Canad. Math. Bull. Vol. **48** (2), 2005 pp. 251–259

The Index Theory Associated to a Non-Finite Trace on a C^*-Algebra

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Abstract. The index theory considered in this paper, a generalisation of the classical Fredholm index theory, is obtained in terms of a non-finite trace on a unital C^*-algebra. We relate it to the index theory of M. Breuer, which is developed in a von Neumann algebra setting, by means of a representation theorem. We show how our new index theory can be used to obtain an index theorem for Toeplitz operators on the compact group $\mathbf{U}(2)$, where the classical index theory does not give any interesting result.

1 Introduction

Index theory for various classes of operators, such as pseudodifferential operators and Toeplitz operators, forms a vast and important aspect of modern operator theory. Usually, the index is the classical Fredholm index. Recall that a norm-bounded linear operator T on a Hilbert space is a *Fredholm* operator if its range is closed and the kernels of T and T^* are finite-dimensional. In this case the index of T is defined to be the difference

$$\operatorname{ind}(T) = \dim(\ker(T)) - \dim(\ker(T^*)).$$

This idea has been generalised and other concepts of index are of increasing importance. We single out for special mention the extension of classical Fredholm theory due to M. Breuer [3, 4], where the index is no longer an integer but may be an arbitrary real number. This index has been used in results of M. Atiyah and A. Connes (see [6] for an exposition of some of this theory) and, as is especially relevant to the considerations of this paper, in results of the author [10] and of L. Coburn, R. G. Douglas, D. Schaeffer and I. M. Singer [5].

Nevertheless, even this more extended concept of index does not cover all cases of interest. We develop in Section 3 an index theory for Toeplitz operators on the compact unitary group $\mathbf{U}(2)$. If one uses the ordinary Fredholm index in this case, then one gets a trivial theory, since the only such Toeplitz operators that are Fredholm all have zero index, as was shown by C. A. Berger and L. A. Coburn in [1]. We exhibit another index theory for these operators, where the index remains integer-valued, but which is non-trivial in that there are Toeplitz operators that are Fredholm in the more extended sense and that have non-zero index.

Our new index theory will, however, in general have real values rather than integer values. In this respect it is like the Breuer index theory and we show, in Proposition 2.2, that the theories are related; however, they are not the same, as is seen by the

Received by the editors July 30, 2003; revised September 23, 2003.
AMS subject classification: 46L, 47B35, 47L80.

example of Toeplitz operators on the matrix group $\mathbf{U}(2)$ referred to in the preceding paragraph. The fundamental feature of our theory is that it is defined in terms of a trace on a C^*-algebra. We previously introduced the elements of this theory in an earlier paper [11]. Here we develop and expand these ideas, relate the theory to the Breuer theory, and give some applications.

2 The Index Associated to a Non-Finite Trace

Suppose that $\mathcal{A}$ is a unital C^*-algebra and let $\tau\colon \mathcal{A}^+ \to [0, +\infty]$ denote a trace on $\mathcal{A}$. The linear span of the elements $a \in \mathcal{A}^+$ such that $\tau(a) < \infty$ is a self-adjoint ideal $\mathcal{M}_\tau$ of $\mathcal{A}$ such that $\mathcal{M}_\tau \cap \mathcal{A}^+ = \{a \in \mathcal{A}^+ \mid \tau(a) < \infty\}$, and there is a unique positive linear extension of τ to $\mathcal{M}_\tau$ that we denote by the same symbol τ. Moreover, if $a \in \mathcal{A}$ and $x \in \mathcal{M}_\tau$, then $\tau(ax) = \tau(xa)$.

In this setting, one can define the Fredholm index of an element of $\mathcal{A}$ by analogy with the case of the C^*-algebra $B(H)$, where H is an infinite-dimensional Hilbert space, where the role of τ is played by the usual trace function $\tau = \mathrm{tr}$, and the new theory one obtains reduces to the usual one in this case.

We recall the basic facts from [11]. Fix a unital C^*-algebra $\mathcal{A}$ and a trace τ on $\mathcal{A}$. We suppose that τ is not finite; that is, $\mathcal{M}_\tau \neq \mathcal{A}$.

An element a of $\mathcal{A}$ is *Fredholm* relative to τ, or τ*-Fredholm*, if there exists an element $b \in \mathcal{A}$ such that $1 - ab$ and $1 - ba$ belong to $\mathcal{M}_\tau$. The element b is then a *partial inverse* of a. The *Fredholm index* of a relative to τ, or τ*-index* of a, is defined by setting $\mathrm{ind}_\tau(a) = \tau(ab - ba)$. This is easily seen to be well-defined. Observe that although the trace can take on arbitrary complex values on $\mathcal{M}_\tau$ (unless it is trivial), this is not the case for the index, which is real-valued.

Let $\mathcal{K}_\tau$ denote the closure of $\mathcal{M}_\tau$ in $\mathcal{A}$; of course, $\mathcal{K}_\tau$ is a proper ideal in $\mathcal{A}$. If $a \in \mathcal{A}$, then it is clear that a is τ-Fredholm if, and only if, it is invertible modulo $\mathcal{K}_\tau$. Hence, if π is the quotient map from $\mathcal{A}$ to $\mathcal{A}/\mathcal{K}_\tau$, the set $\Phi = \Phi(\mathcal{A}, \tau)$ of τ-Fredholm elements of $\mathcal{A}$ is equal to $\pi^{-1}\,\mathrm{Inv}(\mathcal{A}/\mathcal{K}_\tau)$, where $\mathrm{Inv}(\mathcal{A}/\mathcal{K}_\tau)$ denotes the set of invertible elements of $\mathcal{A}/\mathcal{K}_\tau$. It follows immediately that Φ is open in the norm topology of $\mathcal{A}$ and that it is closed under multiplication.

We also have the following results from [11]:

Let $a, b \in \Phi$ and let $x \in \mathcal{K}_\tau$. Then

1. $\mathrm{ind}_\tau(ab) = \mathrm{ind}_\tau(a) + \mathrm{ind}_\tau(b)$;
2. ind_τ is locally constant;
3. $\mathrm{ind}_\tau(a + x) = \mathrm{ind}_\tau(a)$;
4. $\mathrm{ind}_\tau(a) = 0$, if $a = a^*$.

In the classical Fredholm theory, a Fredholm operator of index zero is a compact perturbation of an invertible operator. The analogue of this result does not extend to the general theory.

There is another generalization of Fredholm index theory due to J. Phillips and I. Raeburn that involves a trace, this time on a von Neumann algebra [12]. In the case that the von Neumann algebra is a II_∞-factor, this theory coincides with the better known Fredholm theory of M. Breuer [3, 4] and, in fact, the Phillips–Raeburn theory is obtained from Breuer's theory by making some minor modifications, as is pointed out in [12]. We shall show now that, at least partially, the Phillips–Raeburn theory is a special case of our theory.

Let $\mathcal{R}$ be a *semi-finite von Neumann algebra* on a Hilbert space H; that is, we suppose that $\mathcal{R}$ admits a faithful, normal, semi-finite trace tr. A projection p in $\mathcal{R}$ is tr-*finite* if $\mathrm{tr}(p) < \infty$ and we write $\dim(p) = \mathrm{tr}(p)$ for the *generalized dimension* of p in this case.

Suppose now a is an element of $\mathcal{R}$ and $a = u|a|$ is its polar decomposition, so that the partial isometry u and the positive element $|a|$ belong to $\mathcal{R}$. If R_a is the range projection of a (the projection onto the closure of $a(H)$), then $R_a \in \mathcal{R}$, since $R_a = uu^*$. Note also that $R_{a^*} = u^*u$. If $\ker(a)$ denotes the projection onto the kernel of a, then $\ker(a) = 1 - R_{a^*}$, so $\ker(a) \in \mathcal{R}$ also.

In the Phillips–Raeburn theory an element a of $\mathcal{R}$ is *Fredholm* relative to tr if $\ker(a)$ is tr-finite and there exists a tr-finite projection e in $\mathcal{R}$ such that $(1-e)(H) \subseteq a(H)$. In this case $\ker(a^*)$ is also tr-finite and the *Breuer–Raeburn–Phillips index* is defined by setting

$$\mathrm{ind}_{\mathrm{BPR}}(a) = \dim(\ker(a)) - \dim(\ker(a^*)).$$

It is a theorem of Phillips and Raeburn that an element $a \in \mathcal{R}$ is Fredholm in their sense relative to tr if, and only if, it is invertible modulo $\mathcal{K}_{\mathrm{tr}}$. Hence, their definition of Fredholmness relative to tr coincides with our definition of tr-Fredholmness. It remains to show that their definition of the index also coincides with ours in this case. First, we need to recall that they show that their index has similar properties to the usual Fredholm index: $\mathrm{ind}_{\mathrm{BPR}}(ab) = \mathrm{ind}_{\mathrm{BPR}}(a) + \mathrm{ind}_{\mathrm{BPR}}(b)$, if a and b are tr-Fredholm elements, and $\mathrm{ind}_{\mathrm{BPR}}(a+x) = \mathrm{ind}_{\mathrm{BPR}}(a)$, if $x \in \mathcal{K}_{\mathrm{tr}}$. Also, $\mathrm{ind}_{\mathrm{BPR}}$ is locally constant.

Now let $a \in \mathcal{R}$ be tr-Fredholm and let $a = u|a|$ be its polar decomposition. If $\pi\colon \mathcal{R} \to \mathcal{R}/\mathcal{K}_{\mathrm{tr}}$ is the quotient map, then $\pi(a) = \pi(u)|\pi(a)|$, so that invertibility of $\pi(a)$ implies invertibility of $|\pi(a)| = \pi(|a|)$ and therefore of $\pi(u)$. Hence, $|a|$ and u are tr-Fredholm. Moreover, since $\pi(u)$ is a partial isometry (and invertible), it is a unitary and therefore $1 - uu^*$ and $1 - u^*u$ belong to $\mathcal{K}_{\mathrm{tr}}$. It follows, since $\mathrm{ind}_{\mathrm{tr}}(|a|) = 0$, that $\mathrm{ind}_{\mathrm{tr}}(a) = \mathrm{ind}_{\mathrm{tr}}(u) + \mathrm{ind}_{\mathrm{tr}}(|a|) = \mathrm{ind}_{\mathrm{tr}}(u) = \mathrm{tr}(uu^* - u^*u) = \mathrm{tr}((1-u^*u) - (1-uu^*)) = \mathrm{tr}((1-R_{a^*}) - (1-R_a)) = \mathrm{tr}(\ker(a) - \ker(a^*)) = \dim(\ker(a)) - \dim(\ker(a^*)) = \mathrm{ind}_{\mathrm{BPR}}(a)$.

The preceding remarks have shown the following:

Proposition 2.1 *Let $\mathcal{R}$ be a semi-finite von Neumann algebra and let* tr *be a faithful, normal semi-finite trace on $\mathcal{R}$. If $a \in \mathcal{R}$, then a is Fredholm relative to* tr *in the sense of Phillips and Raeburn if, and only if, a is* tr*-Fredholm in our sense. Moreover, in this case*

$$\mathrm{ind}_{\mathrm{tr}}(a) = \mathrm{ind}_{\mathrm{BPR}}(a) = \mathrm{tr}(\ker(a)) - \mathrm{tr}(\ker(a^*)).$$

Now let us return to the case of a unital C^*-algebra $\mathcal{A}$ and a non-finite trace $\tau\colon \mathcal{A}^+ \to [0,\infty]$.

We need to recall some definitions:

(1) τ is said to be *lower semicontinuous* if, for all $t \in \mathbf{R}^+$, the set

$$\{a \in \mathcal{A}^+ \mid \tau(a) \leq t\}$$

is closed in $\mathcal{A}^+$.

(2) τ is *semi-finite* if, for each element $a \in \mathcal{A}^+$, we have

$$\tau(a) = \sup\{\tau(b) \mid b \in \mathcal{A}^+ \,\&\, b \leq a \,\&\, \tau(b) < +\infty\}.$$

Recall that a *trace representation* of a C^*-algebra $\mathcal{A}$ is a pair (π, tr) consisting of a nondegenerate representation $\pi \colon \mathcal{A} \to B(H)$ of $\mathcal{A}$ on a Hilbert space H and a faithful, normal trace tr on the von Neumann $\mathcal{R} = \pi(A)''$ such that $\pi(\mathcal{A}) \cap \mathcal{N}_{\mathrm{tr}}$ is weakly dense in $\mathcal{R}$ ($\mathcal{N}_{\mathrm{tr}} = \{a \in \mathcal{R} \mid \mathrm{tr}(a^*a) < +\infty\}$). These conditions imply tr is semifinite, so that $\mathcal{R}$ is semifinite. The function $\tau = \mathrm{tr} \circ \pi \colon \mathcal{A}^+ \to [0, +\infty]$ is then a lower semicontinuous, semifinite trace on $\mathcal{A}$. Conversely, every lower semicontinuous, semifinite trace on $\mathcal{A}$ arises from a trace representation in this way. See [7, Section 6.6] for details.

Our next result says that we can always realise our index theory relative to a trace on a C^*-algebra in the special setting of a semi-finite von Neumann algebra considered by Phillips and Raeburn, provided the trace is semi-finite and lower semicontinuous. Thus, the abstract Fredholm theory relative to the original trace can be given a concrete representation in terms of operators on a Hilbert space and the Fredholm elements then have kernels and co-kernels that are small in an appropriate sense, with the difference of their dimensions giving the index, as occurs in the classical theory of Fredholm operators.

Proposition 2.2 *Let τ be a non-finite, semifinite, lower semicontinuous trace on a unital C^*-algebra $\mathcal{A}$ and let (π, tr) be an associated trace representation (so $\tau(a) =$ $\mathrm{tr}(\pi(a))$, for all $a \in \mathcal{A}^+$). Let $\mathcal{R} = \pi(\mathcal{A})''$. If $a \in \mathcal{A}$ is a τ-Fredholm element, then $\pi(a)$ is a Fredholm element of $\mathcal{R}$ relative to* tr *and*

$$\mathrm{ind}_\tau(a) = \mathrm{ind}_{\mathrm{BPR}}(\pi(a)) = \mathrm{tr}(\ker \pi(a)) - \mathrm{tr}(\ker \pi(a^*)).$$

Proof It is clear that $\pi(\mathcal{K}_\tau) \subseteq \mathcal{K}_{\mathrm{tr}}$. It follows that if a is a τ-Fredholm element of $\mathcal{A}$, then $\pi(a)$ is an tr-Fredholm element of $\mathcal{R}$. Moreover, choosing (as we may) an element $b \in \mathcal{A}$ such that $1 - ab$ and $1 - ba$ belong to $\mathcal{M}_\tau$, then

$$\mathrm{ind}_\tau(a) = \tau(ab - ba) = \mathrm{tr}(\pi(a)\pi(b) - \pi(b)\pi(a)) = \mathrm{ind}_{\mathrm{tr}}(\pi(a)),$$

since $1 - \pi(a)\pi(b)$ and $1 - \pi(b)\pi(a)$ belong to $\mathcal{M}_{\mathrm{tr}}$. It now follows immediately from the preceding proposition that $\mathrm{ind}_\tau(a) = \mathrm{ind}_{\mathrm{BPR}}(\pi(a)) = \mathrm{tr}(\ker \pi(a)) -$ $\mathrm{tr}(\ker \pi(a^*))$, as required. ∎

There is an interesting, very simple, consequence of this representation. If $a \in \mathcal{A}$ is normal, then $\mathrm{ind}_\tau(a) = 0$, since $\mathrm{ind}_\tau(a) = \mathrm{tr}(\ker \pi(a)) - \mathrm{tr}(\ker \pi(a^*))$ and $\ker(\pi(a)) = \ker(\pi(a^*))$, because $\pi(a)$ is normal. I do not know how to prove $\mathrm{ind}_\tau(a) = 0$ directly (it may even be the case that it is not necessarily true if the trace τ does not satisfy the hypothesis of the theorem).

We finish this section by pointing out the connection of our index theory with the index coming from K-theory. Recall that if $\mathcal{K}$ is a closed ideal in a unital C^*-algebra

$\mathcal{A}$ and $\pi\colon \mathcal{A} \to \mathcal{C}$ is the canonical map onto the quotient C^*-algebra $\mathcal{C} = \mathcal{A}/\mathcal{K}$, the index map $\partial\colon K_1(\mathcal{C}) \to K_0(\mathcal{K})$ is the connecting homomorphism associated to the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{A} \to \mathcal{C} \to 0.$$

If Φ denotes the semigroup of elements in $\mathcal{A}$ that are invertible modulo $\mathcal{K}$, the map

$$\iota\colon \Phi \to K_0(K), \quad a \mapsto \partial[\pi(a)]$$

has many of the characteristics one expects of a Fredholm-type index, such as $\iota(ab) = \iota(a) + \iota(b)$ and $\iota(a + x) = \iota(a)$, for all $a, b \in \Phi$ and $x \in \mathcal{K}$. Now if $\tau\colon \mathcal{A}^+ \to [0, +\infty]$ is a non-finite trace on $\mathcal{A}$ whose domain of definition $\mathcal{M}_\tau$ is dense in $\mathcal{K}$, then our index ind_τ factors through the index ι. More precisely, if tr_* is the usual extension of tr to a homomorphism $\mathrm{tr}_*\colon K_0(\mathcal{K}) \to \mathbf{R}$, then $\mathrm{ind}_\tau(a) = \mathrm{tr}_*\, \iota(a)$, for all $a \in \Phi$.

This observation fits our index theory within the framework of K-theory, but does not appear to add any particular extra insight. The theory we have developed is, of course, much more elementary than the K-theory index ∂ and also more concrete, and does not require the (often difficult) computation of K-groups to calculate the index.

The referee has kindly pointed out one way in which this factorisation of the tracial index ind_τ through the K-theory index ∂ may be useful. It is known that ∂ may be non-zero on a normal element a. If one can find such an $a \in \mathcal{A}$ and a suitable non-finite trace such that $\mathrm{tr}_*\, \iota(a) \neq 0$, then we would have $\mathrm{ind}_\tau(a) \neq 0$. Although I know of no such example, this reasoning does seem to suggest that $\mathrm{ind}_\tau(a)$ is not always equal to zero for a normal.

3 Application to Toeplitz Operators on $\mathbf{U}(2)$

To give some applications of these ideas we need to consider some preliminary constructions involving traces. We do not attempt to achieve maximal generality here; rather, we prove results that are sufficient for our considerations.

Theorem 3.1 *Let $\mathcal{A}$ be a separable C^*-algebra whose closed commutator ideal $\mathcal{K}$ is proper. Then $\mathcal{A}$ admits a tracial state τ whose left kernel N_τ is equal to $\mathcal{K}$.*

Proof The quotient C^*-algebra $\mathcal{A}/\mathcal{K}$ is non-zero, separable and commutative. Hence, its character space is separable, admitting a dense sequence $(\tau_n)_{n=1}^{\infty}$, say. Then, if $b \in \mathcal{A}/\mathcal{K}$ and $\tau_n(b) = 0$, for all $n \geq 1$, we have $b = 0$.

Now let $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{K}$ be the quotient $*$-homomorphism and define a tracial positive linear functional $\tau\colon \mathcal{A} \to \mathbf{C}$ by setting

$$\tau(a) = \sum_{n=1}^{\infty} \tau_n\big(\pi(a)\big)/2^n,$$

for all $a \in \mathcal{A}$. Clearly, $\tau(a^*a) = 0$ if, and only if, $\tau_n\big(\pi(a^*a)\big) = 0$, for all $n \geq 1$ and this is equivalent to $\pi(a^*a) = 0$; that is, $a \in \mathcal{K}$. Thus, $N_\tau = \mathcal{K}$. By renormalising τ, we get a tracial state with the properties required by the theorem. ∎

Consider a C^*-algebra of the form $\mathcal{A} = K(H) \otimes \mathcal{B}$, where $K(H)$ denotes the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space H and $\mathcal{B}$ is a unital C^*-algebra, and the symbol $\otimes$ denotes the spatial (minimal) C^*-tensor product. If $(e_n)_{n=1}^{\infty}$ is an orthonormal basis for H, we denote by e_{mn} the element of $K(H)$ defined by setting

$$e_{mn}(x) = (x|e_n)e_m,$$

for all $x \in H$. It is well known that every element b in the multiplier algebra $M(\mathcal{A})$ can be written as a sum

$$b = \sum_{m,n=1}^{\infty} e_{mn} \otimes b_{mn},$$

for unique elements b_{mn} in $\mathcal{B}$. The sum converges in the strict topology.

Now suppose τ is a tracial state on $\mathcal{B}$. Then we define a trace $\hat{\tau}\colon M(\mathcal{A})^+ \to [0, +\infty]$ by setting

$$\hat{\tau}(a) = \sum_{m=1}^{\infty} \tau(a_{mm}),$$

for all $a \in M(\mathcal{A})^+$.

If T is a positive element of $K(H)$ and b a positive element of $\mathcal{B}$, then

$$\hat{\tau}(T \otimes b) = \operatorname{tr}(T)\tau(b),$$

where tr is the canonical trace on $K(H)$. For, in this case, $T = \sum_{mn=1}^{\infty} T_{mn}e_{mn}$, where $T_{mn} = (T(e_n)|e_m)$, and therefore

$$\hat{\tau}(T \otimes b) = \hat{\tau}\Big(\sum_{mn} e_{mn} \otimes T_{mn}b\Big) = \sum_{m=1}^{\infty} \tau(T_{mm}b) = \operatorname{tr}(T)\tau(b).$$

For this reason, we shall henceforth denote $\hat{\tau}$ by $\operatorname{tr} \otimes \tau$ and call it the *tensor product* of the traces tr and τ.

We are going to apply these ideas now to the Toeplitz algebra of the unitary matrix group $\mathbf{U}(2)$ of all unitary matrices of order 2. Let A denote the closed subalgebra of $C(\mathbf{U}(2))$ generated by the coordinate functions

$$Z_{ij}\colon \mathbf{U}(2) \to \mathbf{C}, \quad u \mapsto u_{ij},$$

for $i, j = 1, 2$. Let $L^2(\mathbf{U}(2))$ denote the L^2-space relative to the normalised Haar measure of $\mathbf{U}(2)$ and let $H^2(\mathbf{U}(2))$ be the closure of A in $L^2(\mathbf{U}(2))$. If P is the projection of $L^2(\mathbf{U}(2))$ on $H^2(\mathbf{U}(2))$ and $\varphi \in C(\mathbf{U}(2))$, we define the Toeplitz operator $T_\varphi \in B\big(H^2(\mathbf{U}(2))\big)$ by setting $T_\varphi(f) = P(\varphi f)$, for all $f \in H^2(\mathbf{U}(2))$. The Toeplitz algebra $\mathbf{A} = \mathbf{A}(\mathbf{U}(2))$ is the C^*-subalgebra of $B\big(H^2(\mathbf{U}(2))\big)$ generated by all T_φ, for $\varphi \in C(\mathbf{U}(2))$.

This algebra and the operators T_φ were studied intensively in [1], where some surprising results were obtained. It was shown there (see Theorems 20 and 22 of [1]) that although $\mathbf{A}$ contains Fredholm operators of arbitrary index, any Toeplitz operator T_φ that is Fredholm necessarily has zero index and the symbol φ is of the form e^ψ, for some function $\psi \in C\big(\mathbf{U}(2)\big)$. Thus, the usual Fredholm theory is useless for analysing the index theory of the operators T_φ. We shall show, however, that there is a certain trace on $\mathbf{A}$ relative to which the index theory of the T_φ is very satisfactory.

Our first step in developing this index theory is to exhibit the special trace referred to. For this we shall need to recall some results of [1]. It is shown in [1, Theorem 4] that there is faithful representation α of $\mathbf{A}$ on a Hilbert space tensor product $H_1 \otimes H_2$ that maps the closed commutator ideal $\mathbf{K}$ of $\mathbf{A}$ onto the C^*-tensor product $K(H_1) \otimes B$, where B is a certain separable unital C^*-subalgebra of $B(H_2)$ having $K(H_2)$ as its closed commutator ideal. Here H_1 and H_2 are the Hilbert spaces $H^2(\mathbf{T})$ and $L^2(\mathrm{SU}(2))$, respectively, where $H^2(\mathbf{T})$ is the usual Hardy Hilbert space on the unit circle $\mathbf{T}$ and $L^2(\mathrm{SU}(2))$ is the L^2-space of the special unitary group $\mathrm{SU}(2)$ of order 2. Although we shall not need it, let us note in passing that B is the C^*-algebra on $L^2(\mathrm{SU}(2))$ generated by all the pseudo-differential operators of order zero.

Since the closed commutator ideal $K(H_2)$ in B is proper (and B is separable), we may apply Theorem 3.1 to deduce that B admits a tracial state τ whose left kernel N_τ is equal to $K(H_2)$. The tensor product trace $\mathrm{tr} \otimes \tau$ on $M(K(H_1) \otimes B)$ then defines a trace $\sigma = (\mathrm{tr} \otimes \tau)\alpha$ on $\mathbf{A}$. Note in passing that it is clear that $\mathrm{tr} \otimes \tau$ vanishes on $K(H_1 \otimes H_2) = K(H_1) \otimes K(H_2)$, so that $\mathrm{tr} \otimes \tau$ is not the canonical trace on $K(H_1 \otimes H_2)$. Note that if T is a positive, finite-rank operator on H_1 and $b \in B^+$, then $(\mathrm{tr} \otimes \tau)(T \otimes b) = \mathrm{tr}(T)\tau(b) < +\infty$. It follows easily from this, and the fact that $\alpha\mathbf{K} = K(H_1) \otimes B$, that $\mathbf{K} \subseteq \mathcal{K}_\sigma$.

For the index theory we are developing we need $\mathbf{K} = \mathcal{K}_\sigma$ to get the most definitive result. However, it is not clear that this is true for the trace σ that we have constructed. A simple trick gets around this problem: We simply redefine σ so that it remains unchanged on $\mathbf{K}^+$ and we set $\sigma(a) = +\infty$, for all $a \in \mathbf{A}^+ \setminus \mathbf{K}^+$. One easily checks then that with this definition of σ, we have $\mathbf{K} = \mathcal{K}_\sigma$.

Now let $\Delta\colon \mathbf{U}(2) \to \mathbf{C}$ be the determinant function $Z_{11}Z_{22} - Z_{12}Z_{21}$. In [1, Theorem 3] it is shown that $\alpha(T_\Delta) = U \otimes 1$, where U is the unilateral shift on the standard orthonormal basis of H_1. Hence, $\alpha(T_\Delta)$ is invertible modulo the ideal $K(H_1) \otimes B$, since $T_\Delta^* T_\Delta = 1$ and $1 - \alpha(T_\Delta)\alpha(T_\Delta^*) = (1 - UU^*) \otimes 1$. Therefore, T_Δ is invertible modulo $\mathcal{K}_\sigma$. Also, $-\operatorname{ind}_\sigma(T_\Delta) = \sigma(1 - T_\Delta T_\Delta^*) = (\mathrm{tr} \otimes \tau)\big((1 - UU^*) \otimes 1\big) = \mathrm{tr}(1 - UU^*)\tau(1) = 1$; that is, $\operatorname{ind}_\sigma(T_\Delta) = -1$.

We now have sufficient preliminary material gathered to prove the following result:

Theorem 3.2 *Let σ be the trace on $\mathbf{A}A(\mathbf{U}(2))$ constructed above. Let $\varphi \in C(\mathbf{U}(2))$. Then the Toeplitz operator T_φ is Fredholm relative to σ if and only if φ never vanishes. In this case, the corresponding Fredholm index of T_φ is given by $\operatorname{ind}_\sigma(T_\varphi) = -n$, where n is the unique integer such that $\varphi = \Delta^n e^\psi$, for some continuous function $\psi \in C\big(\mathbf{U}(2)\big)$.*

Proof First recall a theorem of Van Kampen [13], building on a result of Bohr [2], that states that if φ is a non-vanishing continuous function on a compact, connected group G, then $\varphi = \chi e^\psi$, for a unique continuous character $\chi\colon G \to \mathbf{T}$ and for some continuous function $\psi\colon G \to \mathbf{C}$. Since the only continuous characters on

https://doi.org/10.4153/CMB-2005-023-1 Published online by Cambridge University Press

the connected compact group $\mathbf{U}(2)$ are the powers Δ^n of the determinant Δ (this is easily verified), we can state the Van Kampen–Bohr result in this case as follows: If $\varphi\colon \mathbf{U}(2) \to \mathbf{C}$ is continuous and never vanishes, then $\varphi = \Delta^n e^{\psi}$, for a unique integer n and some continuous function ψ on $\mathbf{U}(2)$.

Since $\mathcal{K}_\sigma$ is equal to the closed commutator ideal $\mathbf{K}$ of $\mathbf{A}$, if $\varphi \in C(\mathbf{U}(2))$, then T_φ is Fredholm relative to σ if, and only if, $T_\varphi + \mathbf{K}$ is invertible in $\mathbf{A}/\mathbf{K}$. However, the map

$$C\big(\mathbf{U}(2)\big) \to \mathbf{A}/\mathbf{K}, \quad \varphi \mapsto T_\varphi + \mathbf{K},$$

is a $*$-isomorphism [1, Theorem 15]. Hence, T_φ is Fredholm relative to σ if, and only if, φ is invertible in $C(\mathbf{U}(2))$; that is, φ never vanishes.

Now suppose that φ does not vanish and write $\varphi = \Delta^n e^{\psi}$, for some integer n and some function $\psi \in C(\mathbf{U}(2))$. To complete the proof of the theorem we have only to show that $\operatorname{ind}_\sigma(T_\varphi) = -n$. In showing this, we may assume $n \geq 0$ (otherwise replace φ by $\bar{\varphi}$). Then, $\operatorname{ind}_\sigma(T_\varphi) = \operatorname{ind}_\sigma(T_{e^\psi} T_\Delta^n)$, since $\Delta \in A$ (it is easily checked that for any $f \in C(\mathbf{U}(2))$ and $g \in A$, $T_{fg} = T_f T_g$). Hence, $\operatorname{ind}_\sigma(T_\varphi) = \operatorname{ind}_\sigma(T_{e^\psi}) + n \operatorname{ind}_\sigma(T_\Delta) = \operatorname{ind}_\sigma(T_{e^\psi}) - n$, since we saw already that $\operatorname{ind}_\sigma(T_\Delta) = -1$. We have now only to show that $\operatorname{ind}_\sigma(T_{e^\psi}) = 0$, but this is obvious from continuity and local constancy of the index, since the map $t \mapsto \operatorname{ind}_\sigma(T_{e^{t\psi}})$ is a continuous locally constant function on the (connected) closed interval [0,1] whose value at $t = 0$ is $\operatorname{ind}_\sigma(1) = 0$. ∎

If $\varphi\colon \mathbf{T} \to \mathbf{C}$ is a non-vanishing continuous function, then $\varphi = z^n e^{\psi}$, for a unique integer n, where $z\colon \mathbf{T} \to \mathbf{C}$ is the inclusion function and $\psi\colon \mathbf{T} \to \mathbf{C}$ is some continuous function. Clearly, $n = \operatorname{wn}(\varphi)$, the winding number of φ around the origin. The classical Gohberg–Krein index theorem asserts that a Toeplitz operator T_φ on the usual Hardy space $H^2(\mathbf{T})$ of the circle, with continuous symbol φ, is a Fredholm operator (in the usual sense) if and only if φ never vanishes, and in this case $\operatorname{ind}(T_\varphi) = -\operatorname{wn}(\varphi)$ (see [8, Theorem 3.5.15]). This result is a prototype for the Atiyah–Singer index theorem, in which an analytic (Fredholm) index of a pseudodifferential operator is equated to a topological index of its symbol. We can cast Theorem 3.2 into this format also if we define the topological index $\operatorname{ind}(\varphi)$ of a non-vanishing continuous function $\varphi\colon \mathbf{U}(2) \to \mathbf{C}$ to be the unique integer n such that $\varphi = \Delta^n e^{\psi}$, for some continuous function $\psi\colon \mathbf{U}(2) \to \mathbf{C}$. We then get the following:

Theorem 3.3 *Let σ be the trace on $\mathbf{A}(\mathbf{U}(2))$ constructed above. Let $\varphi \in C(\mathbf{U}(2))$. Then the Toeplitz operator T_φ is Fredholm relative to σ if, and only if, φ never vanishes. In this case, the corresponding Fredholm index of T_φ is given by $\operatorname{ind}_\sigma(T_\varphi) = -\operatorname{ind}(\varphi)$.*

Note that although, *a priori*, there is no reason to suppose that the index $\operatorname{ind}_\sigma$ on $\mathbf{A}(\mathbf{U}(2))$ is integer-valued, this is, in fact, the case. This follows easily from the fact that $\operatorname{ind}_\sigma$ is integer-valued on the Toeplitz operators. For, if T is an element of $\mathbf{A}(\mathbf{U}(2))$, then we may write it in the form $T = T_\varphi + K$, for some element $\varphi \in C(\mathbf{U}(2))$ and some element $K \in \mathbf{K}$ [1, Theorem 15]. Hence, if T is σ-Fredholm, so is T_φ and $\operatorname{ind}_\sigma(T) = \operatorname{ind}_\sigma(T_\varphi)$ and the latter is an integer.

The question of extending the index result of the preceding theorem to Toeplitz operators on more general Hardy spaces (such as those considered in [9]) is a challenging one for the future. Two principal ingredients are needed: the existence of an appropriate trace on the Toeplitz algebra and an appropriate concept of topological index for the symbol of a Toeplitz operator. It is not at all clear that one or other ingredient will be found in generality nor is it clear that if they are, an index theorem can be proved. Nevertheless, the ideas of this section should provide some grounds for optimism and also some pointers for tackling this very interesting question. (The index theory developed in [10] also provides grounds for optimism.)

References

[1] C. A. Berger and L. A. Coburn, *Wiener–Hopf operators on U_2*. J. Integral Equations Operat. Theory **2**(1979), 139–173.

[2] H. Bohr, *Über die Argumentvariation einer fastperiodischen Funktion*. Danske Vid. Selsk. **10**(1930), 10.

[3] M. Breuer, *Fredholm theories in von Neumann algebras I*. Math. Ann. **178**(1968), 243–254.

[4] ———, *Fredholm theories in von Neumann algebras II*. Math. Ann. **180**(1969), 313–325.

[5] L. A. Coburn, R. G. Douglas, D. Schaeffer and I. M. Singer, *C^*-algebras of operators on a half-space II. Index theory*. Inst. Hautes Études Sci. Publ. Math. **40**(1971), 69–79.

[6] A. Connes, *Noncommutative Geometry*. Academic Press, San Diego, 1994.

[7] J. Dixmier, *C^*-Algebras*. North Holland, Amsterdam, 1982.

[8] G. J. Murphy, *C^*-algebras and Operator Theory*. Academic Press, San Diego, 1990.

[9] ———, *Toeplitz operators on generalised H^2 spaces*. J. Integral Equations Operat. Theory **15**(1992), 825–852.

[10] ———, *An index theorem for Toeplitz operators*. J. Operator Theory **29**(1993), 97–114.

[11] ———, *Fredholm index and the trace*. Proc. Royal Irish Acad. **94**(1994), 161–166.

[12] J. Phillips and I. Raeburn, *An index theorem for Toeplitz operators with noncommutative symbol space*. J. Funct. Anal. **120**(1994), 239–263.

[13] E. Van Kampen, *On almost periodic functions of constant absolute value*. J. London Math. Soc. **12**(1937), 3–6.

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https://doi.org/10.4153/CMB-2005-023-1 Published online by Cambridge University Press