order to compute the expected values of Λ_v , $\mathbf{Y}_{v, c, i}$, and $\mathbf{R}_{v, c, i}$, we need the following simple proposition.

Proposition 4.2. For every $i \in [m]$, $v \in V(G_i)$, and $c \in L(v)$,

$$\mathbb{P} [\exists u \in N_{G_i}(v) \cap A \text{ such that } \psi(u) = c] = \left(1 - \frac{p}{\Lambda}\right)^D. \tag{5}$$

Moreover, for every $v \in V(G)$ and $c \in L(v)$,

$$\mathbb{P} [c \in L_{A,\psi}(v)] = \left(1 - \frac{p}{\Lambda}\right)^{DC}. \tag{6}$$

Proof. First we prove (5). For each $u \in N_{G_i}(v)$ with $c \in L(u)$, we have $\mathbb{P} [u \in A] = p$ and $\mathbb{P} [\psi(u) = c] = 1/\Lambda$, and the events that $u \in A$ and that $\psi(u) = c$ are independent. Moreover, these events are mutually independent for all such u , so

$$\mathbb{P} [\exists u \in N_{G_i}(v) \cap A \text{ s.t. } \psi(u) = c] = \prod_{u \in N_{G_i}(v) : c \in L(u)} (1 - \mathbb{P} [u \in A] \cdot \mathbb{P} [\psi(u) = c]) = \left(1 - \frac{p}{\Lambda}\right)^D,$$

as desired.

Now we prove (6). For every $v \in V(G)$ and $c \in L(v)$, we have $c \in L_{A,\psi}(v)$ if and only if there is no $u \in N_{G_i}(v) \cap A$ with $\psi(u) = c$ for every i for which $v \in V(G_i)$. By (i), these events are mutually independent for all such i (of which there are C by (ii)), so by (5),

$$\mathbb{P} [c \in L_{A,\psi}(v)] = \prod_{i \in [m] : v \in V(G_i)} \mathbb{P} [\exists u \in N_{G_i}(v) \cap A \text{ s.t. } \psi(u) = c] = \left(1 - \frac{p}{\Lambda}\right)^{DC},$$

as desired. □

Now we compute the expectations of our random variables.

Lemma 4.3. Every vertex $v \in V(G)$ satisfies

$$\mathbb{E} [\mathbf{A}_v] = \left(1 - \frac{p}{\Lambda}\right)^{DC} \Lambda = \Lambda \mathbb{E}(\Lambda, D, C, p). \tag{7}$$

Moreover, for every $i \in [m]$, $v \in V(G_i)$, and $c \in L(v)$,

$$\mathbb{E} [\mathbf{Y}_{v,c,i}] = p \left(1 - \left(1 - \frac{p}{\Lambda}\right)^{DC}\right) D \text{ and} \tag{8}$$

$$\mathbb{E} [\mathbf{R}_{v,c,i}] = (1 - p) \left(1 - \frac{p}{\Lambda}\right)^{D(C-1)} D. \tag{9}$$

Proof. By the linearity of expectation, we have $\mathbb{E} [\mathbf{A}_v] = \sum_{c \in L(v)} \mathbb{P} [c \in L_{A,\psi}(v)]$, so (7) follows from (6).

Again by the linearity of expectation, we have

$$\mathbb{E} [\mathbf{Y}_{v,c,i}] = \sum_{u \in N_{G_i}(v) : c \in L(u)} \mathbb{P} [u \in A \text{ and } u \notin X_{A,\psi}].$$

Note that $\mathbb{P} [u \in A \text{ and } u \notin X_{A,\psi}] = \mathbb{P} [u \in A \text{ and } \psi(u) \notin L_{A,\psi}(u)]$, and the events that $u \in A$ and that $\psi(u) \notin L_{A,\psi}(u)$ are independent. We have $\mathbb{P} [u \in A] = p$ and $\mathbb{P} [\psi(u) \notin L_{A,\psi}(u)] = 1 - (1 - p/\Lambda)^{DC}$ by (6), so (8) follows from the equation above.

By the linearity of expectation, we have

$$\mathbb{E} [\mathbf{R}_{v,c,i}] = \sum_{u \in N_{G_i}(v) : c \in L(u)} \mathbb{P} [u \notin A \text{ and } \psi(w) \neq c \forall w \in N_{G_j}(u) \cap A, j \in [m] \setminus \{i\}].$$

By (5), $\mathbb{P} \left[\psi(w) \neq c \forall w \in N_{G_j}(u) \cap A, j \in [m] \setminus \{i\} \right] = (1 - p/\Lambda)^{D(C-1)}$. Since the events that $u \notin A$ and $\psi(w) \neq c \forall w \in N_{G_j}(u) \cap A, j \in [m] \setminus \{i\}$ are independent, (9) follows from the equation above. \square

Now we need to show that the random variables in Lemma 4.3 are close to their expectation with high probability. We use Theorem 3.3, the ‘exceptional outcomes’ version of Talagrand’s Inequality. To that end, we define an exceptional outcome for each vertex and show that it is unlikely. First, for each vertex $v \in V(G)$ and $c \in L(v)$ and $i \in [m]$, we define the random variable

$$\bullet \mathbf{F}_{v,c,i}(A, \psi) := \left| \{u \in N_{G_i}(v) : \psi(u) = c\} \right|.$$

Then, for each vertex $v \in V(G)$ and $i \in [m]$, we define

$$\bullet \mathbf{\Omega}_{v,i}^* = \{(A, \psi) : \exists u \in \{v\} \cup N_{G_i}(v) \cup N_{G_i}^2(v), \exists c \in L(u), \mathbf{F}_{u,c,i}(A, \psi) \geq \log D\},$$

where $N_{G_i}^2(v)$ is the set of vertices at distance two from v in G_i . To apply Theorem 3.3 to $\mathbf{Y}_{v,c,i}$, we will show that $\mathbf{Y}_{v,c,i}$ has upward (s, δ) -certificates with respect to exceptional outcomes $\mathbf{\Omega}_{v,i}^*$ with $s = 4D$ and $\delta = \log D$. It turns out that we need to consider these exceptional outcomes because, for example, if there is a set of more than $\log D$ vertices in $N_{G_i}(v) \cap A$ that are all assigned the same colour, then changing the outcome of a single trial (in particular the trial determining either $\psi(u)$ or whether $u \in A$ for some $u \in \{v\} \cup N_{G_i}(v) \cup N_{G_i}^2(v)$ for which $\mathbf{F}_{u,c,i}(A, \psi) \geq \log D$ for some $c \in L(u)$) can affect the value of $\mathbf{Y}_{v,c,i}$ by more than $\log D$ (see the final part of the proof of Lemma 4.5 for more details). We remark that the event $\mathbf{\Omega}_{v,i}^*$ contains more outcomes than is necessary; it is simply more convenient to consider $\mathbf{\Omega}_{v,i}^*$, and the argument adapts more easily for correspondence colouring in Section 5.

Now we bound the probability of these exceptional events. For this, we assume without loss of generality that $uv \in E(G)$ only if $L(u) \cap L(v) \neq \emptyset$, so each G_i has maximum degree at most ΛD .

Proposition 4.4. *Every vertex $v \in V(G)$ and $i \in [m]$ satisfies*

$$\mathbb{P} \left[\mathbf{\Omega}_{v,i}^* \right] \leq 11^3 C^3 D^5 (\log D)^{-\log D}.$$

Proof. First we bound the probability that $\mathbf{F}_{u,c,i}$ is too large for each $u \in V(G)$ and $c \in L(u)$, as follows:

$$\mathbb{P} \left[\mathbf{F}_{u,c,i} \geq \log D \right] \leq \sum_{i=\lceil \log D \rceil}^D \binom{D}{i} \left(\frac{1}{\Lambda} \right)^i \leq \sum_{i=\lceil \log D \rceil}^D \left(\frac{eD}{i\Lambda} \right)^i \leq \sum_{i=\lceil \log D \rceil}^D \left(\frac{e}{i} \right)^i.$$

Since each term in the sum is at most $(e/\log D)^{\log D}$ and there are at most D terms, it follows that

$$\mathbb{P} \left[\mathbf{F}_{u,c,i} \geq \log D \right] \leq D^2 (\log D)^{-\log D}.$$

Thus, combining the above inequality with the Union Bound, we have $\mathbb{P} \left[\mathbf{\Omega}_{v,i}^* \right] \leq \Lambda(1 + \Lambda D + \Lambda^2 D^2) (\log D)^{-\log D} \leq 11^3 C^3 D^5 (\log D)^{-\log D}$, as desired. \square

Combining Theorem 3.3 and Proposition 4.4, we show that $\mathbf{\Lambda}_v$, $\mathbf{Y}_{v,c,i}$ and $\mathbf{R}_{v,c,i}$ are close to their expectation with high probability in the following lemma.

Lemma 4.5. *Every vertex $v \in V(G)$ satisfies*

$$\mathbb{P} \left[|\mathbf{\Lambda}_v - \mathbb{E}[\mathbf{\Lambda}_v]| > D^{2/3} \right] \leq \exp(-D^{1/4}). \tag{10}$$

Moreover, if $v \in V(G_i)$ and $c \in L(v)$, then

$$\mathbb{P} [|Y_{v,c,i} - \mathbb{E} [Y_{v,c,i}]| > D^{2/3}] \leq \exp(-D^{1/4}) \quad \text{and} \tag{11}$$

$$\mathbb{P} [|\mathbf{R}_{v,c,i} - \mathbb{E} [\mathbf{R}_{v,c,i}]| > D^{2/3}] \leq \exp(-D^{1/4}). \tag{12}$$

Proof. First we prove (10). We apply Theorem 3.3 with exceptional outcomes $\Omega^* = \emptyset$. To that end, we show that $\Lambda - \Lambda_v$ has upward (s, δ) -certificates with respect to Ω^* , where $s = 2\Lambda$ and $\delta = 1$. Let $(A, \psi) \in \Omega$. For every $c \in L(v) \setminus L_{A,\psi}(v)$, there is at least one neighbour $u \in N_G(v)$ of v such that $u \in A$ and $\psi(u) = c$. Choose one such neighbour, and denote it by u_c . Let $I_{A,\psi}$ index the trials determining whether $u_c \in A$ and $\psi(u_c) = c$ for each $c \in L(v) \setminus L_{A,\psi}(v)$. Now if $(A', \psi') \in \Omega$ differs from (A, ψ) in at most t of the trials indexed by $I_{A,\psi}$ (and differs arbitrarily for trials not indexed by $I_{A,\psi}$), then all but at most t colours in $L(v) \setminus L_{A,\psi}(v)$ are also in $L(v) \setminus L_{A',\psi'}(v)$. Hence, $\Lambda - \Lambda_v(A', \psi') \geq \Lambda - \Lambda_v(A, \psi) - t$, and since $I_{A,\psi} \leq 2\Lambda = s$, it follows that $\Lambda - \Lambda_v$ has upward (s, δ) -certificates, as desired. Therefore by Theorem 3.3 with $t = D^{2/3}$,

$$\mathbb{P} [|\Lambda_v - \mathbb{E} [\Lambda_v]| > D^{2/3}] \leq 4 \exp\left(-\frac{D^{4/3}}{32\Lambda}\right) \leq \exp(-D^{1/4}),$$

as desired.

Now we prove (12). We apply Theorem 3.3 with exceptional outcomes $\Omega^* = \emptyset$. To that end, we show that $D - \mathbf{R}_{v,c,i}$ has upward (s, δ) -certificates with respect to Ω^* , where $s = 3D$ and $\delta = 1$. Let $(A, \psi) \in \Omega$, and note that

$$D - \mathbf{R}_{v,c,i}(A, \psi) = |\{x \in N_{G_i}(v) : c \in L(x)\} \cap (A \cup \bigcup_{w \in A \setminus V(G_i) : \psi(w)=c} N_G(w))|.$$

For every $u \in \{x \in N_{G_i}(v) : c \in L(x)\} \cap \bigcup_{w \in A \setminus V(G_i) : \psi(w)=c} N_G(w)$, there is at least one neighbour $w \in N_{G_i}(u)$ of u where $j \in [m] \setminus \{i\}$ such that $w \in A$ and $\psi(w) = c$. Choose one such neighbour, and denote it by w_u . By (i), the vertices w_u are distinct for different choices of u . Let $I_{A,\psi}$ index the trials determining whether $u \in A$ for each $u \in \{x \in N_{G_i}(v) : c \in L(x)\}$ and whether $w_u \in A$ and $\psi(w_u) = c$ for each $u \in \{x \in N_{G_i}(v) : c \in L(x)\} \cap \bigcup_{w \in A \setminus V(G_i) : \psi(w)=c} N_G(w)$. Now if $(A', \psi') \in \Omega$ differs from (A, ψ) in at most t of the trials indexed by $I_{A,\psi}$ (and differs arbitrarily for trials not indexed by $I_{A,\psi}$), all but at most t vertices in $\{x \in N_{G_i}(v) : c \in L(x)\} \cap (A \cup \bigcup_{w \in A \setminus V(G_i) : \psi(w)=c} N_G(w))$ are in $\{x \in N_{G_i}(v) : c \in L(x)\} \cap (A' \cup \bigcup_{w \in A \setminus V(G_i) : \psi'(w)=c} N_G(w))$. Hence, $D - \mathbf{R}_{v,c,i}(A', \psi') \geq D - \mathbf{R}_{v,c,i}(A, \psi) - t$, and since $I_{A,\psi} \leq 3D = s$, it follows that $\mathbf{R}_{v,c,i}$ has upward (s, δ) -certificates, as desired. Therefore by Theorem 3.3 with $t = D^{2/3}$,

$$\mathbb{P} [|\mathbf{R}_{v,c,i} - \mathbb{E} [\mathbf{R}_{v,c,i}]| > D^{2/3}] \leq 4 \exp\left(-\frac{D^{4/3}}{48D}\right) \leq \exp(-D^{1/4}),$$

as desired.

Finally, we prove (11). We apply Theorem 3.3 with exceptional outcomes $\Omega^* = \Omega_{v,i}^*$. To that end, we show that $Y_{v,c,i}$ has upward (s, δ) -certificates with respect to Ω^* , where $s = 4D$ and $\delta = \log D$. Let $(A, \psi) \in \Omega \setminus \Omega^*$. For every $u \in \{x \in N_{G_i}(v) \cap A : c \in L(x)\} \setminus X_{A,\psi}$, there is at least one neighbour $w \in N_G(u)$ of u such that $w \in A$ and $\psi(w) = \psi(u)$. Choose one such neighbour, and denote it by w_u . Let $I_{A,\psi}$ index the trials determining whether $u \in A$, whether $w_u \in A$, and the assignment of $\psi(u)$ and $\psi(w_u)$ for each $u \in \{x \in N_{G_i}(v) \cap A : c \in L(x)\}$. Since $(A, \psi) \notin \Omega^*$, the multi-set $\{w_u : u \in (N_{G_i}(v) \cap A) \setminus X_{A,\psi}, c \in L(u)\}$ has maximum multiplicity at most $\log D$ (as otherwise, if w appears more than $\log D$ times in this multi-set, then $F_{w,\psi(w),i}(A, \psi) > \log D$). Hence, if $(A', \psi') \in \Omega$ differs from (A, ψ) in at most $t/\log D$ of the trials indexed by $I_{A,\psi}$ (and differs arbitrarily for trials not indexed by $I_{A,\psi}$), then all but at most t vertices in $\{x \in N_{G_i}(v) \cap A : c \in L(x)\} \setminus X_{A,\psi}$ are in $\{x \in N_{G_i}(v) \cap A' : c \in L(x)\} \setminus X_{A',\psi'}$. Hence, $Y_{v,c,i}(A', \psi') \geq Y_{v,c,i}(A, \psi) - t$, and

since $I_{A,\psi} \leq 4D = s$, it follows that $\mathbf{Y}_{v,c,i}$ has upward (s, δ) -certificates, as desired. Therefore by Theorem 3.3 with $t = D^{2/3}$ and Proposition 4.4,

$$\mathbb{P} [|\mathbf{Y}_{v,c,i} - \mathbb{E} [\mathbf{Y}_{v,c,i}]| > D^{2/3}] \leq 4 \exp \left(-\frac{D^{4/3}}{64D \log^2 D} \right) + 44^3 C^3 D^5 (\log D)^{-\log D} \leq \exp (-D^{1/4}),$$

as desired. □

Finally we can prove Lemma 2.1, using Lemmas 4.3 and 4.5.

Proof of Lemma 2.1. First, rather than showing (2.1.1), it suffices to show that $|L'(v)| \geq (1 - p/\Lambda)^{DC} \Lambda - \Lambda^{4/5}$ for every $v \in V(G) \setminus X$, since we can truncate each list until equality holds.

We consider (A, ψ) chosen randomly as described earlier, and we define the following set of bad events for each vertex $v \in V(G)$ and $c \in L(v)$ and $i \in [m]$:

$$\begin{aligned} \mathcal{A}_v &= \{ (A, \psi) : \mathbf{\Lambda}_v(A, \psi) < \mathbb{E} [\mathbf{\Lambda}_v] - \Lambda^{4/5} \}, \\ \mathcal{A}_{v,c,i} &= \{ (A, \psi) : \mathbf{Y}_{v,c,i} > \mathbb{E} [\mathbf{Y}_{v,c,i}] + D^{4/5}/2 \}, \text{ and} \\ \mathcal{A}'_{v,c,i} &= \{ (A, \psi) : \mathbf{R}_{v,c,i}(A, \psi) > \mathbb{E} [\mathbf{R}_{v,c,i}] + D^{4/5}/2 \}. \end{aligned}$$

Letting \mathcal{A} be the union of all such bad events, note that each event in \mathcal{A} is mutually independent of all but at most $(\lceil \Lambda \rceil CD)^4 \leq (11C^2D^2)^4$ other events in \mathcal{A} . By Lemma 4.5, every event in \mathcal{A} occurs with probability at most $\exp (-D^{1/4})$. Therefore by the Lovász Local Lemma, there exists $(A, \psi) \notin \mathcal{A}$.

Now we show that $X := X_{A,\psi}$ and $L' := L_{A,\psi}|_{V(G)\setminus X}$ satisfy the lemma with $\phi := \psi|_X$. Indeed, ϕ is an $L|_X$ -colouring of $G[X]$ and since $X \subseteq A$, we have $L'(v) \subseteq L(v) \setminus \{\phi(u) : u \in N_G(v) \cap X\}$, as required. Moreover, for every $v \in V(G)$, since $(A, \psi) \notin \mathcal{A}_v$, by (7), we have $|L'(v)| \geq (1 - p/\Lambda)^{DC} \Lambda - \Lambda^{4/5}$, as desired, and if $v \in V(G_i)$ and $c \in L(v)$, then since $(A, \psi) \notin \mathcal{A}_{v,c,i} \cup \mathcal{A}'_{v,c,i}$, by (4), (8), and (9), we have (2.1.2), as desired. □

5. Correspondence colouring

In this section we introduce *correspondence colouring* and describe how to generalize Theorem 1.3 to this setting.

Definition 5.1. Let G be a graph with list assignment L .

- If M is a map with domain $E(G)$ where for each $e = uv \in E(G)$, $M(e)$ is a matching of $\{u\} \times L(u)$ and $\{v\} \times L(v)$, we say (L, M) is a correspondence assignment for G .
- An (L, M) -colouring of G is a map ϕ with domain $V(G)$ such that $\phi(u) \in L(u)$ for every $u \in V(G)$, and every $e = uv \in E(G)$ satisfies $(u, \phi(u))(v, \phi(v)) \notin M(e)$. If G has an (L, M) -colouring, then we say G is (L, M) -colourable.
- The correspondence chromatic number of G , also called the *DP*-chromatic number, denoted $\chi_{DP}(G)$, is the minimum k such that G is (L, M) -colourable for every correspondence assignment (L, M) satisfying $|L(v)| \geq k$ for all $v \in V(G)$.

For convenience, if $uv \in E(G)$, $c_1 \in L(u)$, $c_2 \in L(v)$, and $(u, c_1)(v, c_2) \in M(uv)$, we will just write $c_1c_2 \in M(uv)$. For each $v \in V(G)$ and $c \in L(v)$, we will let $N_{G,(L,M)}(v, c) := \{(u, c') : uv \in E(G), cc' \in M(uv)\}$, and we omit the subscript in $N_{G,(L,M)}(v, c)$ when it is clear from the context. Note that if for each $e = uv \in E(G)$ and $c \in L(u) \cap L(v)$, we have $cc \in M(uv)$, then an (L, M) -colouring is an L -colouring. Hence, every graph G satisfies $\chi_\ell(G) \leq \chi_{DP}(G)$.

For a graph G with correspondence assignment (L, M) , we define the *colour degree* of each $v \in V(G)$ and $c \in L(v)$ to be $d_{G,(L,M)}(v, c) := |N_{G,(L,M)}(v, c)|$ and the *maximum colour degree* to be

$\Delta(G, (L, M)) := \max_{v \in V(G)} \max_{c \in L(v)} d_{G, (L, M)}(v, c)$. With only minor modifications to the argument used to prove Theorem 1.3, which we describe below, we can strengthen Theorem 1.3 to the setting of correspondence colouring, as follows.

Theorem 5.2. *For every $C, \varepsilon > 0$, the following holds for all sufficiently large D . Let G_1, \dots, G_m be graphs that*

- (i) *are nearly disjoint and*
- (ii) *satisfy $|\{i \in [m] : v \in V(G_i)\}| \leq C$ for every $v \in \bigcup_{i=1}^m V(G_i)$.*

If (L, M) is a correspondence assignment for $G := \bigcup_{i=1}^m G_i$ satisfying

- (iii) $\Delta(G_i, (L|_{V(G_i)}, M|_{E(G_i)}) \leq D$ *for every $i \in [m]$ and*
- (iv) $|L(v)| \geq (1 + \varepsilon)D$ *for every $v \in V(G)$,*

then G is (L, M) -colourable.

As mentioned, Theorem 5.2 implies that Corollary 1.1 actually holds for the correspondence chromatic number, which in turn implies the main result of [19], that linear and uniform hypergraphs of maximum degree at most D have *correspondence chromatic index* at most $D + o(D)$.

To prove Theorem 5.2, we use the argument presented in Section 2, but with Lemmas 2.1 and 2.3 replaced with the following lemmas, respectively.

Lemma 5.3. *For every $C, \varepsilon > 0$, there exists $D_{5.3}$ such that the following holds for every $D \geq D_{5.3}$. Let G_1, \dots, G_m be graphs, and let (L, M) be a correspondence assignment for $G := \bigcup_{i=1}^m G_i$ satisfying (i)–(iv) (of Theorem 5.2). If $|L(v)| = \lceil \Lambda \rceil$ for every $v \in V(G)$, where $(1 + \varepsilon)D \leq \Lambda \leq 10CD$, and if $\log^{-1} D \geq p \geq \log^{-2} D$, then there exist $X \subseteq V(G)$, an $(L|_X, M|_{E(G[X])})$ -colouring ϕ of $G[X]$, and a correspondence assignment L' for $G - X$ satisfying $L'(v) \subseteq \{c \in L(v) : (v, c) \in (\{v\} \times L(v)) \setminus \bigcup_{u \in X} N(u, \phi(u))\}$ for every $v \in V(G)$, such that*

$$(5.3.1) \quad |L'(v)| = \lceil \Lambda_{\mathbb{E}}(\Lambda, D, C, p) - \Lambda^{4/5} \rceil \text{ for every } v \in V(G) \setminus X \text{ and}$$

$$(5.3.2) \quad \Delta(G_i - X, (L'|_{V(G_i - X)}, M|_{E(G_i - X)})) \leq D_{\mathbb{E}}(\Lambda, D, C, p) + D^{4/5} \text{ for every } i \in [m].$$

Lemma 5.4. *Let G be a graph with correspondence assignment L . If $|L(v)| \geq 8D$ for every $v \in V(G)$ and $\Delta(G, (L, M)) \leq D$, then G is (L, M) -colourable.*

Lemma 5.4, like Lemma 2.3, can be proved with a straightforward application of the Lovász Local Lemma. A stronger result (with ‘ $|L(v)| \geq 8D$ ’ replaced by ‘ $|L(v)| \geq 2D$ ’) also follows easily from a well-known result of Haxell [11] on independent transversals.

Therefore it remains to describe how the argument presented in Section 4 to prove Lemma 2.1 can be modified to obtain Lemma 5.3. We consider the same probability space $(\Omega, \Sigma, \mathbb{P})$ of pairs (A, ψ) , but we instead define $L_{A, \psi}(v) := \{c \in L(v) : (v, c) \in (\{v\} \times L(v)) \setminus \bigcup_{u \in A} N(u, \psi(u))\}$. The definition of $X_{A, \psi}$ and of $\Lambda_v(A, \psi)$ remains the same, but we replace the definitions of $\mathbf{D}_{v, c, i}(A, \psi)$, $\mathbf{Y}_{v, c, i}(A, \psi)$, and $\mathbf{R}_{v, c, i}(A, \psi)$ with the following:

- $\mathbf{D}_{v, c, i}(A, \psi) := \left| \left\{ (u, c') \in N_{G_i, (L|_{V(G_i)}, M|_{E(G_i)})(v, c) : u \notin X_{A, \psi}, c' \in L_{A, \psi}(u) \right\} \right|$,
- $\mathbf{Y}_{v, c, i}(A, \psi) := \left| \left\{ (u, c') \in N_{G_i, (L|_{V(G_i)}, M|_{E(G_i)})(v, c) : u \in A \setminus X_{A, \psi} \right\} \right|$, and
- $\mathbf{R}_{v, c, i}(A, \psi) := \left| \left\{ (u, c') \in N_{G_i, (L|_{V(G_i)}, M|_{E(G_i)})(v, c) : u \notin A \right\} \setminus \bigcup_{u \in A \setminus V(G_i)} N_{G, (L, M)}(u, \psi(u)) \right|$.

The definition of $\Omega_{v,i}^*$ remains the same, but we replace the definition of $F_{v,c,i}(A, \psi)$ with the following:

$$\bullet F_{v,c,i}(A, \psi) := \left| \left\{ (u, c') \in N_{G_i, (L_{V(G_i)}, M_{E(G_i)})}(v, c) : \psi(u) = c' \right\} \right|.$$

The remainder of the proof can be obtained via straightforward modifications, so we omit the details. This completes the proof of Lemma 5.3 and in turn implies Theorem 5.2.

6. Proof of Theorems 1.4 and 1.5

In this section, we prove Theorems 1.4 and 1.5. Recall that $f(m, \mathcal{G})$ is the largest possible chromatic number of the union of at most m nearly disjoint graphs in \mathcal{G} , and \mathcal{G}_n^X is the set of graphs of chromatic number at most n . We begin by providing the construction that certifies that $f(3, \mathcal{G}_n^X) \geq n + 1$.

Proof of Theorem 1.4(ii). Let H_1 and H_2 be complete graphs on $n + 1$ vertices such that $V(H_1) \cap V(H_2) = \{v\}$ for some vertex v , let $u \in V(H_1) \setminus \{v\}$, let $w \in V(H_2) \setminus \{v\}$, let $G_1 := H_1 - uv$, let $G_2 := H_2 - wv$, and let G_3 be the graph consisting of the single edge uw . It is straightforward to check that

- G_1, G_2 , and G_3 are nearly disjoint,
- $\chi(G_i) \leq n$ for every $i \in [3]$, and
- $\chi(G_1 \cup G_2 \cup G_3) = n + 1$,

as desired. □

To prove Theorem 1.4, it remains to prove the bound $f(m, \mathcal{G}_n^X) \leq m + n - 2$ when $m + n$ is sufficiently large. First we note the following immediate consequence of the main result in [17] (namely that the Erdős–Faber–Lovász conjecture holds for all sufficiently large n).

Theorem 6.1 ([17]). *The following holds for all sufficiently large C : If G_1, \dots, G_m are nearly disjoint graphs satisfying $\max\{m, |V(G_1)|, \dots, |V(G_m)|\} \leq C$, then $\chi(\bigcup_{i=1}^m G_i) \leq C$.*

Proof of Theorem 1.4(i). Suppose to the contrary, and let G_1, \dots, G_m be nearly disjoint graphs, each of chromatic number at most n , such that $\chi(\bigcup_{i=1}^m G_i) > m + n - 2$ and $|V(\bigcup_{i=1}^m G_i)|$ is minimum. Note that $n \geq 2$ as otherwise $\chi(\bigcup_{i=1}^m G_i) = 1$. By Theorem 6.1 with $n + m - 2$ playing the role of C , there exists $i \in [m]$ such that $|V(G_i)| > m + n - 2$, and we may assume without loss of generality that $i = 1$. Let $X := V(G_1) \setminus \bigcup_{i=2}^m V(G_i)$. Since G_1, \dots, G_m are nearly disjoint,

$$|X| \geq |V(G_1)| - (m - 1) \geq n. \tag{13}$$

Since $\chi(G_1) \leq n$, there is a partition I_1, \dots, I_n of $V(G_1)$ into independent sets. By (13) and the pigeonhole principle, there exists some $j \in [n]$ such that either $|I_j \cap X| > 1$ or both $I_j \cap X$ and $I_j \setminus X$ are nonempty. We may assume without loss of generality that $j = 1$, so there exist distinct vertices $u, v \in I_1$ such that $u \in X$. Let G'_1 be the graph obtained from G_1 by identifying u and v into a single new vertex. Crucially, G'_1, G_2, \dots, G_m are nearly disjoint, $\chi(G'_1) \leq n$, and $\chi(G'_1 \cup \bigcup_{i=2}^m G_i) \geq \chi(\bigcup_{i=1}^m G_i)$, contradicting the choice of G_1, \dots, G_m to have $|V(\bigcup_{i=1}^m G_i)|$ minimum. □

Now we prove Theorem 1.5. In this proof, we use a Latin square to construct a graph of large chromatic number. An *order- n Latin square* is an $n \times n$ array of n symbols where each row and column contains each symbol exactly once. That is, for an order- n Latin square L , with rows,

columns, and symbols indexed by A, B , and C , respectively, we have $\{L(a, b) : b \in B\} = C$ for every $a \in A$ and $\{L(a, b) : a \in A\} = C$ for every $b \in B$.

Proof of Theorem 1.5. First, let $G \cong K_m$, and since $3 \mid m$, there exist sets $A, B, C \subseteq V(G)$ of size $m/3$ that partition $V(G)$. Since $n \geq m - 1$, we can let G' be the multigraph obtained from G by adding $n - (m - 1) + m/3$ loops to each vertex in $A \cup B$ and $n - (m - 1)$ loops to each vertex in C , and we let H be the line graph of G' . Let L be an order- $(m/3)$ Latin square with rows, columns, and symbols indexed by A, B , and C , respectively, and for each $a \in A$ and $b \in B$, let e_1^{ab} denote the edge in H with ends ab and ac , and let e_2^{ab} denote the edge in H with ends ab and bc , where $c = L(a, b)$. Let

$$H' := H - \bigcup_{(a,b) \in A \times B} \{e_1^{ab}, e_2^{ab}\}.$$

We prove that H' has chromatic number at least $n + m/6$ and is a nearly disjoint union of m graphs of chromatic number n , as desired.

To that end, for each $v \in V(G)$, let $G_v := H'[\{e \in E(G') : e \ni v\}]$. By construction, the graphs in $\{G_v : v \in V(G)\}$ are nearly disjoint, and $H' = \bigcup_{v \in V(G)} G_v$. Moreover, G_v is isomorphic to $K_{n+m/3}$ with a matching of size $m/3$ removed for every $v \in A \cup B$ and is isomorphic to K_n for every $v \in C$, so $\chi(G_v) = n$ for every $v \in V(G)$, as required.

To prove that $\chi(H') \geq n + m/6$, we let ϕ be a proper colouring of H' using at most k colours, and we show that $k \geq n + m/6$, as follows. First, let

$$B_a := \{b \in B : \phi(ab) = \phi(ac), \text{ where } c = L(a, b)\} \text{ for every } a \in A \quad \text{and}$$

$$A_b := \{a \in A : \phi(ab) = \phi(bc), \text{ where } c = L(a, b)\} \text{ for every } b \in B.$$

We claim that the following holds:

- (a) $|B_a| \geq n + m/3 - k$ for every $a \in A$,
- (b) $|A_b| \geq n + m/3 - k$ for every $b \in B$, and
- (c) $\sum_{a \in A} |B_a| + \sum_{b \in B} |A_b| \leq \frac{m^2}{9}$.

Altogether, (a)–(c) imply that $(2m/3)(n + m/3 - k) \leq m^2/9$. Rearranging terms in this inequality, we have $k \geq n + m/6$, as desired. Thus, it remains to prove (c)–(c).

To prove (a), fix some $a \in A$. Since ϕ is a proper colouring, the colours assigned to $\{ab : b \in B_a\}$ and the colours assigned to vertices of G_a not incident to an edge of $\{e_1^{ab} : b \in B_a\}$ are distinct. Thus, since at most k colours are used in total, $|B_a| + n + m/3 - 2|B_a| \leq k$. Rearranging terms in this inequality, we have $|B_a| \geq n + m/3 - k$, as desired. The proof of (b) is essentially the same, so we omit the details.

To prove (c), suppose to the contrary that $\sum_{a \in A} |B_a| + \sum_{b \in B} |A_b| > m^2/9$. In this case, there is a pair $(a, b) \in A \times B$ such that $a \in A_b$ and $b \in B_a$. Letting $c = L(a, b)$, since $a \in A_b$, we have $\phi(ab) = \phi(bc)$, and since $b \in B_a$, we have $\phi(ab) = \phi(ac)$. However, since ϕ is a proper colouring, $\phi(bc) \neq \phi(ac)$, a contradiction. \square

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